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BOUNDED FUNCTION ON WHICH INFINITE ITERATIONS OF WEIGHTED BEREZIN TRANSFORM EXIST

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Abstract. We exhibit some properties of the weighted Berezin transform $T_\alpha f$ on $L^\infty(B_n)$ and on $L^1(B_n)$. As the main result, we prove that if $f \in L^\infty(B_n)$ with $\lim_{k\to\infty} T_\alpha^k f$ exists, then there exist unique $\mathcal{M}$-harmonic function g and $h \in \overline{(I-T_\alpha)L^\infty(B_n)}$ such that $f = g + h$. We also show that of the norm of weighted Berezin operator T_α on $L^1(B_n, \nu)$ converges to 1 as α tends to infinity, where ν is an ordinary Lebesgue measure.

1. Introduction and Preliminaries

Let B_n be the unit ball of $\mathbb{C}^n$ and let ν be the Lebesgue measure on $\mathbb{C}^n$ normalized to $\nu(B_n) = 1$. For $\alpha > 0$, we define a positive measure ν_α by

$$d\nu_\alpha(z) = \frac{\Gamma(n+\alpha+1)}{n!\Gamma(\alpha+1)}(1-|z|^2)^\alpha d\nu(z)$$

so that $\nu_\alpha(B_n) = 1$. For such α and $f \in L^1(B_n, \nu_\alpha)$, the weighted Berezin transform $T_\alpha f$ on B_n is defined by

$$(T_\alpha f)(z) \;=\; \int_{B_n} f\big(\varphi_z(w)\big)\, d\nu_\alpha(w) \quad \text{for } z \in B_n,$$

where $\varphi_a \in \mathrm{Aut}(B_n)$ is the canonical automorphism given by

$$\varphi_a(z) \;=\; \frac{a - Pz - (1-|a|^2)^{1/2} Qz}{1 - \langle z, a\rangle}$$

on which P is the projection into the space spanned by $a \in B_n$ and $Q_z = z - Pz$ as we follow the standard notations of [8]. Equivalently, we can write the weighted Berezin transform

$$(T_\alpha f)(z) \;=\; \int_{B_n} f(w)\, \frac{(1-|z|^2)^{n+1+\alpha}}{|1-\langle z, w\rangle|^{2n+2+2\alpha}}\, d\nu_\alpha(w). \tag{1.1}$$

The invariant Laplacian $\tilde{\Delta}$ is defined for $f \in C^2(B_n)$ by

$$(\tilde{\Delta} f)(z) \;=\; \Delta(f \circ \varphi_z)(0).$$

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The $\mathcal{M}$-harmonic functions on B_n are those for which $\tilde{\Delta} f = 0$. If a function $f \in L^1(B_n, \nu_\alpha)$ is $\mathcal{M}$-harmonic, then $f \circ \psi$ is also $\mathcal{M}$-harmonic for every $\psi \in \mathrm{Aut}(B_n)$, so that f satisfies

$$(T_\alpha f)(z) = f(z) \quad \text{for every } z \in B_n.$$

Conversely, it is proved that a bounded function f in the symmetric domain invariant under the general Berezin transform is harmonic with respect to its intrinsic metric([1], [2], [4]). Especially, $f \in L^\infty(B_n)$ satisfying $T_\alpha f = f$ is $\mathcal{M}$-harmonic([5]).

The invariant measure τ on B_n is defined by

$$d\tau(z) \ = \ (1 - |z|^2)^{-n-1} \, d\nu(z)$$

and satisfies

$$\int_{B_n} f \, d\tau \ = \ \int_{B_n} (f \circ \psi) \, d\tau$$

for every $f \in L^1(\tau)$ and $\psi \in \mathrm{Aut}(B_n)$.

Even though τ is not a finite measure on B_n so that a non-zero constant does not belong to $L^1(\tau)$, T_α on $L^\infty(B_n)$ is the adjoint of T_α on $L^1(\tau)$ in the sense that

$$\int_{B_n} (T_\alpha f) \cdot g \, d\tau \ = \ \int_{B_n} f \cdot (T_\alpha g) \, d\tau \tag{1.2}$$

for $f \in L^1(\tau)$ and $g \in L^\infty(B_n)$. For $1 \le p \le \infty$, we denote $L^p_R(\tau)$ as the subspace of $L^p(B_n, \tau)$ which consists of radial functions.

Previously, in Lemma 2.1 of [5], the author proved that

$$\lim_{k \to \infty} \|T^k_\alpha (I - T_\alpha)\| = 0 \quad \text{on} \quad L^1_R(\tau), \tag{1.3}$$

which plays an important role in this paper.

In the previous papers [5], [6] and [7], the author invesitgated various properties on iterates of T_α on $L^1(\tau)$, $L^\infty(B_n)$ and $L^p(\tau)$ for $1 < p < \infty$. Especially, the Corollary 2.2 of [7] shows that there exists $f \in L^\infty(B_n)$ for which $\lim_{k\to\infty} T^k_\alpha f$ does not exist. Here we answer the question on which $f \in L^\infty(B_n)$, $\lim_{k\to\infty} T^k_\alpha f$ exists.

This paper exhibits some new properties of the weighted Berezin transform $T_\alpha f$ on $L^\infty(B_n)$ and on $L^1(B_n)$. In section 2, as a main result of this paper, we characterize $f \in L^\infty(B_n)$ on which $\lim_{k\to\infty} T^k_\alpha f$ exists.

In section 3, we investigate the limit of the norm of T_α on $L^1(B_n, \nu)$ by showing that T_α is bounded on $L^1(B_n, \nu)$ and its norm converges to 1 as α tends to infinity.

2. The iterations of T_α

This section contains main results of the paper, which characterize $f \in L^\infty(B_n)$ on which $\lim_{k\to\infty} T^k_\alpha f$ exists. Previously, in Corollary 2.2 of [7], the author showed that there exists $f \in L^\infty(B_n)$ for which $\lim_{k\to\infty} T^k_\alpha f$ does not exist pointwise. For such f, it is quite easy to see that $\lim_{k\to\infty} T^k_\alpha f$ does not exist in $L^\infty(B_n)$ either. The proof of the following lemma is almost identical to that of Corollary 2.2 of [7].

LEMMA 2.1. *There exists* $f \in L^\infty(B_n)$ *for which* $\lim_{k\to\infty} T^k_\alpha f$ *does not exist in* $L^\infty(B_n)$.

Proof. Assume that lim $T_\alpha^k \ell$ exists for every $\ell \in L^\infty(B_n)$. Since T_α is self-adjoint on $L^1(\tau)$ in the sense that

$$(2.1) \qquad \int_{B_n} (T_\alpha g) \cdot \ell \; d\tau \;=\; \int_{B_n} g \cdot (T_\alpha \ell) \; d\tau$$

for every $g \in L^1(\tau)$. Now choose any radial function $g \in L^1_R(\tau)$ with $\int_{B_n} g \; d\tau \neq 0$ then

$$\lim_{k\to\infty} \int_{B_n} \left(T_\alpha^k g\right) \, \ell \; d\tau \;=\; \lim_{k\to\infty} \int_{B_n} g \, \left(T_\alpha^k \ell\right) \, d\tau$$

exists for every $\ell \in L^\infty(B_n)$. This means $\{T_\alpha^k g\}$ converges weakly. Therefore, by Proposition 2.1 of [7], we have

$$\lim_{k\to\infty} \|T_\alpha^k g\|_{L^1(\tau)} \;=\; 0.$$

On the other hand, by Proposition 3.2 of [6], we also have

$$(2.2) \qquad \lim_{k\to\infty} \|T_\alpha^k g\|_{L^1(\tau)} \;=\; \left| \int_{B_n} g \; d\tau \right| \neq 0,$$

which is a contradiction. □

The next lemma extends Lemma 2.1 of [5] to the whole $L^\infty(B_n)$, and plays a key role in the proof of the main theorem.

Lemma 2.2. *$\lim_{k\to\infty} \|T_\alpha^k (I - T_\alpha)\| = 0$ on $L^\infty(B_n)$.*

Proof. The Lemma 2.1 of [5] states that

$$(2.3) \qquad \lim_{k\to\infty} \|T_\alpha^k (I - T_\alpha)\| = 0 \quad \text{on} \quad L^1_R(\tau).$$

Since $L^\infty_R(B_n)$ is a dual space of $L^1_R(\tau)$, the spectrum of the self-adjoint operator T_α on $L^\infty_R(B_n)$ is the same as the spectrum of T_α on $L^1_R(B_n, \tau)$. Therefore, we get

$$(2.4) \qquad \lim_{k\to\infty} \|T_\alpha^k (I - T_\alpha)\| = 0 \quad \text{on} \quad L^\infty_R(B_n).$$

For that reason, it is enough to show that the spectrum of T_α on $L^\infty(B_n)$ is the same as the spectrum of T_α on $L^\infty_R(B_n)$. Let λ be in the spectrum of T_α on $L^\infty(B_n)$, then there exists a sequence $\{f_k\}$ in $L^\infty(B_n)$ with $\|f_k\|_\infty = 1$ for which

$$\lim_{k\to\infty} \| \; T_\alpha f_k - \lambda f_k\|_\infty = 0.$$

Let $\phi_k \in \mathrm{Aut}(B_n)$ satisfy $\|R(f_k \circ \phi_k)\|_\infty = 1$ where Rf is the radialization (4.2.1 of [8]) of f. Since T_α and R are contractions on $L^\infty(B_n)$,

$$\begin{aligned} \| \; T_\alpha\big(R(f_k \circ \phi_k)\big) - \lambda R(f_k \circ \phi_k)\|_\infty \;&=\; \| \; R\big(\; T_\alpha(f_k \circ \phi_k)\big) - R(\lambda f_k \circ \phi_k) \; \|_\infty \\ &\leq\; \| \; T_\alpha(f_k \circ \phi_k) \;-\; \lambda f_k \circ \phi_k \; \|_\infty \\ &=\; \| \; (T_\alpha f_k) \circ \phi_k \;-\; \lambda f_k \circ \phi_k \; \|_\infty \\ &=\; \| \; T_\alpha f_k \;-\; \lambda f_k \; \|_\infty \;\to\; 0 \quad \text{as} \quad k \;\to\; \infty \end{aligned}$$

Hence λ is in the spectrum of T_α on $L^\infty_R(B_n)$. Therefore the spectrum of T_α on $L^\infty(B_n)$ is the same as the spectrum of T_α on $L^\infty_R(B_n)$ so that by (2.4) we have

$$(2.5) \qquad \lim_{k\to\infty} \|T_\alpha^k (I - T_\alpha)\| = 0 \quad \text{on} \quad L^\infty(B_n).$$

This completes the proof. □

Next theorem is the main result of the paper.

Theorem 2.3. *Let $f \in L^\infty(B_n)$. If $\lim_{k\to\infty} T_\alpha^k f$ exists, then there exist unique $\mathcal{M}$-harmonic function g and $h \in \overline{(I - T_\alpha)L^\infty(B_n)}$ such that $f = g + h$.*

Proof. If $f \in L^\infty(B_n)$ and $\lim_{k\to\infty} T_\alpha^k f$ exists, then there is $g \in L^\infty(B_n)$ such that

$$\lim_{k\to\infty} \|T_\alpha^k f - g\|_\infty = 0.$$

Hence we have

$$g(z) = \lim_{k\to\infty} (T_\alpha^k f)(z) \quad \text{for } z \in B_n.$$

By by the Dominated Convergence Theorem, we have

$$T_\alpha g = T_\alpha(\lim_{k\to\infty} T_\alpha^k f) = \lim_{k\to\infty} T_\alpha^{k+1} f = g.$$

Which means that $g \in L^\infty(B_n)$ satisfies $T_\alpha g = g$. Therefore, g is $\mathcal{M}$-harmonic.
Define $h = f - g$ so that $f = g + h$. And then we will show that

$$h \in \overline{(I - T_\alpha)L^\infty(B_n)}. \tag{2.6}$$

Let $F \in L^\infty(B_n)^*$ satisfy $F(\ell - T_\alpha \ell) = 0$ for all $\ell \in L^\infty(B_n)$. If we denote

$$[F,\ \ell] = F(\ell) \tag{2.7}$$

then we get

$$0 = F(\ell - T_\alpha \ell) = [F,\ \ell - T_\alpha \ell] = [F - T_\alpha^* F,\ \ell]$$

for all $\ell \in L^\infty(B_n)$. Thus, we have $T_\alpha^* F = F$.
Hence for every $k \geq 0$,

$$[F,\ h] = [F,\ f - g] = [(T_\alpha^*)^k F,\ f - g] = [F,\ T_\alpha^k(f - g)]. \tag{2.8}$$

Since $\lim_{k\to\infty} \|T_\alpha^k f - g\|_\infty = 0$ and $T_\alpha g = g$ we have

$$\lim_{k\to\infty} \|T_\alpha^k (f - g)\|_\infty = 0.$$

By taking the limit $k \to \infty$ in (2.8), we get $[F,\ h] = 0$.

So far we showed that every bounded linear functional on $L^\infty(B_n)$ that vanishes on
$(I - T_\alpha)L^\infty(B_n)$ also vanishes at h. Hence by the Hahn-Banach theorem, we conclude

$$h \in \overline{(I - T_\alpha)L^\infty(B_n)}.$$

Thus, it remains to prove the uniqueness of such decomposition $f = g + h$.

Suppose $f = g_1 + h_1 = g_2 + h_2$ where $g_1,\ g_2$ are $\mathcal{M}$-harmonic and $h_1, h_2 \in \overline{(I - T_\alpha)L^\infty(B_n)}$. Since

$$g_1 - g_2 = h_2 - h_1 \in \overline{(I - T_\alpha)L^\infty(B_n)}, \tag{2.9}$$

$g_1 - g_2$ is an $\mathcal{M}$-harmonic function which belongs to $\overline{(I - T_\alpha)L^\infty(B_n)}$. By Lemma 2.2, we have

$$\lim_{k\to\infty} \|T_\alpha^k (I - T_\alpha)\| = 0 \quad \text{on} \quad L^\infty(B_n). \tag{2.10}$$

Therefore

$$\lim_{k\to\infty} \|T_\alpha^k (g_1 - g_2)\|_\infty = 0.$$

However, since $g_1 - g_2$ is $\mathcal{M}$-harmonic, we get $T_\alpha^k(g_1 - g_2) = g_1 - g_2$ for every $k \geq 0$. So that we have

$$\|g_1 - g_2\|_\infty = 0.$$

Therefore, we get

$$g_1 - g_2 = h_2 - h_1 = 0. \tag{2.11}$$

And this completes the proof of the theorem. □

3. The iterations of T_α

This section investigates the limit of the nrom of T_α on $L^1(B_n, \nu)$. Naturally T_α is an operator on $L^1(B_n, \nu_\alpha)$. However, T_α also can be an operator on $L^1(B_n, \nu)$. Even though we could find the norm of T_α on $L^1(B_n, \nu)$ as a closed form, we can show that T_α is bounded on $L^1(B_n, \nu)$ and its norm converges to 1 as α tends to infinity.

PROPOSITION 3.1. *If $\alpha > 0$ then T_α is bounded on $L^1(B_n, \nu)$ and $\lim_{\alpha\to\infty} \|T_\alpha\| = 1$.*

Proof. If we put $f \equiv 1$ then $T_\alpha f = f$ and $\|f\|_{L^1(\nu)} = 1$. Hence we have

$$\|T_\alpha\| \geq 1. \tag{3.1}$$

On the other hand, since

$$(T_\alpha f)(z) = \frac{\Gamma(n+\alpha+1)}{n!\Gamma(\alpha+1)} \int_{B_n} f(w) \frac{(1-|z|^2)^{n+1+\alpha}(1-|w|^2)^\alpha}{|1-\langle z, w\rangle|^{2n+2+2\alpha}}\, d\nu(w), \tag{3.2}$$

the norm of T_α on $L^1(B_n, \nu)$ is

$$\begin{aligned} \|T_\alpha\| &= \sup_{\|f\|_{L^1(\nu)}=1} \|T_\alpha f\|_{L^1(\nu)} \\ &= \sup_{w\in B_n} \frac{\Gamma(n+\alpha+1)}{n!\Gamma(\alpha+1)}(1-|w|^2)^\alpha \int_{B_n} \frac{(1-|z|^2)^{n+1+\alpha}}{|1-\langle z, w\rangle|^{2n+2+2\alpha}}\, d\nu(z). \end{aligned}$$

To calculate the integral by using the polar coordinate we have

$$\int_{B_n} \frac{(1-|z|^2)^{n+1+\alpha}}{|1-\langle z, w\rangle|^{2n+2+2\alpha}}\, d\nu(z) = 2n \int_0^1 (1-r^2)^{n+1+\alpha} \int_S \frac{1}{|1-\langle r\zeta, w\rangle|^{2n+2+2\alpha}}\, d\sigma(\zeta) r^{2n-1} dr,$$

where S is the unit sphere of $\mathbb{C}^n$ and σ is the rotation-invariant positive Borel measure normalized to be $\sigma(S) = 1$. Using the binomial formula for $|x| < 1$ and $\lambda > 0$

$$\frac{1}{(1-x)^\lambda} = \sum_{k=0}^\infty \frac{\Gamma(k+\lambda)}{k!\Gamma(\lambda)} x^k,$$

the above inner integral is

$$\begin{aligned} \int_S \frac{1}{|1-\langle rw, \zeta\rangle|^{2n+2+2\alpha}}\, d\sigma(\zeta) &= \int_S \left| \sum_{k=0}^\infty \frac{\Gamma(k+n+1+\alpha)}{k!\Gamma(n+1+\alpha)} \langle rw, \zeta\rangle^k \right|^2 d\sigma(\zeta) \\ &= \sum_{k=0}^\infty \left| \frac{\Gamma(k+n+1+\alpha)}{k!\Gamma(n+1+\alpha)} \right|^2 \int_S \left| \langle rw, \zeta\rangle^k \right|^2 d\sigma(\zeta) \\ &= \frac{\Gamma(n)}{\Gamma^2(n+\alpha+1)} \sum_{k=0}^\infty \frac{\Gamma^2(n+\alpha+1+k)|rw|^{2k}}{\Gamma(k+1)\Gamma(n+k)}. \end{aligned}$$

We insert the above series into inner integral of the polar coordinate and integrate term by term to obtain

$$\int_{B_n} \frac{(1-|z|^2)^{n+1+\alpha}}{|1-\langle z,w\rangle|^{2n+2+2\alpha}}\,d\nu(z) \;=\; \frac{\Gamma(n+1)\Gamma(n+\alpha+2)}{\Gamma^2(n+\alpha+1)}\sum_{k=0}^{\infty}\frac{\Gamma^2(n+\alpha+1+k)|w|^{2k}}{\Gamma(k+1)\Gamma(2n+2+\alpha+k)}.$$

Now using a well known inequality $\Gamma^2(a+b) \le \Gamma(a)\Gamma(a+2b)$ for $a, b > 0$, we have

$$\Gamma^2(n+\alpha+1+k) \le \Gamma(k+\alpha)\Gamma(2n+2+k+\alpha).$$

So that

$$\sum_{k=0}^{\infty}\frac{\Gamma^2(n+\alpha+1+k)|w|^{2k}}{\Gamma(k+1)\Gamma(2n+2+\alpha+k)} \le \sum_{k=0}^{\infty}\frac{\Gamma(k+\alpha)\ |w|^{2k}}{\Gamma(k+1)}.$$

Therefore, we have

$$\begin{aligned}
\|T_\alpha\| &= \sup_{w\in B_n}\frac{\Gamma(n+\alpha+1)}{n!\Gamma(\alpha+1)}(1-|w|^2)^\alpha\int_{B_n}\frac{(1-|z|^2)^{n+1+\alpha}}{|1-\langle z,w\rangle|^{2n+2+2\alpha}}\,d\nu(z)\\
&\le \sup_{w\in B_n}\frac{\Gamma(n+\alpha+1)}{n!\Gamma(\alpha+1)}(1-|w|^2)^\alpha\frac{\Gamma(n+1)\Gamma(n+\alpha+2)}{\Gamma^2(n+\alpha+1)}\sum_{k=0}^{\infty}\frac{\Gamma(k+\alpha)\ |w|^{2k}}{\Gamma(k+1)}\\
&= \sup_{w\in B_n}\frac{n+\alpha+1}{\alpha}(1-|w|^2)^\alpha\sum_{k=0}^{\infty}\frac{\Gamma(k+\alpha)\ |w|^{2k}}{\Gamma(\alpha)\Gamma(k+1)}\\
&= \frac{n+\alpha+1}{\alpha},
\end{aligned}$$

where the last equality comes from

$$\sum_{k=0}^{\infty}\frac{\Gamma(k+\alpha)\ |w|^{2k}}{\Gamma(\alpha)\Gamma(k+1)} = \frac{1}{(1-|w|^2)^\alpha}.$$

Combining this with (3.1), we get

$$1 \le \|T_\alpha\| \le \frac{n+\alpha+1}{\alpha}$$

on $L^1(B_n,\nu)$. By taking $\alpha\to\infty$ we complete the proof. □

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