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A REMARK ON STATISTICAL MANIFOLDS WITH TORSION

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Abstract. Consider a Riemannian manifold M equipped with a metric g. In this article, we study a notion for statistical manifolds (M, g, ∇), which can have a non-zero torsion, abbreviated to SMT. Then it turns out that the tensor fields ∇g and $\tilde{\nabla} g$ obtained from two different SMT-connections are different.

1. Introduction

Let M be a manifold with a metric g. Given a linear connection ∇ there exists a unique connection ∇^* such that

$$d(g(X, Y)) = g(\nabla X, Y) + g(X, \nabla^* Y)$$

and we then say that ∇, ∇^* are dual connections with respect to the metric g.

A statistical manifold can be defined using the notion of dual connections, that is, a manifold (M, g, ∇, ∇^*) satisfying

$$T^{\nabla} = T^{\nabla^*} = 0$$

where the torsion of ∇ is given by

$$T^{\nabla}(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y].$$

There are a few equivalent ways in which statistical manifolds have been introduced; for details we refer to [1, 3, 7, 11].

In this article, we consider statistical manifolds whose torsions are not necessarily zero. We will use a notion of "statistical manifolds admitting torsion" as introduced in [6] and abreviate it as "SMT".

The difference between a linear connection ∇ and the Levi-Civita connection ∇^g is a $(2, 1)$-tensor field denoted by A, that is

$$\nabla_X Y = \nabla^g_X Y + A(X, Y). \tag{1}$$

The notation A is also used for the $(3, 0)$-tensor defined by

$$A(X, Y, Z) = g(A(X, Y), Z).$$

In [5], given a SMT (M, g, ∇) with $\nabla = \nabla^g + A$, an equivalent condition for the difference tensor A is computed, see (8). In this article, we will consider the space

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of A satisfying this condition and denote the space by $\mathcal{SMT}$. We also consider the symmetric space of $\mathcal{A}^S$ consisting of $(3,0)$-tensor fields A which are symmetric with respect to the second and third variables.

In the main Theorem, we will then construct a bijection between $\mathcal{SMT}$ and $\mathcal{A}^S$. Here we observe that $\mathcal{A}^S$ is actually the space of ∇g's where (M, g, ∇) is a SMT, so we conclude that $\nabla g \neq \tilde{\nabla} g$ for two different SMT-connections ∇ and $\tilde{\nabla}$.

2. Preliminaries

Let (M, g) be a Riemannian manifold and $\Gamma(M)$, $\Gamma^*(M)$ the set of sections of the tangent bundle TM, T^*M, respectively.

A linear connection ∇ is then a map

$$\nabla : \Gamma(M) \otimes \Gamma(M) \to \Gamma(M)$$

with some properties and gives a way how to transport a vector field along a direction.

A metric connection ∇ is a linear connection, which gives isometries between tangent spaces by parallel transport, that is

$$V(g(X,Y)) = g(\nabla_V X, Y) + g(X, \nabla_V Y). \tag{2}$$

The condition (2) is equivalent to $\nabla g = 0$, since for $(2,0)$- tensor field g

$$(\nabla_V g)(X,Y) = V(g(X,Y)) - g(\nabla_V X, Y) - g(X, \nabla_V Y).$$

The Levi-Civita connection, denoted by ∇^g, is the unique metric connection with torsion $T = 0$.

The difference between a linear connection ∇ and the Levi-Civita connection ∇^g is a $(2,1)$-tensor (field) A, that is, for any tangent vector fields $X, Y \in \Gamma(M)$,

$$\nabla_X Y = \nabla^g_X Y + A(X,Y).$$

Using the same notation, a $(3,0)$-tensor A is defined by

$$A(X,Y,Z) = \langle A(X,Y), Z\rangle.$$

We now consider the case where isometries between tangent spaces are obtained by parallel transports with respect to two connections ∇, ∇^* as follows.

Definition 2.1 (Dual Connections). For a linear connection ∇, the dual connection ∇^* of ∇ with respect to g is defined by

$$Z(g(X,Y)) = g(\nabla_Z X, Y) + g(X, \nabla^*_Z Y)).$$

By the expression (1) let

$$\begin{aligned} \nabla_X Y &= \nabla^g + A(X,Y) & (3)\\ \nabla^*_X Y &= \nabla^g + A^*(X,Y). & (4) \end{aligned}$$

We can then easily check the following.

Lemma 2.2. *Given a linear connection ∇ and its dual connection ∇^* defined as above, the following equality holds:*

$$\langle A(Z,X), Y\rangle + \langle X, A^*(Z,Y)\rangle = A(Z,X,Y) + A^*(Z,Y,X) = 0. \tag{5}$$

So, a linear connection ∇ has a unique dual connection ∇^*

3. Statistical manifolds admitting torsion

A statistical manifold in a classical sense is a torsion-free manifold with some properties.

In [6] a notion for generalized statistical manifolds is introduced. There are some well-known equivalent properties of these statistical manifolds. In this article, we take the following properties as definitions.

Definition 3.1. [2, 3, 6, 8]

(i) A Riemannian manifold (M, g, ∇) is a statistical manifold if

$$(\nabla_X g)(Y, Z) - (\nabla_Y g)(X, Z) = 0, \tag{6}$$

for $X, Y, Z \in \Gamma(TM)$.

(ii) A Riemannian manifold (M, g, ∇) is a statistical manifold admitting torsion, (SMT) for short, if

$$(\nabla_X g)(Y, Z) - (\nabla_Y g)(X, Z) = -g(T^{\nabla}(X, Y), Z), \tag{7}$$

for $X, Y, Z \in \Gamma(TM)$, where T^{∇} is the torsion tensor of ∇.

Considering the difference tensor field A as in (3), we obtain the following result.

Proposition 3.2. [5, 8] *Given a Riemannian manifold (M, g, ∇) the following conditions are equivalent.*

(i) (M, g, ∇, ∇^*) *is a SMT.*

(ii) Let $\nabla = \nabla^g + A$. *Then it holds*

$$A(X, Y, Z) = A(Z, Y, X) \quad \text{for } X, Y, Z \in \Gamma(TM). \tag{8}$$

(iii) $T^{\nabla^*} = 0$.

Here we note that a statistical manifold (M, g, ∇, ∇^*) in a classical sense is the manifold with $T^{\nabla} = T^{\nabla^*} = 0$.

We consider the $(3, 0)$- tensor field A as an element of $\otimes^3 TM$, identifying TM with TM^*. Then by Proposition 3.2 (ii), for the set of SMT's we can consider a space as follows:

$$\mathcal{SMT} = \{A \in \otimes^3 TM | A(X, Y, Z) = A(Z, Y, X)\}.$$

We also take a symmetric space:

$$\mathcal{A}^S = \{A \in \otimes^3 TM | A(X, Y, Z) = A(X, Z, Y)\} = TM \otimes S^2 TM.$$

We will then find a bijection between the above two spaces in the following theorem.

Theorem 3.3. *A bijection between $\mathcal{SMT}$ and $\mathcal{A}^S$ arises from the following:*

For $S \in \mathcal{SMT}$, we associate $G \in \mathcal{A}^S$ by

$$G(X, Y, Z) = S(X, Y, Z) + S(X, Z, Y). \tag{9}$$

And for $G \in \mathcal{A}^S$, we associate $S \in \mathcal{SMT}$ by

$$2S(X, Y, Z) = G(X, Y, Z) - G(Y, Z, X) + G(Z, X, Y).$$

Proof. Given $S \in \mathcal{SMT}$, we get $G \in \mathcal{A}^S$ by

$$G(X, Y, Z) = S(X, Y, Z) + S(X, Z, Y) \in \mathcal{A}^S.$$

Now since $S \in \mathcal{SMT}$,

$$\begin{aligned} & G(X, Y, Z) - G(Y, Z, X) + G(Z, X, Y) \\ = \; & S(X, Y, Z) + S(X, Z, Y) - S(Y, Z, X) - S(Y, X, Z) \\ & + S(Z, X, Y) + S(Z, Y, X) \\ = \; & 2S(X, Y, Z). \end{aligned}$$

We note that the above (9) gives a linear map for each T_xM, $x \in M$.

Finally, the elements of $\mathcal{SMT}$ and $\mathcal{A}^S$ are symmetric with respect to two variables, namely, first and third ones for $\mathcal{SMT}$, second and third ones for $\mathcal{A}^S$. So, $\mathcal{SMT}$ and $\mathcal{A}^S$ have the same dimension.

We now conclude that the mapping (9) is a bijection from $\mathcal{SMT}$ to $\mathcal{A}^S$. □

Corollary 3.4. *Two different SMT-connections ∇ and $\tilde{\nabla}$ give two different and tensor fields ∇g and $\tilde{\nabla} g$.*

Proof. For $\nabla = \nabla^g + A$, recall that

$$\nabla g = A(X, Y, Z) + A(X, Z, Y). \tag{10}$$

So, by the bijection in Theorem 3.3, we have two different tensor fields ∇g and $\tilde{\nabla} g$ for two different SMT- connection ∇, $\tilde{\nabla}$.

Here (10) follows from

$$\begin{aligned} \nabla g(X, Y, Z) &= (\nabla_X g)(Y, Z) \\ &= X(g(Y, Z)) - g(\nabla_X Y, Z) - g(Y, \nabla_X Z) \\ &= \nabla^g g(X, Y, Z) + A(X, Y, Z) + A(X, Z, Y) \\ &= A(X, Y, Z) + A(X, Z, Y). \end{aligned}$$

□

Remark 3.5. Since $\otimes^2 TM = \Lambda^2 TM \oplus S^2 TM$ where the tensor product Λ^2 and S^2 is skew-symmetric and symmetric tensor products, respectively, we have

$$\otimes^3 TM = \mathcal{A}^M \oplus \mathcal{A}^S$$

with

$$\mathcal{A}^M = \{A \in \otimes^3 TM | A(X, Y, Z) = -A(X, Z, Y)\} = TM \otimes \Lambda^2 TM.$$

So, from the bijection in Theorem 3.3 we also have a bijection between $\mathcal{SMT}$ and $\otimes^3 TM/\mathcal{A}^M$. Note that $\mathcal{A}^M$ is the space of A's of metric connections ∇, that is, linear connections with $\nabla g = 0$.

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