

SOME RESULTS RELATED TO NON-DEGENERATE LINEAR TRANSFORMATIONS ON EUCLIDEAN JORDAN ALGEBRAS

K. SARAVANAN*, V. PIRAMANANTHAM, AND R. THEIVARAMAN

ABSTRACT. This article deals with non-degenerate linear transformations on Euclidean Jordan algebras. First, we study non-degenerate for cone invariant, copositive, Lyapunov-like, and relaxation transformations. Further, we study that the non-degenerate is invariant under principal pivotal transformations and algebraic automorphisms.

1. Introduction

A matrix $A \in \mathbb{R}^{n \times n}$ is non-degenerate if every principal minors of A is nonzero. It is equivalent to say that $u * (Au) = 0$ implies $u = 0$ (see [8]). The class of non-degenerate matrix has been well studied in literature due to wide applications in linear complementarity problems (see, for example [7]). From given element $q \in \mathbb{R}^n$ and matrix $A \in \mathbb{R}^{n \times n}$, the problem is to find an element $u \in \mathbb{R}^n$ such that

$$u \geq 0, v = Au + q \geq 0, \langle u, v \rangle = 0.$$

This problem is called the standard linear complementarity problem, $LCP(A, q)$. The solution set in the standard linear complementarity problem [1] is finite if and only if the corresponding matrix is non-degenerate. Different types of matrices have been studied for the existence and uniqueness of solutions of the linear complementarity problem in the literature (see [1]). In the literature linear complementarity problem associated with Euclidean Jordan algebra over a symmetric cone has been studied for the past fifteen years (see [3, 5, 6]).

In this article, we continue our study in the circumstance of Euclidean Jordan algebras about the non-degenerate linear transformations. We focus on copositive transformations, Lyapunov-like transformations, automorphism invariance and relaxation transformations in particular. Especially, our contributions of this paper include:

1. Showing that strict copositivity, R_0 property and cone non-degenerate transformations are equivalent for cone invariant transformations.
2. Proving that cone non-degenerate transformations are equivalent to non-degenerate transformations for Lyapunov-like transformation.

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* Corresponding author.

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3. Studying that the non-degenerate for some special linear transformations.

A summary of the paper is provided here. Section 2 provides a quick overview of Euclidean Jordan algebras. Section 3 examines the cone non-degenerate for copositive transformations, cone invariant transformations, and the Lyapunov-like transformation. In section 4, we discuss some special linear transformation about non-degenerate.

2. Preliminaries

We review a few Euclidean Jordan algebraic notions, properties, and results within this section. Most of these can be found in [3, 4, 6].

DEFINITION 2.1. A Finite dimensional inner product space $(V, \langle \cdot, \cdot \rangle)$ over \mathbb{R} is said to be a Euclidean Jordan algebra if there is a bilinear map from the cartesian product $V \times V$ into V represented by $(u, v) \mapsto u \circ v$ such that it satisfies the requirements given below:

- (i) $u \circ v = v \circ u$ for all $u, v \in V$.
- (ii) $u \circ (u^2 \circ v) = u^2 \circ (u \circ v)$ for all $u, v \in V$.
- (iii) $\langle u \circ v, w \rangle = \langle v, u \circ w \rangle$ for all $u, v, w \in V$.

REMARK 2.2. Suppose, if there exist an element $e \in V$ such that $u \circ e = u$ for all $u \in V$, then we say that e is an unit element of V . In the above definition, the product $u \circ v$ is referred to as Jordan product in V . In V , the set of squares $K := \{u \circ u : u \in V\}$ is a symmetric cone [4]. Also, for any element $u \in V$, we write $u \geq 0$ if and only if $u \in K$. For any element $u \in V$, we write $u = u^+ - u^-$, $u^+, u^- \geq 0$ and $u^+ \circ u^- = 0$.

DEFINITION 2.3. An element $d \in V$ is said to an *idempotent* if $d^2 = d$. We say that a nonzero element $d \in V$ is a *primitive idempotent* if d cannot be represented as the sum of two nonzero idempotents.

REMARK 2.4. (i) A *Jordan frame* in V is a collection $\{e_1, e_2, \dots, e_s\}$ of primitive idempotents having the following properties

$$e_k \circ e_l = 0 \text{ for } k \neq l, \text{ and } \sum_{k=1}^s e_k = e.$$

(ii) The Spectral Decomposition (refer to [4]): For any $u \in V$, we can find a real numbers $\lambda_1, \dots, \lambda_s$ and $\{e_1, e_2, \dots, e_s\}$ as a Jordan frame in V such that

$$u = \lambda_1 e_1 + \dots + \lambda_s e_s.$$

In this case the representation $\lambda_1 e_1 + \dots + \lambda_s e_s$ is called the spectral decomposition of u , where $\lambda_i \in \mathbb{R}$ is an eigenvalue of u . Throughout this paper, we fix a Jordan frame $\{e_1, e_2, \dots, e_s\}$ otherwise stated.

(iii) For any $u \in V$, we can define a linear map $L_u : V \rightarrow V$ as $L_u(w) = u \circ w$. If $L_u L_v = L_v L_u$, then we say u and v operator commute. It is understood that u operator commute with v if and only if u and v have their spectral decompositions with respect to a common Jordan frame ([4], Lemma X.2.2).

Some standard examples are listed below.

EXAMPLE 2.5. Consider \mathbb{R}^n with the Jordan product and usual inner product respectively which are represented as follows:

$$u \circ v = u * v \text{ and } \langle u, v \rangle = \sum_{i=1}^n u_i v_i,$$

where $u * v$ denotes the component wise product of u and v . Then \mathbb{R}^n is a Euclidean Jordan algebra with symmetric cone is \mathbb{R}_+^n , the nonnegative orthant.

EXAMPLE 2.6. Let $S^n = \{U \in \mathbb{R}^{n \times n} \mid U = U^T\}$. Then, the Euclidean Jordan algebra is the set S^n associated with the Jordan product and inner product respectively given by

$$U \circ V := \frac{1}{2}(UV + VU) \text{ and } \langle U, V \rangle := \text{trace}(UV).$$

Moreover, the related symmetric cone is S_+^n , the set of all $n \times n$ positive semi-definite matrices.

Now, we recall the proposition from [3] as follows:

PROPOSITION 2.7. Consider for any $u, v \in V$. Then, the following conditions are equivalent:

- (i) $u \geq 0, v \geq 0$, and $\langle u, v \rangle = 0$
- (ii) $u \geq 0, v \geq 0$, and $u \circ v = 0$.

In any case, u and v operator commute.

Throughout this article, we always take V to be a Euclidean Jordan algebra and K to be the associated symmetric cone. Now, we recall some definitions as follows.

DEFINITION 2.8. Let $L : V \rightarrow V$ be a linear transformation. We say that L is/has

- (i) Strictly copositive on K if $\langle u, L(u) \rangle > 0$ for all $0 \neq u \in K$.
- (ii) Copositive on K if $\langle u, L(u) \rangle \geq 0$ for all $u \in K$.
- (iii) Non-degenerate if $u \in V$

$$u \text{ operator commutes with } L(u) \text{ and } u \circ L(u) = 0 \implies u = 0.$$

- (iv) Cone non-degenerate if

$$u \in K, u \text{ operator commutes with } L(u) \text{ and } u \circ L(u) = 0 \implies u = 0.$$

- (v) The strictly semi-monotone(SSM) property if

$$u \in K, u \text{ operator commutes with } L(u) \text{ and } u \circ L(u) \leq 0 \implies u = 0.$$

- (vi) The R_0 property if

$$u \in K, L(u) \in K \text{ and } u \circ L(u) = 0 \implies u = 0.$$

- (vii) The globally unique solvable(GUS) property on K if $\text{LCP}(L, q)$ has only one solution for all $q \in K$;

- (viii) Lyapunov-like transformation if

$$u, v \in K, \text{ and } \langle u, v \rangle = 0 \implies \langle L(u), v \rangle = 0.$$

- (ix) Cone invariant if $L(K) \subseteq K$.

3. Main Results

In this section, we study a non-degenerate transformations on Euclidean Jordan Algebras. This section is divided into two parts. The first one deals with non-degenerate transformations for cone invariant and copositive linear transformations, and the other one deals with non-degenerate on Lyapunov-like transformation; both are related to the Euclidean Jordan algebras.

3.1. Non-degenerate transformations for cone invariant and copositive linear transformations: In this subsection, we first characterize the cone non-degenerate transformations for cone invariant transformations.

THEOREM 3.1. *Consider a linear transformation $L : V \rightarrow V$ which is cone invariant. Then the following three statements are equivalent:*

- (i) L is cone non-degenerate.
- (ii) L is strictly copositive on K .
- (iii) L has the R_0 property.

Proof. (i) \Rightarrow (ii): Suppose L is not strictly copositive on K , then we can find a nonzero element $u \in K$ such that $\langle u, L(u) \rangle \leq 0$. However, since $u \geq 0$, $L(u) \geq 0$, we can rule out $\langle u, L(u) \rangle < 0$. By proposition 1, $u \circ L(u) = 0$ and u and $L(u)$ operator commute. By item (i) $u = 0$, which is not possible.

(ii) \Rightarrow (i): Let $u \in K$ such that u operator commutes with $L(u)$ and $u \circ L(u) = 0$. Then the Jordan frame will exist as $\{e_1, e_2, \dots, e_s\}$ such that

$$u = \sum_{i=1}^s u_i e_i \text{ and } L(u) = \sum_{i=1}^s v_i e_i.$$

From $u \circ L(u) = 0$, we have $\sum_{i=1}^s u_i v_i e_i = 0$. This implies that $u_i v_i = 0$ for all i . Now it is enough to prove that $u = 0$. If $u \neq 0$, then $\langle u, L(u) \rangle > 0$ as L is strictly copositive on K . This indicates that

$$0 < \langle u, L(u) \rangle = \sum_{i=1}^s u_i v_i \|e_i\|^2 = 0$$

which is not possible. Therefore $u = 0$.

(ii) \iff (iii): It follows from proposition 3.1, [9]. □

The following theorem provides that the SSM property is equivalent to the GUS on K and cone non-degenerate for copositive transformation.

THEOREM 3.2. *Let $L : V \rightarrow V$ be a linear transformation that is copositive on K . Then the following are equivalent:*

- (i) L is cone non-degenerate.
- (ii) L has the SSM property.
- (iii) L has the GUS property on K .

Proof. (i)⇒(ii): Let us take an element $u \geq 0$ such that $u \circ L(u) \leq 0$ and u operator commutes with $L(u)$. Since u and $L(u)$ operator commute and $u \geq 0, L(u) \geq 0$, we have $u \circ L(u) \geq 0$. Hence, $u \circ L(u) = 0$. By item(i), $u = 0$.

(ii)⇒(i): It is obvious.

(i)⇒(iii): Assume that L is cone non-degenerate. Now we want to show that 0 is the only solution to LCP(L, q) for any $q \geq 0$. Suppose $u \in K$ is a solution of LCP(L, q). Then

$$u \geq 0, L(u) + q \geq 0 \text{ and } \langle u, L(u) + q \rangle = 0.$$

Therefore, by Proposition 1, u operator commutes with $L(u) + q$ and $u \circ (L(u) + q) = 0$. $\langle u, L(u) + q \rangle = 0$ implies that $\langle u, L(u) \rangle + \langle u, q \rangle = 0$. From copositivity of L on K , we have $\langle u, q \rangle = 0$ and $\langle u, L(u) \rangle = 0$. Then u operator commutes with q and $u \circ q = 0$ by Proposition 1. This implies that $u \circ L(u) = 0$. Hence $u = 0$ as L is non-degenerate on K . This arguments says that LCP(L, q) has the unique solution for all $q \in K$.

(iii)⇒(ii): It is follows from [6]. □

3.2. Non-degenerate on Lyapunov-like transformation. Matrix $A \in \mathbb{R}^{n \times n}$ is non-degenerate, then it is invertible. But, the invertible matrix need not be non-degenerate. See the following example:

EXAMPLE 3.3. Consider a matrix $A = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 1 & 2 & 1 \end{pmatrix}$. $\text{Det}(A) = -2$. It is invertible.

But the principle minors of A are $A_{11} = -3, A_{22} = 0, A_{33} = 1$. Hence A is not non-degenerate matrix.

The following result tells us that for a Lyapunov-like transformation, the invertible transformation is equivalent to the non-degenerate linear transformation.

THEOREM 3.4. *A Lyapunov-like transformations $L : V \rightarrow V$ is non-degenerate on V if and only if L is invertible .*

Proof. It is sufficient to show the reverse part. Suppose that L is invertible. Let us consider an element $u \in V$ such that $u \circ L(u) = 0$ and u operator commutes with $L(u)$.

Then we can find a Jordan frame $\{e_1, e_2, \dots, e_s\}$ such that $u = \sum_{i=1}^s u_i e_i$ and $L(u) = \sum_{i=1}^s v_i e_i$. This implies that $\sum_{i=1}^s u_i L(e_i) = \sum_{i=1}^s v_i e_i$. Taking inner product with e_j on both sides, we get $u_j \langle e_j, L(e_j) \rangle = v_j \|e_j\|^2$. Since $u_j v_j = 0$, we have if $u_j \neq 0$, then $v_j = 0$ and if $u_j = 0$, then $v_j \|e_j\|^2 = 0$ implying $v_j = 0$ for all j . In any case, $L(u) = 0 \Rightarrow u = 0$ as L is invertible. □

The relationship between cone non-degenerate and the non-degenerate transformations for cone invariant transformation is shown by the results below.

THEOREM 3.5. *Let $L : V \rightarrow V$ be a Lyapunov-like transformation that is cone invariant. Then the following two statement are equivalent.*

- (i) L is non-degenerate on K .
- (ii) L is non-degenerate on V .

Proof. (i) \implies (ii). Assume that L is non-degenerate on K . Let $u \circ L(u) = 0$ and u operator commute with $L(u)$. Then the Jordan frame $\{e_1, e_2, \dots, e_s\}$ will exist such that

$$u = \sum_{i=1}^s u_i e_i \text{ and } L(u) = \sum_{i=1}^s v_i e_i.$$

Since $u \circ L(u) = 0$, we have $u_i v_i = 0$ for all i .

We have

$$\langle u, L(u) \rangle = \sum_{i=1}^s u_i v_i \|e_i\|^2 = 0.$$

We know that

$$u = u^+ - u^- \text{ and } L(u) = L(u^+) - L(u^-),$$

which implies that

$$\langle u, L(u) \rangle = \langle u^+ - u^-, L(u^+) - L(u^-) \rangle = 0.$$

This gives that

$$\langle u^+, L(u^+) \rangle + \langle u^-, L(u^-) \rangle - \langle u^+, L(u^-) \rangle - \langle u^-, L(u^+) \rangle = 0.$$

Since L is Lyapunov-like, we have $\langle u^-, L(u^+) \rangle = 0$ and $\langle u^+, L(u^-) \rangle = 0$ which imply $\langle u^+, L(u^+) \rangle = 0$ and $\langle u^-, L(u^-) \rangle = 0$ as L is also cone invariant.

Therefore $u^+ \circ L(u^+) = 0$ and $u^- \circ L(u^-) = 0$. Since $u^+ \geq 0, L(u^+) \geq 0$ and $\langle u^+, L(u^+) \rangle = 0$, u^+ and $L(u^+)$ operator commute by Proposition 1. Similarly, $u^- \geq 0, L(u^-) \geq 0$ and $\langle u^-, L(u^-) \rangle = 0$, u^- and $L(u^-)$ operator commute. Hence $u^+ = 0$ and $u^- = 0$ by L is non-degenerate on K . This implies that $u = 0$.

(ii) \implies (i) is obvious. \square

4. The non-degenerate on some special transformations

In this segment, we prove some results related to the relaxation transformation and automorphism invariance.

4.1. The relaxation transformation. In this subsection, we can define the relaxation transformation $R_D : V \rightarrow V$ in a following way. Then for any $u \in V$, the decomposition of Peirce with the Jordan frame $\{e_1, e_2, \dots, e_s\}$ in V as,

$$u = \sum_{i=1}^s u_i e_i + \sum_{i < j} u_{ij}.$$

Then

$$R_D(u) = \sum_{i=1}^s v_i e_i + \sum_{i < j} u_{ij},$$

where

$$[v_1, v_2, \dots, v_s]^T = D[u_1, u_2, \dots, u_s]^T.$$

The generalized version of this transformation has been established by Gowda and Tao [12]. Gowda and Tao [6, 12] have investigated several relationships between the

characteristics of D and R_D . We now characterize the non-degeneracy of R_D in terms of the matrix D .

THEOREM 4.1. *If R_D is non-degenerate on V , then D is non-degenerate matrix.*

Proof. Let $u * Du = 0$, where $u^T = [u_1, u_2, \dots, u_s]$. Let $v = Du = [v_1, v_2, \dots, v_s]^T$. We claim that $u = 0$. We define $w = \sum_{i=1}^s u_i e_i$. Then

$$R_D(w) = \sum_{i=1}^s v_i e_i \text{ and } w \circ R_D(w) = \sum_{i=1}^s u_i v_i e_i = 0.$$

Note that w and $R_D(w)$ operator commute. Since $R_D(w)$ is non-degenerate, this implies that $w = 0$ and hence $u = 0$. Therefore, D is a non-degenerate matrix. \square

By using the following example, the converse of the above theorem need not be true.

EXAMPLE 4.2. Let $V = S^2$ be the set of all 2×2 real symmetric matrices. Consider elements $E_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, $E_2 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ and $D = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$

It is easy to show that D is non-degenerate.

Now we define R_D with respect to the Jordan frame $\{E_1, E_2\}$ as

$$R_D(U) = -aE_1 + (-c)E_2 + \begin{bmatrix} 0 & b \\ b & 0 \end{bmatrix} = \begin{bmatrix} -a & b \\ b & -c \end{bmatrix}$$

Where

$$U = \begin{bmatrix} a & b \\ b & c \end{bmatrix} = aE_1 + cE_2 + \begin{bmatrix} 0 & b \\ b & 0 \end{bmatrix}.$$

And

$$U \circ R_D(U) = \frac{1}{2}[UR_D(U) + R_D(U)U].$$

For $W = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} > 0$, $W \circ R_D(W) = 0$ and W operator commutes with $R_D(W)$ Hence $R_D(W)$ is not non-degenerate.

4.2. Automorphism Invariance.

DEFINITION 4.3. Let $\Lambda : V \rightarrow V$ be an invertible linear transformation. Λ is referred to as an algebra automorphism if $\Lambda(u \circ v) = \Lambda(u) \circ \Lambda(v)$ for all $u, v \in V$. Here $Aut(V)$ -set of all automorphisms of V .

We define transformations \tilde{L} by $\tilde{L} := \Lambda^T L \Lambda$. We say that a property P is invariant under the automorphisms of the algebra if \tilde{L} has property P whenever L has property P .

The non-degenerate in Euclidean Jordan algebras is shown in this subsection to be invariant under algebra automorphism. We recall the following result from Proposition 4.2 in [5], which will be used in sequel.

LEMMA 4.4. *Let $\{e_1, e_2, \dots, e_s\}$ be a Jordan frame in V and $\Lambda \in Aut(V)$. Then $\{\Lambda(e_1), \Lambda(e_2), \dots, \Lambda(e_s)\}$ is a Jordan frame in V and there exist positive numbers $\theta_1, \theta_2, \dots, \theta_s$ such that $(\Lambda^T)^{-1}(e_i) = \theta_i \Lambda(e_i)$ for all i .*

THEOREM 4.5. *Non-degenerate linear transformations are invariant under algebra automorphism of V .*

Proof. Let $\Lambda \in \text{Aut}(V)$. Suppose L is non-degenerate on V . It is enough to show that $\Lambda^T L \Lambda$ is non-degenerate. Let u operator commute with $\Lambda^T L \Lambda(u)$ and $u \circ \Lambda^T L \Lambda(u) = 0$. Then there exist a Jordan frame $\{e_1, e_2, \dots, e_s\}$ such that

$$u = \sum_{i=1}^s u_i e_i \text{ and } \Lambda^T L \Lambda = \sum_{i=1}^s v_i e_i.$$

Since $u \circ \Lambda^T L \Lambda(u) = 0$, we have $u_i v_i = 0$. Then we can find positive number θ_i and a Jordan frame $\{\Lambda(e_1), \Lambda(e_2), \dots, \Lambda(e_s)\}$ in V , such that $(\Lambda^T)^{-1}(e_i) = \theta_i \Lambda(e_i)$ for all i by the above lemma. Thus we have $\Lambda(u) = \sum_{i=1}^s u_i \Lambda(e_i)$ and $L(\Lambda(u)) = \sum_{i=1}^s v_i \theta_i \Lambda(e_i)$. This means that $\Lambda(u)$ and $L(\Lambda(u))$ operator commute. Since $u_i v_i = 0$ for all i , then $\Lambda(u) \circ L(\Lambda(u)) = \sum_{i=1}^s u_i v_i \theta_i \Lambda(e_i) = 0$. This implies $\Lambda(u) = 0$ as L is non-degenerate. This implies $u = 0$ as Λ is invertible. Hence $\Lambda^T L \Lambda$ is non-degenerate on V . \square

4.3. The invariance of principal pivotal transformation. Let V_1 and V_2 are two Euclidean Jordan algebras. Then their cartesian product $V = V_1 \times V_2$ is also a Euclidean Jordan algebras. Now let us think about a linear transformation L from V to itself such that L to be expressed in a block form uniquely as

$$L = \begin{bmatrix} P & Q \\ R & S \end{bmatrix},$$

where each entry acts as a linear operator in the following order:

$$P : V_1 \rightarrow V_1, \quad Q : V_2 \rightarrow V_1, \quad R : V_1 \rightarrow V_2, \quad S : V_2 \rightarrow V_2.$$

Assuming P is invertible, we can define the principal pivotal transformation [13] of L as

$$L^* = \begin{bmatrix} P^{-1} & -P^{-1}Q \\ RP^{-1} & L/P \end{bmatrix}.$$

where $L/P = S - RP^{-1}Q$ is the Schur complement of P in L .

Note that

$$\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = L^* \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \iff \begin{bmatrix} u_1 \\ v_2 \end{bmatrix} = L \begin{bmatrix} v_1 \\ u_2 \end{bmatrix}.$$

THEOREM 4.6. *L^* holds the non-degenerate whenever L possess the non-degenerate on V .*

Proof. Suppose

$$\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \text{ and } L^* \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

operator commute and

$$\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \circ L^* \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = 0.$$

Then u_i operator commutes with v_i for $i = 1, 2$. This implies that

$$\begin{bmatrix} v_1 \\ u_2 \end{bmatrix} \text{ and } \begin{bmatrix} u_1 \\ v_2 \end{bmatrix} = L \begin{bmatrix} v_1 \\ u_2 \end{bmatrix}$$

operator commute and

$$\begin{bmatrix} v_1 \\ u_2 \end{bmatrix} \circ L \begin{bmatrix} v_1 \\ u_2 \end{bmatrix} = 0.$$

Since L is non-degenerate, therefore $\begin{bmatrix} v_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

From $v_1 = 0, u_2 = 0$, we get

$$\begin{bmatrix} u_1 \\ v_2 \end{bmatrix} = L \begin{bmatrix} v_1 \\ u_2 \end{bmatrix} = L \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

This indicates that $\begin{bmatrix} u_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$. Therefore L^* is non-degenerate. □

5. Conclusion

In this paper, we first characterized the cone non-degenerate of co-positive and cone invariant transformations. We proved that non-degenerate transformations of a cone invariant Lyapunov-like transformation coincides with the cone non-degenerate. In addition, we showed that non-degenerate under relaxation transformation. Finally we proved that the non-degenerate is invariant under principal pivotal transformation.

6. Author Contribution Statements

For making this article, all the author's contributed equally.

7. Competing Interests

All the author's said that they have no competing interests.

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K. Saravanan

Department of Mathematics, PSG College of Technology,
Coimbatore-641004, Tamilnadu, India
E-mail: ksv.maths@psgtech.ac.in

V. Piramanantham

Department of Mathematics, Bharathidasan University,
Tiruchirappalli-620 024, Tamilnadu, India
E-mail: vpm@bdu.ac.in

R. Theivaraman

Department of Mathematics, Bharathidasan University,
Tiruchirappalli-620 024, Tamilnadu, India
E-mail: deivaraman@gmail.com