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NON-LINEAR PRODUCT $\mathscr{L}\mathscr{M}^* - \mathscr{M}\mathscr{L}^*$ ON PRIME $*-$ALGEBRAS

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Abstract. In this paper, we explore the additivity of the map $\Omega : \mathscr{A} \to \mathscr{A}$ that satisfies

$$\Omega\left([\mathscr{L}, \mathscr{M}]_*\right) = \left[\Omega\left(\mathscr{M}\right), \mathscr{L}\right]_* + \left[\mathscr{M}, \Omega\left(\mathscr{L}\right)\right]_*,$$

where $[\mathscr{L}, \mathscr{M}]_* = \mathscr{L}\mathscr{M}^* - \mathscr{M}\mathscr{L}^*$, for all $\mathscr{L}, \mathscr{M} \in \mathscr{A}$, a prime $*-$algebra with unit $\mathscr{I}$. Additionally we show that if $\Omega(\alpha\mathscr{I})$ is self-adjoint operator for $\alpha \in \{1, i\}$, then $\Omega = 0$.

1. Introduction

Let $\mathscr{A}$ be a $*-$algebra. For any $\mathscr{L}, \mathscr{M} \in \mathscr{A}$, the $*-$Jordan ($*-$Lie) product presented as $\mathscr{L}\diamondsuit\mathscr{M} = \mathscr{L}\mathscr{M} + \mathscr{M}\mathscr{L}^*$ $\left([\mathscr{L}, \mathscr{M}]_* = \mathscr{L}\mathscr{M} - \mathscr{M}\mathscr{L}^*\right)$. These products have garnered a significant amount of attention, and allusions show a widening interest in literature [1–6, 9, 11]. Notice that a map $\Omega : \mathscr{A} \to \mathscr{A}$ is referred to as a reverse derivation if $\Omega(\mathscr{L}+\mathscr{M}) = \Omega(\mathscr{L})+\Omega(\mathscr{M})$ and $\Omega(\mathscr{L}\mathscr{M}) = \Omega(\mathscr{M})\mathscr{L}+\mathscr{M}\Omega(\mathscr{L})$ for all $\mathscr{L}, \mathscr{M} \in \mathscr{A}$. A map Ω is additive $*-$derivation if it is an additive derivation and $\Omega(\mathscr{L}^*) = \Omega(\mathscr{L})^*$. Define a new product $[\mathscr{L}, \mathscr{M}]_* = \mathscr{L}\mathscr{M}^* - \mathscr{M}\mathscr{L}^*$. In 2020, Taghavi and Razeghi [10] studied $[\mathscr{L}, \mathscr{M}]_* = \mathscr{L}^*\mathscr{M} - \mathscr{M}^*\mathscr{L}$ product on $*-$algebras. In particular, they established that the map $\Omega : \mathscr{A} \to \mathscr{A}$ satisfying $\Omega(\mathscr{L}\mathscr{M}) = \Omega(\mathscr{L})\mathscr{M} + \mathscr{L}\Omega(\mathscr{M})$ is a $*-$derivation. The study of non-linear preserving problems is one of the premier areas in matrix theory as well as operator theory. It was Martindale [7] who first asked the question that when are multiplicative/nonadditive maps additive? He answered his question for a multiplicative isomorphism of a ring under the existence of a family of idempotent elements of rings which satisfies some conditions. This spawned a wealth of diverse methods on different algebraic structures like operator algebras, von Neumann algebras, Banach algebras etcetera to establish a variety of interesting results concerning nonadditive maps. Possibly the most obvious approach is the method of algebraic decompositions. A wealth of fundamentally different methods to deal with nonlinear mappings can be found in [1–8, 11] and references therein.

Our main objective of this manuscript is to explore the structure of a non-linear maps on prime $*-$algebras satisfying $\Omega\left([\mathscr{L}, \mathscr{M}]_*\right) = \left[\Omega\left(\mathscr{M}\right), \mathscr{L}\right]_* + \left[\mathscr{M}, \Omega\left(\mathscr{L}\right)\right]_*$, where $[\mathscr{L}, \mathscr{M}]_* = \mathscr{L}\mathscr{M}^* - \mathscr{M}\mathscr{L}^*$, for all $\mathscr{L}, \mathscr{M} \in \mathscr{A}$. We systematize the proof of above theorem in two parts. Firstly, we prove the additivity of Ω by using several claims.

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Secondly, we shall provide numerous constructive facts to elaborate the essertion of our main theorem.

2. Main results

THEOREM 2.1. *Let $\mathscr{A}$ be a prime $*-$algebra with unit $\mathscr{I}$ and a nontrivial projection. Then the map $\Omega : \mathscr{A} \to \mathscr{A}$ that satisfies*

$$\Omega([\mathscr{L},\mathscr{M}]_*) = [\Omega(\mathscr{M}),\mathscr{L}]_* + [\mathscr{M},\Omega(\mathscr{L})]_*, \tag{2.1}$$

where $[\mathscr{L},\mathscr{M}]_* = \mathscr{L}\mathscr{M}^* - \mathscr{M}\mathscr{L}^*$, for all $\mathscr{L},\mathscr{M} \in \mathscr{A}$ is additive.

Proof. Let $\mathscr{P}_1$ be a nontrivial projection in $\mathscr{A}$ and $\mathscr{P}_2 = \mathscr{I}_{\mathscr{A}} - \mathscr{P}_1$. Denote $\mathscr{A}_{ij} = \mathscr{P}_i\mathscr{A}\mathscr{P}_j$, $i,j = 1,2$, then $\mathscr{A} = \sum_{i,j=1}^{2}\mathscr{A}_{ij}$. For every $\mathscr{L} \in \mathscr{A}$, we may write $\mathscr{L} = \mathscr{L}_{11} + \mathscr{L}_{12} + \mathscr{L}_{21} + \mathscr{L}_{22}$. From now on, if we mention $\mathscr{L}_{ij}$, it means that $\mathscr{L}_{ij} \in \mathscr{A}_{ij}$. To illustrate the additivity of Ω on $\mathscr{A}$, we take the aforementioned partition of $\mathscr{A}$ and present several claims that prove Ω is additive on each $\mathscr{A}_{ij}$, $i,j = 1,2$.

Several claims are used to prove the preceding theorem.

Claim 1. $\Omega(0) = 0$.
Take $\mathscr{L} = \mathscr{M} = 0$, then

$$\Omega(0) = \Omega([0,0]_*) = [\Omega(0),0]_* + [0,\Omega(0)]_* = 0.$$

Claim 2. $\Omega(i\mathscr{L}) = i\Omega(\mathscr{L}) + \mathscr{L}^*K$, where $K = \Omega(i\mathscr{I}) - i\Omega(\mathscr{I})$.
Consider

$$\Omega([-i\mathscr{L},\mathscr{I}]_*) = \Omega([\mathscr{L},i\mathscr{I}]_*).$$

So, we have

$$[\Omega(\mathscr{I}),(-i\mathscr{L})]_* + [\mathscr{I},\Omega(-i\mathscr{L})]_* = [\Omega(i\mathscr{I}),\mathscr{L}]_* + [(i\mathscr{I}),\Omega(\mathscr{L})]_*$$

$$\begin{aligned} i\mathscr{L}^*\Omega(\mathscr{I}) + i\mathscr{L}\Omega(\mathscr{I})^* + \Omega(-i\mathscr{L})^* \ &- \ \Omega(-i\mathscr{L}) \\ &= \ \mathscr{L}^*\Omega(i\mathscr{I}) - \Omega(i\mathscr{I})^*\mathscr{L} + i\Omega(\mathscr{L})^* + i\Omega(\mathscr{L}). \end{aligned} \tag{2.2}$$

Consider

$$\Omega([-i\mathscr{L},i\mathscr{I}]_*) = \Omega([\mathscr{I},\mathscr{L}]_*).$$

So, we have

$$[\Omega(i\mathscr{I}),(-i\mathscr{L})]_* + [(i\mathscr{I}),\Omega(-i\mathscr{L})]_* = [\Omega(\mathscr{L}),\mathscr{I}]_* + [\mathscr{L},\Omega(\mathscr{I})]_*$$

$$\begin{aligned} i\mathscr{L}^*\Omega(i\mathscr{I}) + i\mathscr{L}\Omega(i\mathscr{I})^* + i\Omega(-i\mathscr{L})^* \ &+ \ i\Omega(-i\mathscr{L}) \\ &= \ \Omega(\mathscr{L}) - \Omega(\mathscr{L})^* + \Omega(\mathscr{I})^*\mathscr{L} - \mathscr{L}^*\Omega(\mathscr{I}). \end{aligned}$$

Equivalently we obtain

$$\begin{aligned} -\mathscr{L}^*\Omega(i\mathscr{I}) - \mathscr{L}\Omega(i\mathscr{I})^* - \Omega(-i\mathscr{L})^* \ &- \ \Omega(-i\mathscr{L}) \\ &= \ i\Omega(\mathscr{L}) - i\Omega(\mathscr{L})^* + i\Omega(\mathscr{I})^*\mathscr{L} - i\mathscr{L}^*\Omega(\mathscr{I}). \end{aligned} \tag{2.3}$$

By adding equations (2.2) and (2.3), we have

$$i\Omega(\mathscr{L}) + \Omega(-i\mathscr{L}) = i\mathscr{L}^*\Omega(\mathscr{I}) - \mathscr{L}^*\Omega(i\mathscr{I}).$$

Substiting $i\mathscr{L}$ instead of $\mathscr{L}$ in the above equation, we get

$$i\Omega(i\mathscr{L}) + \Omega(\mathscr{L}) = \mathscr{L}^*\Omega(\mathscr{I}) + i\mathscr{L}^*\Omega(i\mathscr{I})$$

$$\Omega(i\mathscr{L}) = i\Omega(\mathscr{L}) + \mathscr{L}^*(\Omega(i\mathscr{I}) - i\Omega(\mathscr{I}))$$

So $\Omega(i\mathscr{L}) = i\Omega(\mathscr{L}) + \mathscr{L}^*K$, where $K = \Omega(i\mathscr{I}) - i\Omega(\mathscr{I})$.

Claim 3. $\Omega(-\mathscr{L}) = -\Omega(\mathscr{L})$.
By considering $\Omega(i\mathscr{L}) = i\Omega(\mathscr{L}) + \mathscr{L}^*K$ and applying $i\mathscr{L}$ instead of $\mathscr{L}$, we have

$$\begin{aligned} \Omega(-\mathscr{L}) &= i\Omega(i\mathscr{L}) - i\mathscr{L}^*K \\ \Omega(-\mathscr{L}) &= i(i\Omega(\mathscr{L}) + \mathscr{L}^*K) - i\mathscr{L}^*K \\ \Omega(-\mathscr{L}) &= -\Omega(\mathscr{L}) + i\mathscr{L}^*K - i\mathscr{L}^*K \\ \Omega(-\mathscr{L}) &= -\Omega(\mathscr{L}). \end{aligned} \tag{2.4}$$

Claim 4. For each $\mathscr{L}_{11} \in \mathscr{A}_{11}, \mathscr{L}_{12} \in \mathscr{A}_{12}$, we have

$$\Omega(\mathscr{L}_{11} + \mathscr{L}_{12}) = \Omega(\mathscr{L}_{11}) + \Omega(\mathscr{L}_{12}).$$

Let $\mathscr{T} = \Omega(\mathscr{L}_{11} + \mathscr{L}_{12}) - \Omega(\mathscr{L}_{11}) - \Omega(\mathscr{L}_{12})$, we will prove that $\mathscr{T} = 0$. For $\mathscr{X}_{21} \in \mathscr{A}_{21}$, we can write that

$$\begin{aligned} & [\Omega(\mathscr{X}_{21}), (\mathscr{L}_{11} + \mathscr{L}_{12})]_* + [\mathscr{X}_{21}, \Omega(\mathscr{L}_{11} + \mathscr{L}_{12})]_* \\ = \; & \Omega([(\mathscr{L}_{11} + \mathscr{L}_{12}), \mathscr{X}_{21}]_*) \\ = \; & \Omega([\mathscr{L}_{11}, \mathscr{X}_{21}]_*) + \Omega([\mathscr{L}_{12}, \mathscr{X}_{21}]_*) \\ = \; & [\Omega(\mathscr{X}_{21}), \mathscr{L}_{11}]_* + [\mathscr{X}_{21}, \Omega(\mathscr{L}_{11})]_* + [\Omega(\mathscr{X}_{21}), \mathscr{L}_{12}]_* + [\mathscr{X}_{21}, \Omega(\mathscr{L}_{12})]_* \\ = \; & [\mathscr{X}_{21}, (\Omega(\mathscr{L}_{11}) + \Omega(\mathscr{L}_{12}))]_* + [\Omega(\mathscr{X}_{21}), (\mathscr{L}_{11} + \mathscr{L}_{12})]_*. \end{aligned}$$

So, we obtain

$$[\mathscr{X}_{21}, \mathscr{T}]_* = 0.$$

Since $\mathscr{T} = \mathscr{T}_{11} + \mathscr{T}_{12} + \mathscr{T}_{21} + \mathscr{T}_{22}$, we have

$$\mathscr{X}_{21}\mathscr{T}_{21}^* + \mathscr{X}_{21}\mathscr{T}_{11}^* - \mathscr{T}_{21}\mathscr{X}_{21}^* - \mathscr{T}_{11}\mathscr{X}_{21}^* = 0.$$

From the above equation and primeness of $\mathscr{A}$, we have $\mathscr{T}_{11} = 0$, and

$$\mathscr{X}_{21}\mathscr{T}_{21}^* - \mathscr{T}_{21}\mathscr{X}_{21}^* = 0. \tag{2.5}$$

Alternatively, by substituting $i\mathscr{X}_{21}$ for $\mathscr{X}_{21}$ in the preceding equation, we obtain

$$-\mathscr{X}_{21}\mathscr{T}_{21}^* - \mathscr{T}_{21}\mathscr{X}_{21}^* = 0. \tag{2.6}$$

From (2.5) and (2.6), we get $\mathscr{T}_{21}\mathscr{X}_{21}^* = 0$. Since $\mathcal{A}$ is prime, then $\mathscr{T}_{21} = 0$. It is suffices to show that $\mathscr{T}_{12} = \mathscr{T}_{22} = 0$. For this purpose take $\mathscr{X}_{12} \in \mathscr{A}_{12}$, we write

$$\begin{aligned} & \Omega([[(\mathscr{L}_{11} + \mathscr{L}_{12}), \mathscr{X}_{12}]_*, \mathscr{P}_1]_*) \\ = \; & [\Omega(\mathscr{P}_1), [(\mathscr{L}_{11} + \mathscr{L}_{12}), \mathscr{X}_{12}]_*]_* + [\mathscr{P}_1, \Omega([(\mathscr{L}_{11} + \mathscr{L}_{12}), \mathscr{X}_{12}]_*)]_* \\ = \; & [\Omega(\mathscr{P}_1), [(\mathscr{L}_{11} + \mathscr{L}_{12}), \mathscr{X}_{12}]_*]_* + [\mathscr{P}_1, [\mathscr{X}_{12}, \Omega(\mathscr{L}_{11} + \mathscr{L}_{12})]_*]_* \\ & + [\mathscr{P}_1, [\Omega(\mathscr{X}_{12}), (\mathscr{L}_{11} + \mathscr{L}_{12})]_*]_* \\ = \; & [\Omega(\mathscr{P}_1), [\mathscr{L}_{11}, \mathscr{X}_{12}]_*]_* + [\Omega(\mathscr{P}_1), [\mathscr{L}_{12}, \mathscr{X}_{12}]_*]_* \\ & + [\mathscr{P}_1, [\Omega(\mathscr{X}_{12}), \mathscr{L}_{11}]_*]_* + [\mathscr{P}_1, [\Omega(\mathscr{X}_{12}), \mathscr{L}_{12}]_*]_* \\ & + [\mathscr{P}_1, [\mathscr{X}_{12}, \Omega(\mathscr{L}_{11} + \mathscr{L}_{12})]_*]_*. \end{aligned}$$

So, we showed that

$$\Omega([[(\mathscr{L}_{11}+\mathscr{L}_{12}),\mathscr{X}_{12}]_*,\mathscr{P}_1]_*) = [\Omega(\mathscr{P}_1),[\mathscr{L}_{11},\mathscr{X}_{12}]_*]_* + [\Omega(\mathscr{P}_1),[\mathscr{L}_{12},\mathscr{X}_{12}]_*]_* + [\mathscr{P}_1,[\Omega(\mathscr{X}_{12}),\mathscr{L}_{11}]_*]_* + [\mathscr{P}_1,[\Omega(\mathscr{X}_{12}),\mathscr{L}_{12}]_*]_* + [\mathscr{P}_1,[\mathscr{X}_{12},\Omega(\mathscr{L}_{11}+\mathscr{L}_{12})]_*]_*. \tag{2.7}$$

Since $[[\mathscr{L}_{12},\mathscr{X}_{12}]_*,\mathscr{P}_1]_*=0$, we hvae

$$\begin{aligned}
&= [\Omega(\mathscr{P}_1),[\mathscr{L}_{11},\mathscr{X}_{12}]_*]_* + [\mathscr{P}_1,\Omega([\mathscr{L}_{11},\mathscr{X}_{12}]_*)]_* + [\Omega(\mathscr{P}_1),[\mathscr{L}_{12},\mathscr{X}_{12}]_*]_* \\
&\quad +[\mathscr{P}_1,\Omega([\mathscr{L}_{12},\mathscr{X}_{12}]_*)]_* \\
&= [\Omega(\mathscr{P}_1),[\mathscr{L}_{11},\mathscr{X}_{12}]_*]_* + [\mathscr{P}_1,([\Omega(\mathscr{X}_{12}),\mathscr{L}_{11}]_* + [\mathscr{X}_{12},\Omega(\mathscr{L}_{11})]_*)]_* \\
&\quad +[\Omega(\mathscr{P}_1),[\mathscr{L}_{12},\mathscr{X}_{12}]_*]_* + [\mathscr{P}_1,([\mathscr{X}_{12},\Omega(\mathscr{L}_{12})]_* + [\Omega(\mathscr{X}_{12}),\mathscr{L}_{12}]_*)]_* \\
&= [\Omega(\mathscr{P}_1),[\mathscr{L}_{11},\mathscr{X}_{12}]_*]_* + [\Omega(\mathscr{P}_1),[\mathscr{L}_{12},\mathscr{X}_{12}]_*]_* + [\mathscr{P}_1,[\Omega(\mathscr{X}_{12}),\mathscr{L}_{11}]_*]_* \\
&\quad +[\mathscr{P}_1,\Omega(\mathscr{X}_{12}),[\mathscr{L}_{12}]_*]_* + [\mathscr{P}_1,[\mathscr{X}_{12},\Omega(\mathscr{L}_{11})]_*]_* + [\mathscr{P}_1,[\mathscr{X}_{12},\Omega(\mathscr{L}_{12})]_*]_*.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\Omega([[\mathscr{L}_{11}+\mathscr{L}_{12},\mathscr{X}_{12}]_*,\mathscr{P}_1]_*) &= [\Omega(\mathscr{P}_1),[\mathscr{L}_{11},\mathscr{X}_{12}]_*]_* + [\Omega(\mathscr{P}_1),[\mathscr{L}_{12},\mathscr{X}_{12}]_*]_* \\
&\quad +[\mathscr{P}_1,[\Omega(\mathscr{X}_{12}),\mathscr{L}_{12}]_* + [\mathscr{P}_1,[\Omega(\mathscr{X}_{12}),\mathscr{L}_{12}]_*]_* \\
&\quad +[\mathscr{P}_1,[\mathscr{X}_{12},\Omega(\mathscr{L}_{11})]_*]_* + [\mathscr{P}_1,[\mathscr{X}_{12},\Omega(\mathscr{L}_{12})]_*]_*.
\end{aligned} \tag{2.8}$$

From (2.7) and (2.8), we have

$$[\mathscr{P}_1,[\mathscr{X}_{12},\Omega(\mathscr{L}_{11}+\mathscr{L}_{12})]_*]_* = [\mathscr{P}_1,[\mathscr{X}_{12},\Omega(\mathscr{L}_{11})]_*]_* + [\mathscr{P}_1,[\mathscr{X}_{12},\Omega(\mathscr{L}_{12})]_*]_*.$$

It follows that $[\mathscr{P}_1,[\mathscr{X}_{12},\mathscr{T}]_*]_*=0$, so $\mathscr{X}_{12}\mathscr{T}_{22}^*-\mathscr{T}_{22}\mathscr{X}_{12}^*=0$. We have $\mathscr{T}_{22}\mathscr{X}_{12}^*=0$ or $\mathscr{P}_1\mathscr{X}\mathscr{T}_{22}=0$ for all $\mathscr{X}\in\mathscr{A}$, then we have $\mathscr{T}_{22}=0$. Similarly, we can show that $\mathscr{T}_{12}=0$ by applying $\mathscr{P}_2$ instead of $\mathscr{P}_1$ in above.

Claim 5. For each $\mathscr{L}_{11}\in\mathscr{A}_{11},\mathscr{L}_{12}\in\mathscr{A}_{12},\mathscr{L}_{21}\in\mathscr{A}_{21}$ and $\mathscr{L}_{22}\in\mathscr{A}_{22}$, we have

$$\Omega(\mathscr{L}_{11}+\mathscr{L}_{12}+\mathscr{L}_{21}) = \Omega(\mathscr{L}_{11})+\Omega(\mathscr{L}_{12})+\Omega(\mathscr{L}_{21})$$

$$\Omega(\mathscr{L}_{12}+\mathscr{L}_{21}+\mathscr{L}_{22}) = \Omega(\mathscr{L}_{12})+\Omega(\mathscr{L}_{21})+\Omega(\mathscr{L}_{22}).$$

We show that $\mathscr{T}=\Omega(\mathscr{L}_{11}+\mathscr{L}_{12}+\mathscr{L}_{21})-\Omega(\mathscr{L}_{11})-\Omega(\mathscr{L}_{12})-\Omega(\mathscr{L}_{21})=0$. For $\mathscr{X}_{21}\in\mathscr{A}_{21}$, we have

$$\begin{aligned}
[\Omega(\mathscr{X}_{21}),(\mathscr{L}_{11}+\mathscr{L}_{12}+\mathscr{L}_{21})]_* &+ [\mathscr{X}_{21},\Omega(\mathscr{L}_{11}+\mathscr{L}_{12}+\mathscr{L}_{21})]_* \\
&= \Omega([(\mathscr{L}_{11}+\mathscr{L}_{12}+\mathscr{L}_{21}),\mathscr{X}_{21}]_*) \\
&= \Omega([(\mathscr{L}_{11}+\mathscr{L}_{12}),\mathscr{X}_{21}]_*)+\Omega([\mathscr{L}_{21},\mathscr{X}_{21}]_*) \\
&= \Omega([\mathscr{L}_{11},\mathscr{X}_{21}]_*)+\Omega([\mathscr{L}_{12},\mathscr{X}_{21}]_*)+\Omega([\mathscr{L}_{21},\mathscr{X}_{21}]_*) \\
&= [\mathscr{X}_{21},(\Omega(\mathscr{L}_{11})+\Omega(\mathscr{L}_{12})+\Omega(\mathscr{L}_{21}))]_* \\
&\quad +[\Omega(\mathscr{X}_{21}),(\mathscr{L}_{11}+\mathscr{L}_{12}+\mathscr{L}_{21})]_*.
\end{aligned}$$

It follows that $[\mathscr{X}_{21},\mathscr{T}]_*=0$. Since $\mathscr{T}=\mathscr{T}_{11}+\mathscr{T}_{12}+\mathscr{T}_{21}+\mathscr{T}_{22}$, we have

$$\mathscr{X}_{21}\mathscr{T}_{21}^*+\mathscr{X}_{21}\mathscr{T}_{11}^*-\mathscr{T}_{21}\mathscr{X}_{21}^*-\mathscr{T}_{11}\mathscr{X}_{21}^*=0.$$

Therefore,$\mathscr{T}_{11} = \mathscr{T}_{21} = 0$.
From Claim 4, we obtain

$$\begin{aligned}
[\Omega(\mathscr{X}_{12}), (\mathscr{L}_{11} + \mathscr{L}_{12} + \mathscr{L}_{21})]_* &+ [\mathscr{X}_{12}, \Omega(\mathscr{L}_{11} + \mathscr{L}_{12} + \mathscr{L}_{21})]_* \\
&= \Omega([(\mathscr{L}_{11} + \mathscr{L}_{12} + \mathscr{L}_{21}), \mathscr{X}_{12}]_*) \\
&= \Omega([(\mathscr{L}_{11} + \mathscr{L}_{12}), \mathscr{X}_{12}]_*) + \Omega([\mathscr{L}_{21}, \mathscr{X}_{12}]_*) \\
&= \Omega([\mathscr{L}_{11}, \mathscr{X}_{12}]_*) + \Omega([\mathscr{L}_{12}, \mathscr{X}_{12}]_*) + \Omega([\mathscr{L}_{21}, \mathscr{X}_{12}]_*) \\
&= [\mathscr{X}_{12}, (\Omega(\mathscr{L}_{11}) + \Omega(\mathscr{L}_{12}) + \Omega(\mathscr{L}_{21}))]_* \\
&\quad +[\Omega(\mathscr{X}_{12}), (\mathscr{L}_{11} + \mathscr{L}_{12} + \mathscr{L}_{21})]_*.
\end{aligned}$$

Hence,

$$\mathscr{X}_{12}^*\mathscr{T}_{12} + \mathscr{X}_{12}^*\mathscr{T}_{11} - \mathscr{T}_{12}^*\mathscr{X}_{12} - \mathscr{T}_{11}^*\mathscr{X}_{12} = 0.$$

Then,$\mathscr{T}_{11} = \mathscr{T}_{12} = 0$. Similarly

$$\Omega(\mathscr{L}_{11} + \mathscr{L}_{12} + \mathscr{L}_{21}) = \Omega(\mathscr{L}_{11}) + \Omega(\mathscr{L}_{12}) + \Omega(\mathscr{L}_{21}).$$

Claim 6. For each $\mathscr{L}_{11} \in \mathscr{A}_{11}, \mathscr{L}_{12} \in \mathscr{A}_{12}, \mathscr{L}_{21} \in \mathscr{A}_{21}$ and $\mathscr{L}_{22} \in \mathscr{A}_{22}$, we have

$$\Omega(\mathscr{L}_{11} + \mathscr{L}_{12} + \mathscr{L}_{21} + \mathscr{L}_{22}) = \Omega(\mathscr{L}_{11}) + \Omega(\mathscr{L}_{12}) + \Omega(\mathscr{L}_{21}) + (\mathscr{L}_{22}).$$

We show that

$$\mathscr{T} = \Omega(\mathscr{L}_{11} + \mathscr{L}_{12} + \mathscr{L}_{21} + \mathscr{L}_{22}) - \Omega(\mathscr{L}_{11}) - \Omega(\mathscr{L}_{12}) - \Omega(\mathscr{L}_{21}) - (\mathscr{L}_{22}) = 0.$$

From Claim 5, we have

$$\begin{aligned}
& [\Omega(\mathscr{X}_{12}), (\mathscr{L}_{11} + \mathscr{L}_{12} + \mathscr{L}_{21} + \mathscr{L}_{22})]_* + [\mathscr{X}_{12}, \Omega(\mathscr{L}_{11} + \mathscr{L}_{12} + \mathscr{L}_{21} + \mathscr{L}_{22})]_* \\
= \ & \Omega([(\mathscr{L}_{11} + \mathscr{L}_{12} + \mathscr{L}_{21} + \mathscr{L}_{22}), \mathscr{X}_{12}]_*) \\
= \ & \Omega([(\mathscr{L}_{11} + \mathscr{L}_{12} + \mathscr{L}_{21}), \mathscr{X}_{12}]_*) + \Omega([\mathscr{L}_{22}, \mathscr{X}_{12}]_*) \\
= \ & \Omega([\mathscr{L}_{11}, \mathscr{X}_{12}]_*) + \Omega([\mathscr{L}_{12}, \mathscr{X}_{12}]_*) + \Omega([\mathscr{L}_{21}, \mathscr{X}_{12}]_*) \\
& +\Omega([\mathscr{L}_{22}, \mathscr{X}_{12}]_*) \\
= \ & [\mathscr{X}_{12}, (\Omega(\mathscr{L}_{11}) + \Omega(\mathscr{L}_{12}) + \Omega(\mathscr{L}_{21}) + \Omega(\mathscr{L}_{22}))]_* \\
& +[\Omega(\mathscr{X}_{12}), (\mathscr{L}_{11} + \mathscr{L}_{12} + \mathscr{L}_{21} + \mathscr{L}_{22})]_*.
\end{aligned}$$

So, $[\mathscr{X}_{12}, \mathscr{T}]_* = 0$. It follows that

$$\mathscr{X}_{12}\mathscr{T}_{12}^* + \mathscr{X}_{12}\mathscr{T}_{22}^* - \mathscr{T}_{12}\mathscr{X}_{12}^* - \mathscr{T}_{22}\mathscr{X}_{12}^* = 0.$$

Then $\mathscr{T}_{22} = \mathscr{T}_{12} = 0$
Similarly, by applying $\mathscr{X}_{21}$ instead of $\mathscr{X}_{12}$ in above, we obtain $\mathscr{T}_{22} = \mathscr{T}_{21} = 0$.

Claim 7. For each $\mathscr{L}_{ij}, \mathscr{M}_{ij} \in \mathscr{A}_{ij}$ such that $i \neq j$, we have

$$\Omega(\mathscr{L}_{ij} + \mathscr{M}_{ij}) = \Omega(\mathscr{L}_{ij}) + \Omega(\mathscr{M}_{ij}).$$

It is easy to show that

$$(\mathscr{P}_i + \mathscr{L}_{ij})(\mathscr{P}_j + \mathscr{M}_{ij}) - (\mathscr{P}_j + \mathscr{M}_{ij}^*)(\mathscr{P}_i + \mathscr{L}_{ij}^*) = \mathscr{L}_{ij} + \mathscr{M}_{ij} - \mathscr{L}_{ij}^* - \mathscr{M}_{ij}^*.$$

So, we can write

$$\begin{aligned}
& \Omega(\mathscr{L}_{ij}+\mathscr{M}_{ij})+\Omega(-\mathscr{L}_{ij}^*-\mathscr{M}_{ji}^*) \\
= & \Omega([(\mathscr{P}_i+\mathscr{L}_{ij}),(\mathscr{P}_j+\mathscr{M}_{ij}^*)]_*) \\
= & [\Omega(\mathscr{P}_j+\mathscr{M}_{ij}^*),(\mathscr{P}_i+\mathscr{L}_{ij})]_*+[(\mathscr{P}_j+\mathscr{M}_{ij}^*),\Omega(\mathscr{P}_i+\mathscr{L}_{ij})]_* \\
= & [(\Omega(\mathscr{P}_j)+\Omega(\mathscr{M}_{ij}^*)),(\mathscr{P}_i+\mathscr{L}_{ij})]_* \\
& +[(\mathscr{P}_j+\mathscr{M}_{ij}^*),(\Omega(\mathscr{P}_i)+\Omega(\mathscr{L}_{ij}))]_* \\
= & [\Omega(\mathscr{M}_{ij}^*),\mathscr{P}_i]_*+[\mathscr{M}_{ij}^*,\Omega(\mathscr{P}_i)]_*+[\Omega(\mathscr{P}_j),\mathscr{L}_{ij}]_*+[\mathscr{P}_j,\Omega(\mathscr{L}_{ij})]_* \\
= & \Omega([\mathscr{P}_i,\mathscr{M}_{ij}^*]_*)+\Omega([\mathscr{L}_{ij},\mathscr{P}_j]_*) \\
= & \Omega(\mathscr{M}_{ij})-\Omega(\mathscr{M}_{ij}^*)+\Omega(\mathscr{L}_{ij})-\Omega(\mathscr{L}_{ij}^*)
\end{aligned}$$

Therefore, we show that

$$(2.9)\ \Omega(\mathscr{L}_{ij}+\mathscr{M}_{ij})+\Omega(-\mathscr{L}_{ij}^*-\mathscr{M}_{ji}^*) \;=\; \Omega(\mathscr{M}_{ij})-\Omega(\mathscr{M}_{ij}^*)-\Omega(\mathscr{L}_{ij}^*)+\Omega(\mathscr{L}_{ij})$$

By an easy computation, we can write

$$(i\mathscr{P}_i+i\mathscr{L}_{ij})(\mathscr{P}_j+\mathscr{M}_{ij})-(\mathscr{P}_j+\mathscr{M}_{ij}^*)(-i\mathscr{P}_i-i\mathscr{L}_{ij}^*)=i\mathscr{L}_{ij}+i\mathscr{M}_{ij}+i\mathscr{L}_{ij}^*+i\mathscr{M}_{ij}^*.$$

Then, we have

$$\begin{aligned}
& \Omega(\mathscr{L}_{ij}+\mathscr{M}_{ij})+\Omega(i\mathscr{L}_{ij}^*+i\mathscr{M}_{ji}^*) \\
= & \Omega([(i\mathscr{P}_i+i\mathscr{L}_{ij}),(\mathscr{P}_j+\mathscr{M}_{ij}^*)]_*) \\
= & [\Omega(\mathscr{P}_j+\mathscr{M}_{ij}^*),(i\mathscr{P}_i+i\mathscr{L}_{ij})]_*+[(\mathscr{P}_j+\mathscr{M}_{ij}^*),\Omega(i\mathscr{P}_i+i\mathscr{L}_{ij})]_* \\
= & [(\Omega(\mathscr{P}_j)+\Omega(\mathscr{M}_{ij}^*)),(i\mathscr{P}_i+i\mathscr{L}_{ij})]_* \\
& +[(\mathscr{P}_j+\mathscr{M}_{ij}^*),(\Omega(i\mathscr{P}_i)+\Omega(i\mathscr{L}_{ij}))]_* \\
= & [\Omega(\mathscr{M}_{ij}^*),i\mathscr{P}_i]_*+[\mathscr{M}_{ij}^*,\Omega(i\mathscr{P}_i)]_*+[\Omega(\mathscr{P}_j),i\mathscr{L}_{ij}]_*+[\mathscr{P}_j,\Omega(i\mathscr{L}_{ij})]_* \\
= & \Omega([i\mathscr{P}_i,\mathscr{M}_{ij}^*]_*)+\Omega([i\mathscr{L}_{ij},\mathscr{P}_j]_*) \\
= & \Omega(i\mathscr{M}_{ij})+\Omega(i\mathscr{M}_{ij}^*)+\Omega(\mathscr{L}_{ij})+\Omega(i\mathscr{L}_{ij}^*)
\end{aligned}$$

We showed that

$$\Omega(i\mathscr{L}_{ij}+i\mathscr{M}_{ij})+\Omega(i\mathscr{L}_{ij}^*+i\mathscr{M}_{ji}^*) \;=\; \Omega(i\mathscr{M}_{ij})+\Omega(i\mathscr{M}_{ij}^*)+\Omega(i\mathscr{L}_{ij}^*)+\Omega(i\mathscr{L}_{ij})$$

From Claims 2, 3 and the above equation, we have

$$(2.10)\Omega(\mathscr{L}_{ij}+\mathscr{M}_{ij})+\Omega(\mathscr{L}_{ij}^*+\mathscr{M}_{ji}^*) \;=\; \Omega(\mathscr{M}_{ij})+\Omega(\mathscr{M}_{ij}^*)+\Omega(\mathscr{L}_{ij}^*)+\Omega(\mathscr{L}_{ij}).$$

By adding equations (2.10) and (2.9), we obtain

$$\Omega(\mathscr{L}_{ij}+\mathscr{M}_{ij})=\Omega(\mathscr{L}_{ij})+\Omega(\mathscr{M}_{ij}).$$

Claim 8. For each $\mathscr{L}_{ii},\mathscr{M}_{ii}\in\mathscr{A}_{ii}$ such that $1\le i\le 2$, we have

$$\Omega(\mathscr{L}_{ii}+\mathscr{M}_{ii})=\Omega(\mathscr{L}_{ii})+\Omega(\mathscr{M}_{ii}).$$

We show that $\mathscr{T} = \Omega(\mathscr{L}_{ii} + \mathscr{M}_{ii}) - \Omega(\mathscr{L}_{ii}) - \Omega(\mathscr{M}_{ii}) = 0$. We can write that

$$\begin{aligned}
& [\Omega(\mathscr{P}_j), (\mathscr{L}_{ii} + \mathscr{M}_{ii})]_* + [\mathscr{P}_j, \Omega(\mathscr{L}_{ii} + \mathscr{M}_{ii})]_* \\
= \; & \Omega([(\mathscr{L}_{ii} + \mathscr{M}_{ii}), \mathscr{P}_j]_*) \\
= \; & \Omega([\mathscr{L}_{ii}, \mathscr{P}_j]_*) + \Omega([\mathscr{M}_{ii}, \mathscr{P}_j]_*) \\
= \; & [\Omega(\mathscr{P}_j), \mathscr{L}_{ii}]_* + [\mathscr{P}_j, \Omega(\mathscr{L}_{ii})]_* + [\Omega(\mathscr{P}_j), \mathscr{M}_{ii}]_* \\
& + [\mathscr{P}_j, \Omega(\mathscr{M}_{ii})]_* \\
= \; & [\Omega(\mathscr{P}_j), (\mathscr{L}_{ii} + \mathscr{M}_{ii})]_* + [\mathscr{P}_j, (\Omega(\mathscr{L}_{ii}) + \Omega(\mathscr{M}_{ii}))]_*.
\end{aligned}$$

So, we have

$$[\mathscr{P}_j, \mathscr{T}]_* = 0.$$

Therefore, we obtain $\mathscr{T}_{ij} = \mathscr{T}_{ji} = \mathscr{T}_{jj} = 0$ On the other hand, for every $\mathscr{X}_{ji} \in \mathscr{A}_{ji}$, we have

$$\begin{aligned}
& [\Omega(\mathscr{X}_{ji}), (\mathscr{L}_{ii} + \mathscr{M}_{ii})]_* + [\mathscr{X}_{ij}, \Omega(\mathscr{L}_{ii} + \mathscr{M}_{ii})]_* \\
= \; & \Omega([(\mathscr{L}_{ii} + \mathscr{M}_{ii}), \mathscr{X}_{ji}]_*) \\
= \; & \Omega([\mathscr{L}_{ii}, \mathscr{X}_{ji}]_*) + \Omega([\mathscr{M}_{ii}, \mathscr{X}_{ij}]_*) \\
= \; & [\Omega(\mathscr{X}_{ji}), \mathscr{L}_{ii}]_* + [\mathscr{X}_{ji}, \Omega(\mathscr{L}_{ii})]_* + [\Omega(\mathscr{X}_{ji}), \mathscr{M}_{ii}]_* \\
& + [\mathscr{X}_{ji}, \Omega(\mathscr{M}_{ii})]_* \\
= \; & [\Omega(\mathscr{X}_{ji}), (\mathscr{L}_{ii} + \mathscr{M}_{ii})]_* + [\mathscr{X}_{ji}, (\Omega(\mathscr{L}_{ii}) + \Omega(\mathscr{M}_{ii}))]_*.
\end{aligned}$$

So,

$$[(\Omega(\mathscr{L}_{ii} + \mathscr{M}_{ii}) - \Omega(\mathscr{L}_{ii}) - \Omega(\mathscr{M}_{ii})), \mathscr{X}_{ji}]_* = 0.$$

It follows that $[\mathscr{X}_{ji}, \mathscr{T}]_* = 0$ or $\mathscr{X}_{ji}\mathscr{T}_{ii} = 0$. By knowing that $\mathscr{A}$ is prime, we have $\mathscr{T}_{ii} = 0$. Hence, the additivity of Ω comes from the above claims. □

In the rest of this paper we show that $\Omega = 0$.

THEOREM 2.2. *Taking reference to the preceding theorem, if $\Omega(\alpha\mathscr{I})$ is self-adjoint operator for $\alpha \in \{1, i\}$, then $\Omega = 0$.*

Proof. Several claims are used to verify the above theorem.

Claim 9. $\Omega(i\mathscr{I}) = \Omega(\mathscr{I}) = 0$.
Consider $\Omega([i\mathscr{I}, \mathscr{I}]_*) = [\Omega(\mathscr{I}), i\mathscr{I}]_* + [\mathscr{I}, \Omega(i\mathscr{I})]_*$ that imply

$$\begin{aligned}
\Omega(2i\mathscr{I}) &= -i\Omega(\mathscr{I}) - i\Omega(\mathscr{I})^* + \Omega(i\mathscr{I})^* - \Omega(i\mathscr{I}) \\
2\Omega(i\mathscr{I}) &= -2i\Omega(\mathscr{I}) \qquad (2.11)
\end{aligned}$$

By taking the adjoint of above equation we have $\Omega(i\mathscr{I}) = \Omega(\mathscr{I}) = 0$.

Claim 10. Ω preserves $*$.
Since $\Omega(i\mathscr{I}) = \Omega(\mathscr{I}) = 0$, then we can write

$$\Omega([\mathscr{I}, (i\mathscr{L})]_*) = [\Omega(i\mathscr{L}), \mathscr{I}]_* + [(i\mathscr{L}), \Omega(\mathscr{I})]_*$$

$$\Omega(i\mathscr{L} + i\mathscr{L}^*) = \Omega(i\mathscr{L})^* - \Omega(i\mathscr{L})$$

Substiting $i\mathscr{L}$ instead of $\mathscr{L}$ in the above equation, we get

$$\Omega(\mathscr{L}^* - \mathscr{L}) = \Omega(\mathscr{L}) - \Omega(\mathscr{L})^* \qquad (2.12)$$

Replace $\mathscr{L}$ by $\mathscr{L}^*$ in (2.12), we have

$$\Omega(\mathscr{L} - \mathscr{L}^*) = \Omega(\mathscr{L}^*) - \Omega(\mathscr{L}^*)^* \qquad (2.13)$$

Adding (2.12) and (2.13), we get

$$\begin{aligned} \Omega(0) &= \Omega(\mathscr{L}) - \Omega(\mathscr{L})^* + \Omega(\mathscr{L}^*) - \Omega(\mathscr{L}^*)^* \\ 0 &= \Omega(\mathscr{L} + \mathscr{L}^*) - \Omega(\mathscr{L})^* - \Omega(\mathscr{L}^*)^* \qquad (2.14) \end{aligned}$$

Replace $\mathscr{L}$ by $i\mathscr{L}$ in (2.14), we obtain

$$\begin{aligned} 0 &= \Omega(i\mathscr{L} - i\mathscr{L}^*) - \Omega(i\mathscr{L})^* - \Omega(-i\mathscr{L}^*)^* \\ 0 &= i\Omega(\mathscr{L} - \mathscr{L}^*) + i\Omega(\mathscr{L})^* + i\Omega(-\mathscr{L}^*)^* \\ 0 &= \Omega(\mathscr{L} - \mathscr{L}^*) + \Omega(\mathscr{L})^* - \Omega(\mathscr{L}^*)^* \qquad (2.15) \end{aligned}$$

By adding (2.14) and (2.15), we obtain

$$\Omega(\mathscr{L}) = \Omega(\mathscr{L}^*)^*$$

Therefore, Ω preserves $*$.

Claim 11. We prove that $\Omega = 0$.
For every $\mathscr{L}, \mathscr{M} \in \mathscr{A}$, we have

$$\begin{aligned} \Omega(\mathscr{L}\mathscr{M} - \mathscr{M}^*\mathscr{L}^*) &= \Omega([\mathscr{L}, \mathscr{M}^*]_*) \\ &= [\Omega(\mathscr{M}^*), \mathscr{L}]_* + [\mathscr{M}^*, \Omega(\mathscr{L})]_* \\ &= \Omega(\mathscr{M}^*)\mathscr{L}^* - \mathscr{L}\Omega(\mathscr{M}^*)^* + \mathscr{M}^*\Omega(\mathscr{L})^* - \Omega(\mathscr{L})\mathscr{M} \\ \Omega(\mathscr{L}\mathscr{M} - \mathscr{M}^*\mathscr{L}^*) &= \Omega(\mathscr{M})^*\mathscr{L}^* - \mathscr{L}\Omega(\mathscr{M}) + \mathscr{M}^*\Omega(\mathscr{L})^* - \Omega(\mathscr{L})\mathscr{M} \qquad (2.16) \end{aligned}$$

Replace $\mathscr{M}$ by $i\mathscr{M}$ in (2.16) and using Claims 2 and 9, we obtain

$$\Omega(\mathscr{L}\mathscr{M} + \mathscr{M}^*\mathscr{L}^*) = -\Omega(\mathscr{M})^*\mathscr{L}^* - \mathscr{L}\Omega(\mathscr{M}) - \mathscr{M}^*\Omega(\mathscr{L})^* - \Omega(\mathscr{L})\mathscr{M} \qquad (2.17)$$

By adding (2.16) and (2.17), we have

$$\Omega(\mathscr{L}\mathscr{M}) = -\mathscr{L}\Omega(\mathscr{M}) - \Omega(\mathscr{L})\mathscr{M}$$

Taking $\mathscr{M} = \mathscr{I}$, we see that $\Omega(\mathscr{L}) = -\Omega(\mathscr{L})$ which gives $\Omega(\mathscr{L}) = 0$ and hence $\Omega = 0$. This completes the proof. □

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