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BERWALD AND DOUGLAS SPACES OF A FINSLER SPACE WITH AN EXPONENTIAL FORM OF (α, β)- METRIC

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Abstract. In the present paper, we have undertaken a study of Berwald space and Douglas space in a Finsler space with exponential form of (α, β)-metric. We have examined the conditions under which this metric will be a Berwald and Douglas space.

1. Introduction

We consider an n-dimensional Finsler space $F^n = (M^n, L)$, i.e., a pair consisting of an n-dimensional differentiable manifold M^n equipped with a fundamental function $L(x^i, y^i)$.

Definition 1.1. A Finsler space F^n of dimension n is a differentiable manifold such that the length s of the curve $x^i(t)$ of F^n is defined by $s = \int L(x^i, y^i)dt$, where $y^i = \frac{dx^i}{dt}$ is a tangent vector. The fundamental function $L(x^i, y^i) = L(x^1, ...x^n, y^1, ...y^n)$ is supposed to be differentiable on manifold M^n and satisfy the following conditions:-
(i) Positively homogeneous: $L(x^i, py^i) = pL(x^i, y^i)$, $p > 0$.
(ii) Positive: $L(x^i, y^i) > 0$, $y^i \neq 0$ on differentiable manifold $(M^n - 0)$.
(iii)Positive definite metric $g_{ij} = \frac{1}{2}\frac{\partial^2 L}{\partial y^i \partial y^j} > 0$.

The interesting and significant examples of an (α, β)- metrics are Randers metric $(\alpha + \beta)$, Kropina metric $\frac{\alpha^2}{\beta}$ and Matsumoto metric [9] $\frac{\alpha^2}{(\alpha - \beta)}$. The notion of an (α, β)-metric was introduced by M. Matsumoto [8] and has been studied by many authors [3, 13, 16].

Definition 1.2. A Finsler metric $L(\alpha, \beta)$ in a differentiable manifold M^n is called an (α, β)-metric if L is a positively homogeneous function of degree one of a Riemannian metric $\alpha = (a_{ij}(x)y^i y^j)^{\frac{1}{2}}$ and a one-form $\beta = b_i(x)y^i$ on M^n.

In 1928, L. Berwald [4] defined an affinely connected Finsler space if the connection coefficients G^i_{jk} of Berwald connection $B\Gamma$ are functions of position coordinate x^i alone with respect to the linear connection and defined the equation of geodesics, which is given by

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$$\frac{d^2x^i}{ds^2} + G^i_{jk}(x)\frac{dx^j}{ds}\frac{dx^k}{ds} = 0.$$

Definition 1.3. A Finsler space is called a Berwald space if the connection coefficients G^i_{jk} of BΓ are functions of position x^i alone in any coordinate system.

The notion of a Douglas space was introduced by S. Bacso and M. Matsumoto [3] as a generalization of a Berwald space from the point of view of geodesic equations. It is remarkable that a Finsler space is a Douglas space if and only if the Douglas tensor vanishes identically. Afterwards, the conditions for Finsler spaces to be Berwald space and Douglas space with different special kind of (α,β)-metrics have been studied by many authors [5, 12, 13, 17].

In 2020, Tripathi [15] considered a special type of Finsler exponential (α,β)-metric and studied the basic properties of Finsler space with this significant metric and various hypersurfaces with this metric. The present paper is devoted to study the condition for a Finsler space with the (α,β)-metric $L = \alpha e^{\frac{\beta}{\alpha}} + \beta e^{-\frac{\beta}{\alpha}}$ to be a Berwald space and Douglas space.

2. Preliminaries

In this paper we consider an n-dimensional Finsler space $F^n = \{M^n, L(\alpha,\beta)\}$, that is, a pair consisting of an n-dimensional differentiable manifold M^n equipped with a Fundamental function L as a special Finsler Space with the metric

$$L(\alpha,\beta) = \alpha e^{\frac{\beta}{\alpha}} + \beta e^{-\frac{\beta}{\alpha}}, \tag{1}$$

where $\alpha = \sqrt{\alpha_{ij}(x)y^iy^j}$ is a Riemannian metric and $\beta = b_i(x)y^i$ a differential one form β.
Differentiating equation (1) partially with respect to α and β are given by

$$(2)\begin{cases} L_\alpha = \dfrac{\alpha^2 e^{\frac{\beta}{\alpha}} - \alpha\beta e^{\frac{\beta}{\alpha}} + \beta^2 e^{\frac{-\beta}{\alpha}}}{\alpha^2}, \quad L_\beta = \dfrac{\alpha e^{\frac{\beta}{\alpha}} + \alpha e^{\frac{-\beta}{\alpha}} - \beta e^{\frac{-\beta}{\alpha}}}{\alpha}, \\ L_{\alpha\alpha} = \dfrac{\alpha\beta^2 e^{\frac{\beta}{\alpha}} - 2\alpha\beta^2 e^{\frac{-\beta}{\alpha}} + \beta^3 e^{\frac{-\beta}{\alpha}}}{\alpha^4}, \quad L_{\beta\beta} = \dfrac{\alpha e^{\frac{\beta}{\alpha}} - 2\alpha e^{\frac{-\beta}{\alpha}} + \beta e^{\frac{-\beta}{\alpha}}}{\alpha^2}, \\ L_{\alpha\beta} = \dfrac{-\alpha\beta e^{\frac{\beta}{\alpha}} + 2\alpha\beta e^{\frac{-\beta}{\alpha}} - \beta^2 e^{\frac{-\beta}{\alpha}}}{\alpha^3}, \quad L_{\beta\alpha} = \dfrac{-\alpha\beta e^{\frac{\beta}{\alpha}} + 2\alpha\beta e^{\frac{-\beta}{\alpha}} - \beta^2 e^{\frac{-\beta}{\alpha}}}{\alpha^3}, \\ L_{\alpha\alpha\alpha} = \dfrac{-3\alpha^2\beta^2 e^{\frac{\beta}{\alpha}} - \alpha\beta^3 e^{\frac{\beta}{\alpha}} + 6\alpha^2\beta^2 e^{\frac{-\beta}{\alpha}} - 6\alpha\beta^3 e^{\frac{-\beta}{\alpha}} + \beta^4 e^{\frac{-\beta}{\alpha}}}{\alpha^6}, \\ L_{\alpha\alpha\beta} = \dfrac{2\alpha^2\beta e^{\frac{\beta}{\alpha}} + \alpha\beta^2 e^{\frac{\beta}{\alpha}} - 4\alpha^2\beta e^{\frac{-\beta}{\alpha}} + 5\alpha\beta^2 e^{\frac{-\beta}{\alpha}} - \beta^3 e^{\frac{-\beta}{\alpha}}}{\alpha^5}. \end{cases}$$

Since an exponential form of Finsler (α,β)-metric (1) is a positive homogeneous function of degree one in α and β. Therefore from Eulers theorem of homogeneous functions, we determine the following relations from (2).

$$(3)\quad \begin{cases} L_\alpha\alpha + L_\beta\beta = L, L_{\alpha\alpha}\alpha + L_{\alpha\beta}\beta = 0, \\ L_{\beta\alpha}\alpha + L_{\beta\beta}\beta = 0, L_{\alpha\alpha\alpha}\alpha + L_{\alpha\alpha\beta}\beta = -L_{\alpha\alpha}. \end{cases}$$

On the other hand, the geodesics of a Finsler space $F^n=(M^n,L)$ are given by the system of differential equations including the function

$$4G^i = g^{ij}(y^r\dot{\partial}_j\partial_r L^2 - \partial_j L^2),$$

where $G^i = g^{ij}G_j$ is a positive homogeneous in y^i of degree two and the fundamental tensor is given by $g_{ij}=\frac{1}{2}\partial_i\partial_j L^2$.

The covariant differentiation with respect to the Levi-Civita connection $\gamma^i_{jk}(x)$ of the associated Riemannian space $R^n=(M^n,\alpha)$ of Finsler space with $F^n=\{M^n,L(\alpha,\beta)\}$ ([1], [10]) is denoted by the symbol ;. We put $(a^{ij})=(a_{ij})^{-1}$, and use the symbols as follows:

$$r_{ij}=\frac{1}{2}(b_{i;j}+b_{j;i}),\ s_{ij}=\frac{1}{2}(b_{i;j}-b_{j;i}),\ r^i_j=a^{ir}r_{rj},\ s^i_j=a^{ir}s_{rj},\ r_j=b_r r^r_j,$$
$$s_j=b_r s^r_j,\ b^i=a^{ir}b_r,\ b^2=a^{rs}b_r b_s.$$

According to [11], if $\beta^2 L_\alpha+\alpha\gamma^2 L_{\alpha\alpha}\neq 0$, where $\gamma^2=b^2\alpha^2-\beta^2$, then the function $G^i(x,y)$ of F^n with an (α,β)-metric is written in the form

$$2G^i=\gamma^i_{00}+2B^i,$$

$$B^i=\frac{\alpha L_\beta}{L_\alpha}s^i_0+C^*\left\{\frac{\beta L_\beta}{\alpha L}y^i-\frac{\alpha L_{\alpha\alpha}}{L_\alpha}\left(\frac{1}{\alpha}y^i-\frac{\alpha}{\beta}b^i\right)\right\},\tag{4}$$

where $L_\alpha=\frac{\partial L}{\partial\alpha}$, $L_\beta=\frac{\partial L}{\partial\beta}$, $L_{\alpha\alpha}=\frac{\partial^2 L}{\partial\alpha\partial\alpha}$, the subscipt 0 means the contraction by y^i and we put

$$C^*=\frac{\alpha\beta(r_{00}L_\alpha-2s_0\alpha L_\beta)}{2(\beta^2L_\alpha+\alpha\gamma^2L_{\alpha\alpha})}.\tag{5}$$

Note : We shall denote the homogeneous polynomial in y^i of degree r by the symbol $hp(r)$ for brevity. For example, γ^i_{00} is a homogeneous polynomial in y^i of degree 2 by the symbol $hp(2)$.

From the equation (5) the Berwald connection $B\Gamma=(G^i_{jk},G^i_j,0)$ of F^n with an (α,β)-metric is given by

$$G^i_j=\dot{\partial}_j G^i=\gamma^i_{0j}+B^i_j,$$
$$G^i_{jk}=\dot{\partial}_k G^i_j=\gamma^i_{jk}+B^i_{jk},$$

where we put $B^i_j=\dot{\partial}_j B^i$ and $B^i_{jk}=\dot{\partial}_k B^i_j$. $B^i(x,y)$ is called the *difference vector* [11]. On account of [11], B^i_{jk} is determined by

$$L_\alpha B^t_{ji}y^jy_t+\alpha L_\beta(B^t_{ji}b_t-b_{j;i})y^j=0,\tag{6}$$

where $y_k=a_{ik}y^i$.

A Finsler space F^n with an (α,β)-metric is a Douglas space if and only if $B^{ij}=B^iy^j-B^jy^i$ is $hp(3)$ [3]. From latter of (5) B^{ij} is written as follows:

$$B^{ij}=\frac{\alpha L_\beta}{L_\alpha}(s^i_0y^j-s^j_0y^i)+\frac{\alpha^2L_{\alpha\alpha}}{\beta L_\alpha}C^*(b^iy^j-b^jy^i).\tag{7}$$

Lemma 2.1. [2] *If $\alpha^2\equiv 0(mod\beta)$, that is, $a_{ij}(x)y^iy^j$ contains $b_i(x)y^i$ as a factor, then the dimension is equal to two and b^2 vanishes. In this case we have $\delta=d_i(x)y^i$ which satisfies $\alpha^2=\beta\delta$ and $d_ib^i=2$.*

3. The condition to be a Berward space

In this section, we have obtained the condition for a Finsler space F^n with an exponential (α, β)-metric $L(\alpha, \beta) = \alpha e^{\frac{\beta}{\alpha}} + \beta e^{-\frac{\beta}{\alpha}}$ to be a Berwald space.

From equations (2) and (6), we have

$$(\alpha^2 e^{\frac{\beta}{\alpha}} - \alpha\beta e^{\frac{\beta}{\alpha}} + \beta^2 e^{\frac{-\beta}{\alpha}})B^t_{ji}y^j y_t + \alpha^2(\alpha e^{\frac{\beta}{\alpha}} + \alpha e^{\frac{-\beta}{\alpha}} - \beta e^{\frac{-\beta}{\alpha}})(B^t_{ji}b_t - b_{j;i})y^j = 0. \tag{8}$$

Suppose that F^n is a Berwald space, i.e., $B^i_{jk} = B^i_{jk}(x)$. Separating (8) into rational and irrational parts of y^i as

$$(\alpha^2 e^{\frac{\beta}{\alpha}} + \beta^2 e^{\frac{-\beta}{\alpha}})B^t_{ji}y^j y_t - (\alpha^2\beta e^{\frac{-\beta}{\alpha}})(B^t_{ji}b_t - b_{j:i})y^j = 0, \tag{9}$$

and

$$(-\beta e^{\frac{\beta}{\alpha}})B^t_{ji}y^j y_t + (\alpha^2 e^{\frac{\beta}{\alpha}} + \alpha^2 e^{\frac{-\beta}{\alpha}})(B^t_{ji}b_t - b_{j:i})y^j = 0. \tag{10}$$

Equations (9) and (10) can be expressed as a matrix of homogeneous linear equations: AX=0,

$$\begin{bmatrix} \alpha^2 e^{\frac{\beta}{\alpha}} + \beta^2 e^{\frac{-\beta}{\alpha}} & -\alpha^2\beta e^{\frac{-\beta}{\alpha}} \\ -\beta e^{\frac{\beta}{\alpha}} & \alpha^2 e^{\frac{-\beta}{\alpha}} + \alpha^2 e^{\frac{\beta}{\alpha}} \end{bmatrix} \begin{bmatrix} B^t_{ji}y^j y_t \\ (B^t_{ji}b_t - b_{j;i})y^j \end{bmatrix} = 0. \tag{11}$$

where

$$A = \begin{bmatrix} \alpha^2 e^{\frac{\beta}{\alpha}} + \beta^2 e^{\frac{-\beta}{\alpha}} & -\alpha^2\beta e^{\frac{-\beta}{\alpha}} \\ -\beta e^{\frac{\beta}{\alpha}} & \alpha^2 e^{\frac{-\beta}{\alpha}} + \alpha^2 e^{\frac{\beta}{\alpha}} \end{bmatrix} \tag{12}$$

and

$$|A| = \alpha^4 e^{\frac{2\beta}{\alpha}} + \alpha^4 + \alpha^2\beta^2 e^{\frac{-2\beta}{\alpha}} \neq 0. \tag{13}$$

Which implies

$$B^t_{ji}y^j y_t = 0, \tag{14}$$

and

$$(B^t_{ji}b_t - b_{j;i})y^j = 0. \tag{15}$$

Contracting equation (15) y_t, we have

$$(B^t_{ji}b_t - b_{j;i})y^j y_t = 0, \tag{16}$$

$$\begin{aligned} &\Rightarrow B^t_{ji}b_t y^j y_t - b_{j;i}y^j y_t = 0 \\ &\Rightarrow (B^t_{ji}y^j y_t)b_t - b_{j;i}y^j y_t = 0 \end{aligned} \tag{17}$$

Using equation (14) in (17), we have

$$b_{j;i} = 0.$$

Conversely, if $b_{j;i} = 0$ in equation (15), then we have

$$B^t_{ji} = 0.$$

Thus, we have

THEOREM 3.1. *A Finsler space with an exponential (α,β)-metric $L(\alpha,\beta) = \alpha e^{\frac{\beta}{\alpha}} + \beta e^{-\frac{\beta}{\alpha}}$ is a Berwald space if and only if $b_{i;j} = 0$.*

4. The condition to be a Douglas space

The present section is devoted to find the condition for a Finsler space F^n with an exponential (α,β)-metric (1) to be a Douglas space. In Finsler space F^n with metric (1), we have

$$C^* = \tfrac{\alpha^2\beta}{2}\{\tfrac{W}{\Omega}\},$$

where
$W = r_{00}(\alpha^2 e^{\frac{\beta}{\alpha}} - \alpha\beta e^{\frac{\beta}{\alpha}} + \beta^2 e^{\frac{-\beta}{\alpha}}) - 2S_0\alpha^2(\alpha e^{\frac{\beta}{\alpha}} + \alpha e^{\frac{-\beta}{\alpha}} - \beta e^{\frac{-\beta}{\alpha}})$,
$\Omega = \alpha\beta^2(\alpha^2 e^{\frac{\beta}{\alpha}} - \alpha\beta e^{\frac{\beta}{\alpha}} + \beta^2 e^{\frac{-\beta}{\alpha}}) + \alpha^2 b^2(\alpha\beta^2 e^{\frac{\beta}{\alpha}} - 2\alpha\beta^2 e^{\frac{-\beta}{\alpha}} + \beta^3 e^{\frac{-\beta}{\alpha}})$
$- \beta^2(\alpha\beta^2 e^{\frac{\beta}{\alpha}} - 2\alpha\beta^2 e^{\frac{-\beta}{\alpha}} + \beta^3 e^{\frac{-\beta}{\alpha}}) \neq 0.$

In view of [11], if $\Omega = \beta^2 L_\alpha + \alpha\gamma^2 L_{\alpha\alpha} \neq 0$, then the function $G^i(x,y)$ of F^n with an (α,β)-metric is written in the form

$$2G^i = \gamma^i_{00} + 2B^i,$$

where

$$B^i = \frac{\alpha L_\beta s^i_0}{L_\alpha} + C^*\left\{\frac{\beta L_\beta y^i}{\alpha L} - \frac{\alpha L_{\alpha\alpha}}{L_\alpha}(\frac{y^i}{\alpha} - \frac{\alpha b^i}{\beta})\right\}.$$

The vector $B^i(x,y)$ is called the difference vector. Hence B^{ij} is written as

$$B^{ij} = \frac{\alpha L_\beta}{L_\alpha}(s^i_0 y^j - s^j_0 y^i) + \frac{\alpha^2 L_{\alpha\alpha}}{\beta L_\alpha}C^*(b^i y^j - b^j y^i). \tag{18}$$

Substituting (2) in above equation, we get

$$\begin{cases}
(2\alpha^5\beta^3 e^{\frac{2\beta}{\alpha}} - 4\alpha^4\beta^4 e^{\frac{2\beta}{\alpha}} + 8\alpha^3\beta^5 + 2\alpha^5\beta^3 b^2 e^{\frac{2\beta}{\alpha}} - 4\alpha^5\beta^3 b^2 + 6\alpha^4\beta^4 b^2\\
-10\alpha^2\beta^6 - 2\alpha^4\beta^4 b^2 e^{\frac{2\beta}{\alpha}} + 2\alpha^2\beta^6 e^{\frac{2\beta}{\alpha}} + 6\alpha\beta^7 e^{\frac{-2\beta}{\alpha}} - 4\alpha^3\beta^5 b^2 e^{\frac{-2\beta}{\alpha}}\\
+2\alpha^2\beta^6 b^2 e^{\frac{-2\beta}{\alpha}} - 2\beta^8 e^{\frac{-2\beta}{\alpha}})B^{ij} + (-2\alpha^6\beta^3 e^{\frac{2\beta}{\alpha}} + 2\alpha^5\beta^4 e^{\frac{2\beta}{\alpha}}\\
-6\alpha^4\beta^5 - 2\alpha^6\beta^3 b^2 e^{\frac{2\beta}{\alpha}} + 2\alpha^6\beta^3 b^2 + 2\alpha^4\beta^5 e^{\frac{2\beta}{\alpha}} - 2\alpha^6\beta^3 + 4\alpha^5\beta^4\\
-2\alpha^4\beta^5 e^{\frac{-2\beta}{\alpha}} + 4\alpha^6\beta^3 b^2 e^{\frac{-2\beta}{\alpha}} - 6\alpha^5\beta^4 b^2 e^{\frac{-2\beta}{\alpha}} - 4\alpha^4\beta^5 e^{\frac{-2\beta}{\alpha}}\\
+8\alpha^3\beta^6 e^{\frac{-2\beta}{\alpha}} + 2\alpha^4\beta^5 b^2 e^{\frac{-2\beta}{\alpha}} - 2\alpha^2\beta^7 e^{\frac{-2\beta}{\alpha}})(S^i_0 y^j - S^j_0 y^i)\\
+r_{00}(-\alpha^5\beta^3 e^{\frac{2\beta}{\alpha}} + \alpha^4\beta^4 e^{\frac{2\beta}{\alpha}} + 2\alpha^5\beta^3 - 3\alpha^4\beta^4\\
+2\alpha^3\beta^5 e^{\frac{-2\beta}{\alpha}} - \alpha^2\beta^6 e^{\frac{-2\beta}{\alpha}}) + s_0(2\alpha^6\beta^3 e^{\frac{2\beta}{\alpha}}\\
-2\alpha^6\beta^3 - 4\alpha^6\beta^3 e^{\frac{-2\beta}{\alpha}} + 6\alpha^5\beta^4 e^{\frac{-2\beta}{\alpha}}\\
-2\alpha^4\beta^5 e^{\frac{-2\beta}{\alpha}})(b^i y^j - b^j y^i) = 0.
\end{cases} \tag{19}$$

If F^n is a Douglas space, then B^{ij} are hp(8). Separating the rational and irrational terms in equation (19), we can write as

$$(20)\quad \begin{cases} (-4\alpha^4\beta^4e^{\frac{2\beta}{\alpha}}+6\alpha^4\beta^4b^2-10\alpha^2\beta^6-2\alpha^4\beta^4b^2e^{\frac{2\beta}{\alpha}}+2\alpha^2\beta^6e^{\frac{2\beta}{\alpha}} \\ +2\alpha^2\beta^6b^2e^{\frac{-2\beta}{\alpha}}-2\beta^8e^{\frac{-2\beta}{\alpha}})B^{ij}+(-2\alpha^6\beta^3e^{\frac{2\beta}{\alpha}}-6\alpha^4\beta^5 \\ -2\alpha^6\beta^3b^2e^{\frac{2\beta}{\alpha}}+2\alpha^6\beta^3b^2+2\alpha^4\beta^5e^{\frac{2\beta}{\alpha}}-2\alpha^6\beta^3-2\alpha^4\beta^5e^{\frac{-2\beta}{\alpha}} \\ +4\alpha^6\beta^3b^2e^{\frac{-2\beta}{\alpha}}-4\alpha^4\beta^5e^{\frac{-2\beta}{\alpha}}+2\alpha^4\beta^5b^2e^{\frac{-2\beta}{\alpha}}-2\alpha^2\beta^7e^{\frac{-2\beta}{\alpha}}) \\ \times(s^i_0y^j-s^j_0y^i)+r_{00}(\alpha^4\beta^4e^{\frac{2\beta}{\alpha}}-3\alpha^4\beta^4-\alpha^2\beta^6e^{\frac{-2\beta}{\alpha}}) \\ +s_0(2\alpha^6\beta^3e^{\frac{2\beta}{\alpha}}-2\alpha^6\beta^3-4\alpha^6\beta^3e^{\frac{-2\beta}{\alpha}}-2\alpha^4\beta^5e^{\frac{-2\beta}{\alpha}}) \\ \times(b^iy^j-b^jy^i)=0. \end{cases}$$

and

$$(21)\quad \begin{cases} (2\alpha^4\beta^3e^{\frac{2\beta}{\alpha}}+8\alpha^2\beta^5+2\alpha^4\beta^3b^2e^{\frac{2\beta}{\alpha}}-4\alpha^4\beta^3b^2+6\beta^7e^{\frac{-2\beta}{\alpha}} \\ -4\alpha^2\beta^5b^2e^{\frac{-2\beta}{\alpha}})B^{ij}+(2\alpha^4\beta^4e^{\frac{2\beta}{\alpha}}+4\alpha^4\beta^4-6\alpha^4\beta^4b^2e^{\frac{-2\beta}{\alpha}} \\ +8\alpha^2\beta^6e^{\frac{-2\beta}{\alpha}})(S^i_0y^j-S^j_0y^i)+r_{00}(-\alpha^4\beta^3e^{\frac{2\beta}{\alpha}}+2\alpha^4\beta^3 \\ +2\alpha^2\beta^5e^{\frac{-2\beta}{\alpha}})+6s_0\alpha^4\beta^4e^{\frac{-2\beta}{\alpha}})(b^iy^j-b^jy^i)=0. \end{cases}$$

Eliminating B^{ij} from equations (20) and (21), we have

$$(22)\quad A(s^i_0y^j-s^j_0y^i)+B(b^iy^j-b^jy^i)=0,$$

where

$$(23)\quad \begin{cases} A=(2\alpha^4\beta^3e^{\frac{2\beta}{\alpha}}+8\alpha^2\beta^5+2\alpha^4\beta^3b^2e^{\frac{2\beta}{\alpha}}-4\alpha^4\beta^3b^2+6\beta^7e^{\frac{-2\beta}{\alpha}} \\ -4\alpha^2\beta^5b^2e^{\frac{-2\beta}{\alpha}})\times(-2\alpha^4\beta^3e^{\frac{2\beta}{\alpha}}-6\alpha^2\beta^5-2\alpha^4\beta^3b^2e^{\frac{2\beta}{\alpha}} \\ +2\alpha^4\beta^3b^2+2\alpha^2\beta^5e^{\frac{2\beta}{\alpha}}-2\alpha^4\beta^3-2\alpha^2\beta^5e^{\frac{-2\beta}{\alpha}}+4\alpha^4\beta^3b^2e^{\frac{-2\beta}{\alpha}} \\ -4\alpha^2\beta^5e^{\frac{-2\beta}{\alpha}}+2\alpha^2\beta^5b^2e^{\frac{-2\beta}{\alpha}}-2\beta^7e^{\frac{-2\beta}{\alpha}})-(-4\alpha^4\beta^4e^{\frac{2\beta}{\alpha}} \\ +6\alpha^4\beta^4b^2-10\alpha^2\beta^6-2\alpha^4\beta^4b^2e^{\frac{2\beta}{\alpha}}+2\alpha^2\beta^6e^{\frac{2\beta}{\alpha}} \\ +2\alpha^2\beta^6b^2e^{\frac{-2\beta}{\alpha}}-2\beta^8e^{\frac{-2\beta}{\alpha}})\times(2\alpha^2\beta^4e^{\frac{2\beta}{\alpha}} \\ +4\alpha^2\beta^4-6\alpha^2\beta^4b^2e^{\frac{-2\beta}{\alpha}}+8\beta^6e^{\frac{-2\beta}{\alpha}}), \\ B=(2\alpha^4\beta^3e^{\frac{2\beta}{\alpha}}+8\alpha^2\beta^5+2\alpha^4\beta^3b^2e^{\frac{2\beta}{\alpha}}-4\alpha^4\beta^3b^2+6\beta^7e^{\frac{-2\beta}{\alpha}} \\ -4\alpha^2\beta^5b^2e^{\frac{-2\beta}{\alpha}})\times r_{00}(\alpha^2\beta^4e^{\frac{2\beta}{\alpha}}-3\alpha^2\beta^4-\beta^6e^{\frac{-2\beta}{\alpha}}) \\ +s_0(2\alpha^4\beta^3e^{\frac{2\beta}{\alpha}}-2\alpha^4\beta^3-4\alpha^4\beta^3e^{\frac{-2\beta}{\alpha}}-2\alpha^2\beta^5e^{\frac{-2\beta}{\alpha}}) \\ -(-4\alpha^4\beta^4e^{\frac{2\beta}{\alpha}}+6\alpha^4\beta^4b^2-10\alpha^2\beta^6-2\alpha^4\beta^4b^2e^{\frac{2\beta}{\alpha}}+2\alpha^2\beta^6e^{\frac{2\beta}{\alpha}} \\ +2\alpha^2\beta^6b^2e^{\frac{-2\beta}{\alpha}}-2\beta^8e^{\frac{-2\beta}{\alpha}})\times\{r_{00(-}\alpha^2\beta^3e^{\frac{2\beta}{\alpha}}+2\alpha^2\beta^3 \\ +2\beta^5e^{\frac{-2\beta}{\alpha}})+6s_0\alpha^2\beta^4e^{\frac{-2\beta}{\alpha}})\}. \end{cases}$$

Contraction of (22) by b_iy_j leads to

$$(24)\quad As_0\alpha^2+B(b^2\alpha^2-\beta^2)=0.$$

The term of (24) which seemingly does not contain α^2 is $2\beta^{15}r_{00}$. Hence we have $hp(15)v_{15}$ such that

$$(25)\quad r_{00}\beta^{15}=\alpha^2v_{15}.$$

Here, we divide our discussion into following the three cases:
(1) $v_{15}=0$,
(2) $v_{15}\neq 0, \alpha^2 \not\equiv 0 \pmod{\beta}$,

(3) $v_{15}\neq 0, \alpha^2\cong 0 \pmod{\beta}$.

Case 1: In case of $V_{15}=0$; from(25), $r_{00}=0$, and (24) is reduced to

$$s_0\{A+B_1(b^2\alpha^2-\beta^2)\}=0, \tag{26}$$

where

$$\begin{cases} B_1=(2\alpha^4\beta^3e^{\frac{2\beta}{\alpha}}+8\alpha^2\beta^5+2\alpha^4\beta^3b^2e^{\frac{2\beta}{\alpha}}-4\alpha^4\beta^3b^2+6\beta^7e^{\frac{-2\beta}{\alpha}} \\ -4\alpha^2\beta^5b^2e^{\frac{-2\beta}{\alpha}})\times(2\alpha^2\beta^3e^{\frac{2\beta}{\alpha}}-2\alpha^2\beta^3-4\alpha^2\beta^3e^{\frac{-2\beta}{\alpha}}-2\beta^5e^{\frac{-2\beta}{\alpha}}) \\ -((-4\alpha^4\beta^4e^{\frac{2\beta}{\alpha}}+6\alpha^4\beta^4b^2-10\alpha^2\beta^6-2\alpha^4\beta^4b^2e^{\frac{2\beta}{\alpha}} \\ +2\alpha^2\beta^6e^{\frac{2\beta}{\alpha}}+2\alpha^2\beta^6b^2e^{\frac{-2\beta}{\alpha}}-2\beta^8e^{\frac{-2\beta}{\alpha}})\times(6\beta^4e^{\frac{-2\beta}{\alpha}}). \end{cases} \tag{27}$$

If $A+B_1(b^2\alpha^2-\beta^2)=0$ in equation (26), then the term of this equation that does not contain α^2 is $4\beta^{14}$. Therefore, there exists $hp(12)v_{12}$ such that $4\beta^{14}=\alpha^2v_{12}$.

In this case, if $\alpha^2\not\cong 0 \pmod{\beta}$, then we have $v_{12}=0$, which leads to a contradiction. Therefore $A+B_1(b^2\alpha^2-\beta^2)\neq 0$. If $\alpha^2\cong 0 \pmod{\beta}$, then $A+B_1(b^2\alpha^2-\beta^2)=0$ is written as following,

$$\begin{cases} [(2\delta^2e^{\frac{2\beta}{\alpha}}+8\beta\delta+6\beta^2e^{\frac{-2\beta}{\alpha}})\times(-2\delta^2e^{\frac{2\beta}{\alpha}}-6\beta\delta+2\beta\delta e^{\frac{2\beta}{\alpha}} \\ -2\delta^2-2\delta\beta e^{\frac{-2\beta}{\alpha}}-4\delta\beta e^{\frac{-2\beta}{\alpha}}-2\beta^2e^{\frac{-2\beta}{\alpha}})-(-4\beta\delta^2e^{\frac{2\beta}{\alpha}} \\ -10\beta^2\delta+2\beta^2\delta e^{\frac{2\beta}{\alpha}}-2\beta^3e^{\frac{-2\beta}{\alpha}})\times(2\delta e^{\frac{2\beta}{\alpha}}+4\delta+8\beta e^{\frac{-2\beta}{\alpha}})] \\ -[(2\delta^2e^{\frac{2\beta}{\alpha}}+8\beta\delta+6\beta^2e^{\frac{-2\beta}{\alpha}})\times(2\delta\beta e^{\frac{2\beta}{\alpha}}-2\beta\delta-4\beta\delta e^{\frac{-2\beta}{\alpha}} \\ -2\beta^2e^{\frac{-2\beta}{\alpha}})-(-4\delta^2e^{\frac{2\beta}{\alpha}}-10\beta\delta+2\beta\delta e^{\frac{2\beta}{\alpha}}-2\beta^2e^{\frac{-2\beta}{\alpha}}) \\ \times(6\beta^2e^{\frac{-2\beta}{\alpha}})]=0. \end{cases} \tag{28}$$

This is provided that $b^2=0$ and $\alpha^2=\beta\delta$ by Lemma (2.1). The term of (28), which does not contain β, is $-8\delta^4$. Therefore there exists $hp(3)v_3$ such that $-8\delta^4=\beta v_3$, which leads to a contradiction. Hence $A+B_1(b^2\alpha^2-\beta^2)\neq 0$.
Therefore, in any case of $\alpha^2\not\cong 0 \pmod{\beta}$ or $\alpha^2\cong 0 \pmod{\beta}$, we see that $A+B_1(b^2\alpha^2-\beta^2)\neq 0$.Thus $s_0=0$ from (26). Put $s_0=0$ and $r_{00}=0$ in (22), we have

$$A(s^i_0y^j-s^j_0y^i)=0. \tag{29}$$

If A=0, then we have from (23)

$$\begin{cases} A=(2\alpha^4\beta^3e^{\frac{2\beta}{\alpha}}+8\alpha^2\beta^5+2\alpha^4\beta^3b^2e^{\frac{2\beta}{\alpha}}-4\alpha^4\beta^3b^2+6\beta^7e^{\frac{-2\beta}{\alpha}} \\ -4\alpha^2\beta^5b^2e^{\frac{-2\beta}{\alpha}})\times(-2\alpha^4\beta^3e^{\frac{2\beta}{\alpha}}-6\alpha^2\beta^5-2\alpha^4\beta^3b^2e^{\frac{2\beta}{\alpha}}+2\alpha^4\beta^3b^2 \\ +2\alpha^2\beta^5e^{\frac{2\beta}{\alpha}}-2\alpha^4\beta^3-2\alpha^2\beta^5e^{\frac{-2\beta}{\alpha}}+4\alpha^4\beta^3b^2e^{\frac{-2\beta}{\alpha}}-4\alpha^2\beta^5e^{\frac{-2\beta}{\alpha}} \\ +2\alpha^2\beta^5b^2e^{\frac{-2\beta}{\alpha}}-2\beta^7e^{\frac{-2\beta}{\alpha}})-(-4\alpha^4\beta^4e^{\frac{2\beta}{\alpha}}+6\alpha^4\beta^4b^2-10\alpha^2\beta^6 \\ -2\alpha^4\beta^4b^2e^{\frac{2\beta}{\alpha}}+2\alpha^2\beta^6e^{\frac{2\beta}{\alpha}}+2\alpha^2\beta^6b^2e^{\frac{-2\beta}{\alpha}}-2\beta^8e^{\frac{-2\beta}{\alpha}})\times \\ (2\alpha^2\beta^4e^{\frac{2\beta}{\alpha}}+4\alpha^2\beta^4-6\alpha^2\beta^4b^2e^{\frac{-2\beta}{\alpha}}+8\beta^6e^{\frac{-2\beta}{\alpha}})=0. \end{cases} \tag{30}$$

The term of (29) that does not contain α^2 is $4\beta^{14}$. Thus, there exists $hp(12)v_{12}$ such that $4\beta^{14}=\alpha^2v_{12}$. From this equation, we have $v_{12}=0$, which leads to a contradiction. So $A\neq 0$. Thus we have from (29)

$$s^i_0y^j-s^j_0y^i=0. \tag{31}$$

Transvection of (31) by y_j gives $s^i_0 = 0$. Finally $r_{ij} = s_{ij} = 0$ are concluded, that is, $b_{i;j} = 0$.

Case 2 : In case of $v_{15} \neq 0, \alpha^2 \not\cong 0 \pmod{\beta}$: In this case,(26) shows that there exists a function $h = h(x)$ satisfying

$$r_{00} = h\alpha^2. \tag{32}$$

Substituting (32) in (24), we have

$$(33)\quad \begin{cases} As_0 + [(2\alpha^4\beta^3 e^{\frac{2\beta}{\alpha}} + 8\alpha^2\beta^5 + 2\alpha^4\beta^3 b^2 e^{\frac{2\beta}{\alpha}} - 4\alpha^4\beta^3 b^2 + 6\beta^7 e^{\frac{-2\beta}{\alpha}} \\ -4\alpha^2\beta^5 b^2 e^{\frac{-2\beta}{\alpha}}) \times (h\alpha^2\beta^4 e^{\frac{2\beta}{\alpha}} - 3h\alpha^2\beta^4 - h\beta^6 e^{\frac{-2\beta}{\alpha}} + 2s_0\alpha^2\beta^3 e^{\frac{2\beta}{\alpha}} \\ -2s_0\alpha^2\beta^3 - 4s_0\alpha^2\beta^3 e^{\frac{-2\beta}{\alpha}} - 2s_0\beta^5 e^{\frac{-2\beta}{\alpha}}) - (-4\alpha^4\beta^4 e^{\frac{2\beta}{\alpha}} + 6\alpha^4\beta^4 b^2 \\ -10\alpha^2\beta^6 - 2\alpha^4\beta^4 b^2 e^{\frac{2\beta}{\alpha}} + 2\alpha^2\beta^6 e^{\frac{2\beta}{\alpha}} + 2\alpha^2\beta^6 b^2 e^{\frac{-2\beta}{\alpha}} - 2\beta^8 e^{\frac{-2\beta}{\alpha}}) \\ (-h\alpha^2\beta^3 e^{\frac{2\beta}{\alpha}} + 2h\alpha^2\beta^3 + 2h\beta^5 e^{\frac{-2\beta}{\alpha}} + 6s_0\beta^4 e^{\frac{-2\beta}{\alpha}})](b^2\alpha^2 - \beta^2) = 0. \end{cases}$$

The terms of (33) that do not contain α^2 are $(4s_0 + 2\beta h)\beta^{14}$. Hence there exists $hp(13)v_{13}$ such that $(4s_0 + 2\beta h)\beta^{14} = \alpha^2 v_{13}$. Since $\alpha \not\cong 0 \pmod{\beta}$, we must have $v_{13} = 0$. Thus we have

$$(4s_0 + 2\beta h)\beta^{14} = 0, \tag{34}$$

which implies $4s_i + 2b_i h = 0$. Transvecting this equation by b^i, we obtain $hb^2 = 0$. If $b^2 = 0$, we get from (24) and (34)

$$(35)\quad \begin{cases} -h/2[(2\alpha^4 e^{\frac{2\beta}{\alpha}} + 8\alpha^2\beta^2 + 6\beta^4 e^{\frac{-2\beta}{\alpha}}) \times (-2\alpha^4 e^{\frac{2\beta}{\alpha}} - 6\alpha^2\beta^2 \\ +2\alpha^2\beta^2 e^{\frac{2\beta}{\alpha}} - 2\alpha^4 - 2\alpha^2\beta^2 e^{\frac{-2\beta}{\alpha}} - 4\alpha^2\beta^2 e^{\frac{-2\beta}{\alpha}} - 2\beta^4 e^{\frac{-2\beta}{\alpha}}) \\ -(-4\alpha^4\beta e^{\frac{2\beta}{\alpha}} - 10\alpha^2\beta^3 + 2\alpha^2\beta^3 e^{\frac{2\beta}{\alpha}} - 2\beta^5 e^{\frac{-2\beta}{\alpha}}) \times \\ (2\alpha^2\beta e^{\frac{2\beta}{\alpha}} + 4\alpha^2\beta + 8\beta^3 e^{\frac{-2\beta}{\alpha}})] - [(2\alpha^4 e^{\frac{2\beta}{\alpha}} + 8\alpha^2\beta^2 + 6\beta^4 e^{\frac{-2\beta}{\alpha}}) \times \\ (h\alpha^2\beta^2 e^{\frac{2\beta}{\alpha}} - 3h\alpha^2\beta^2 - h\beta^4 e^{\frac{-2\beta}{\alpha}} - h\alpha^2\beta^2 e^{\frac{2\beta}{\alpha}} + h\alpha^2\beta^2 + 2h\alpha^2\beta^2 e^{\frac{-2\beta}{\alpha}} \\ +h\beta^4 e^{\frac{-2\beta}{\alpha}}) - ((-4\alpha^4\beta^2 e^{\frac{2\beta}{\alpha}} - 10\alpha^2\beta^4 + 2\alpha^2\beta^4 e^{\frac{2\beta}{\alpha}} - 2\beta^6 e^{\frac{-2\beta}{\alpha}}) \\ \times(-h\alpha^2 e^{\frac{2\beta}{\alpha}} + 2h\alpha^2 + 2h\beta^2 e^{\frac{-2\beta}{\alpha}} - 3h\beta^2 e^{\frac{-2\beta}{\alpha}})] = 0. \end{cases}$$

The term of (35) that does not contain β is $4h\alpha^8$. Therefore, there exists $hp(7)v_7$ such that $4h\alpha^8 = \beta v_7$. Since $\alpha^2 \not\cong 0 \pmod{\beta}$, we have v_7=0, which leads to a contradiction. Thus we have h=0 and hence $s_0 = 0, r_{00} = 0$ from (34) and (32) respectively. Therefore (22) is reduced to $A(s^i_0 y^j - s^j_0 y^i) = 0$.
Since A$\neq$ 0, we have $s^i_0 y^j - s^j_0 y^i = 0$. Transvection of this equation by y_j gives $s^i_0 = 0$. Finally $r_{ij} = s_{ij} = 0$ are concluded, that is, $b_{i;j}$=0.
Case 3 In case of $v_{15} \neq 0, \alpha^2 \cong 0 \pmod{\beta}$: In this case,Lemma (2.1) shows that n=2, $b^2 = 0$ and $\alpha^2 = \beta\delta, \delta = d_i(x)y^i$. Therefore (22) gives

$$A'(s^i_0 y^j - s^j_0 y^i) + B'(b^i y^j - b^j y^i) = 0, \tag{36}$$

where

$$(37)\begin{cases} A' = [(2\delta^2 e^{\frac{2\beta}{\alpha}} + 8\beta\delta + 6\beta^2 e^{\frac{-2\beta}{\alpha}}) \times (-2\delta^2 e^{\frac{2\beta}{\alpha}} - 6\beta\delta + 2\beta\delta e^{\frac{2\beta}{\alpha}} - 2\delta^2 \\ -2\delta\beta e^{\frac{-2\beta}{\alpha}} - 4\delta\beta e^{\frac{-2\beta}{\alpha}} - 2\beta^2 e^{\frac{-2\beta}{\alpha}}) - (-4\beta\delta^2 e^{\frac{2\beta}{\alpha}} - 10\beta^2\delta + 2\beta^2\delta e^{\frac{2\beta}{\alpha}} \\ -2\beta^3 e^{\frac{-2\beta}{\alpha}}) \times (2\delta e^{\frac{2\beta}{\alpha}} + 4\delta + 8\beta e^{\frac{-2\beta}{\alpha}})], \\ B' = r_{00}[(2\delta^2 e^{\frac{2\beta}{\alpha}} + 8\beta\delta + 6\beta^2 e^{\frac{-2\beta}{\alpha}}) \times (\delta e^{\frac{2\beta}{\alpha}} - 3\delta - \beta e^{\frac{-2\beta}{\alpha}}) \\ -((-4\delta^2 e^{\frac{2\beta}{\alpha}} - 10\beta\delta + 2\beta\delta e^{\frac{2\beta}{\alpha}} - 2\beta^2 e^{\frac{-2\beta}{\alpha}}) \times (-\delta e^{\frac{2\beta}{\alpha}} + 2\delta + 2\beta e^{\frac{-2\beta}{\alpha}})] \\ +s_0[(2\delta^2 e^{\frac{2\beta}{\alpha}} + 8\beta\delta + 6\beta^2 e^{\frac{-2\beta}{\alpha}}) \times (2\delta^2 e^{\frac{2\beta}{\alpha}} - 2\delta^2 - 4\delta^2 e^{\frac{-2\beta}{\alpha}} - 2\beta\delta e^{\frac{-2\beta}{\alpha}}) \\ -(6\delta e^{\frac{-2\beta}{\alpha}}) \times (-4\delta^2\beta e^{\frac{2\beta}{\alpha}} - 10\beta^2\delta + 2\beta^2\delta e^{\frac{2\beta}{\alpha}} - 2\beta^3 e^{\frac{-2\beta}{\alpha}})]. \end{cases}$$

Transvection of (36) by $b_i y_j$ leads to

$$(38)\quad\begin{cases} s_0\delta[(2\delta^2 e^{\frac{2\beta}{\alpha}} + 8\beta\delta + 6\beta^2 e^{\frac{-2\beta}{\alpha}}) \times (-2\delta^2 e^{\frac{2\beta}{\alpha}} - 6\beta\delta + 2\beta\delta e^{\frac{2\beta}{\alpha}} \\ -2\delta^2 - 2\delta\beta e^{\frac{-2\beta}{\alpha}} - 4\delta\beta e^{\frac{-2\beta}{\alpha}} - 2\beta^2 e^{\frac{-2\beta}{\alpha}}) - (-4\beta\delta^2 e^{\frac{2\beta}{\alpha}} \\ -10\beta^2\delta + 2\beta^2\delta e^{\frac{2\beta}{\alpha}} - 2\beta^3 e^{\frac{-2\beta}{\alpha}}) \times (2\delta e^{\frac{2\beta}{\alpha}} + 4\delta \\ +8\beta e^{\frac{-2\beta}{\alpha}})] - \beta r_{00}[(2\delta^2 e^{\frac{2\beta}{\alpha}} + 8\beta\delta + 6\beta^2 e^{\frac{-2\beta}{\alpha}}) \\ \times(\delta e^{\frac{2\beta}{\alpha}} - 3\delta - \beta e^{\frac{-2\beta}{\alpha}}) - ((-4\delta^2 e^{\frac{2\beta}{\alpha}} - 10\beta\delta + 2\beta\delta e^{\frac{2\beta}{\alpha}} \\ -2\beta^2 e^{\frac{-2\beta}{\alpha}}) \times (-\delta e^{\frac{2\beta}{\alpha}} + 2\delta + 2\beta e^{\frac{-2\beta}{\alpha}})] + s_0[(2\delta^2 e^{\frac{2\beta}{\alpha}} + 8\beta\delta \\ +6\beta^2 e^{\frac{-2\beta}{\alpha}}) \times (2\delta^2 e^{\frac{2\beta}{\alpha}} - 2\delta^2 - 4\delta^2 e^{\frac{-2\beta}{\alpha}} - 2\beta\delta e^{\frac{-2\beta}{\alpha}}) \\ -(6\delta e^{\frac{-2\beta}{\alpha}}) \times (-4\delta^2\beta e^{\frac{2\beta}{\alpha}} - 10\beta^2\delta + 2\beta^2\delta e^{\frac{2\beta}{\alpha}} - 2\beta^3 e^{\frac{-2\beta}{\alpha}})] = 0. \end{cases}$$

The term of (38), which does not contain β is $-8\delta^5 s_0$. Thus there exists hp(5) v_5 such that $-8s_0\delta^5 = \beta v_5$, which implies $s_0 = k\beta$, where there exists a function k=k(x).

On the other hand,(25) gives $r_{00}\beta^{14} = \delta v_{15}$, which must be reduced to

$$(39)\qquad r_{00} = \delta v_1,$$

where v_1 is a hp(1). Substituting $s_0 = k\beta$ and (39) in (38), we have

$$(40)\quad\begin{cases} K\delta[(2\delta^2 e^{\frac{2\beta}{\alpha}} + 8\beta\delta + 6\beta^2 e^{\frac{-2\beta}{\alpha}}) \times (-2\delta^2 e^{\frac{2\beta}{\alpha}} - 6\beta\delta + 2\beta\delta e^{\frac{2\beta}{\alpha}} \\ -2\delta^2 - 2\delta\beta e^{\frac{-2\beta}{\alpha}} - 4\delta\beta e^{\frac{-2\beta}{\alpha}} - 2\beta^2 e^{\frac{-2\beta}{\alpha}}) - (-4\beta\delta^2 e^{\frac{2\beta}{\alpha}} \\ -10\beta^2\delta + 2\beta^2\delta e^{\frac{2\beta}{\alpha}} - 2\beta^3 e^{\frac{-2\beta}{\alpha}}) \times (2\delta e^{\frac{2\beta}{\alpha}} + 4\delta + 8\beta e^{\frac{-2\beta}{\alpha}})] \\ -\delta v_1[(2\delta^2 e^{\frac{2\beta}{\alpha}} + 8\beta\delta + 6\beta^2 e^{\frac{-2\beta}{\alpha}}) \times (\delta e^{\frac{2\beta}{\alpha}} - 3\delta - \beta e^{\frac{-2\beta}{\alpha}}) \\ -((-4\delta^2 e^{\frac{2\beta}{\alpha}} - 10\beta\delta + 2\beta\delta e^{\frac{2\beta}{\alpha}} - 2\beta^2 e^{\frac{-2\beta}{\alpha}})\times \\ (-8e^{\frac{2\beta}{\alpha}} + 2\delta + 2\beta e^{\frac{-2\beta}{\alpha}})] + k\beta[(2\delta^2 e^{\frac{2\beta}{\alpha}} + 8\beta\delta + 6\beta^2 e^{\frac{-2\beta}{\alpha}}) \\ \times(2\delta^2 e^{\frac{2\beta}{\alpha}} - 2\delta^2 - 4\delta^2 e^{\frac{-2\beta}{\alpha}} - 2\beta\delta e^{\frac{-2\beta}{\alpha}}) - (6\delta e^{\frac{-2\beta}{\alpha}}) \\ \times(-4\delta^2\beta e^{\frac{2\beta}{\alpha}} - 10\beta^2\delta + 2\beta^2\delta e^{\frac{2\beta}{\alpha}} - 2\beta^3 e^{\frac{-2\beta}{\alpha}})] = 0. \end{cases}$$

The term of (40), which does not contain β is $-8\delta^5 k$. Therefore, there exists hp(4) v_4 such that $-8\delta^5 k = \beta v_4$, which leads to a contradiction. Thus we have $k = 0, s_0 = 0$ and $r_{00} = 0$. From (36), we get

$$(41)\qquad A'(s^i_0 y^j - s^j_0 y^i) = 0.$$

If

$$(42)\quad \begin{cases} A' = [(2\delta^2 e^{\frac{2\beta}{\alpha}} + 8\beta\delta + 6\beta^2 e^{\frac{-2\beta}{\alpha}}) \times (-2\delta^2 e^{\frac{2\beta}{\alpha}} - 6\beta\delta + 2\beta\delta e^{\frac{2\beta}{\alpha}} - 2\delta^2 \\ -2\delta\beta e^{\frac{-2\beta}{\alpha}} - 4\delta\beta e^{\frac{-2\beta}{\alpha}} - 2\beta^2 e^{\frac{-2\beta}{\alpha}}) - (-4\beta\delta^2 e^{\frac{2\beta}{\alpha}} - 10\beta^2\delta + 2\beta^2\delta e^{\frac{2\beta}{\alpha}} \\ -2\beta^3 e^{\frac{-2\beta}{\alpha}}) \times (2\delta e^{\frac{2\beta}{\alpha}} + 4\delta + 8\beta e^{\frac{-2\beta}{\alpha}})] = 0. \end{cases}$$

The term of (42), which does not contain β is $-8\delta^4$. Therefore, there exists hp(3) v_3 such that $-8\delta^4 k = \beta v_3$, which leads to a contradiction. Thus $A' \neq 0$ in (42). Hence we get $s^i_0 y^j - s^j_0 y^i = 0$. Transvection of this equation by y_j gives $s^i_0 = 0$. Thus $s_{ij} = r_{ij} = 0$ are concluded, that is, $b_{i;j} = 0$.
Conversely if $b_{i;j} = 0$, then we obtain $B^{ij} = 0$ from (7). Hence F^n is a Douglas space. Consequently we have

Theorem 4.1. *An n-dimensional Finsler space F^n with an exponential (α, β)-metric $L(\alpha, \beta) = \alpha e^{\frac{\beta}{\alpha}} + \beta e^{-\frac{\beta}{\alpha}}$ is a Douglas space, if and only if*

1. $\alpha^2 \not\equiv 0(mod\beta) : b_{j;i} = 0$.
2. $\alpha^2 \equiv 0(mod\beta) : n = 2$ and $b_{j;i} = 0$, where $\alpha^2 = \beta\delta$, $\delta = d_i(x)y^i$ and $k = k(x)$.

Theorem 4.2. *If an n-dimensional Finsler space F^n with an exponential (α, β)-metric $L(\alpha, \beta) = \alpha e^{\frac{\beta}{\alpha}} + \beta e^{-\frac{\beta}{\alpha}}$ is a Douglas space, then F^n is a Berwald space.*

5. Conclusion

In the present paper, we have studied Berwald space and Douglas space of a Finsler space with an exponential form of (α, β)-metric such as $L(\alpha, \beta) = \alpha e^{\frac{\beta}{\alpha}} + \beta e^{-\frac{\beta}{\alpha}}$, and examined with various conditions for this metric will be a Berwald and Douglas space. In the future work, we can also find the condition for weakly Berwald/Landsberg spaces and results related to conformal change in the Finsler space F^n with an exponential (α, β)-metric, which is defined in equation (1).

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