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ON BETA PRODUCT OF HESITANCY FUZZY GRAPHS AND INTUITIONISTIC HESITANCY FUZZY GRAPHS

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ABSTRACT. The degree of hesitancy of a vertex in a hesitancy fuzzy graph depends on the degree of membership and non-membership of the vertex. We define a new class of hesitancy fuzzy graph, the intuitionistic hesitancy fuzzy graph in which the degree of hesitancy of a vertex is independent of the degree of its membership and non-membership. We introduce the idea of β-product of a pair of hesitancy fuzzy graphs and intuitionistic hesitancy fuzzy graphs and prove certain results based on this product.

1. Introduction

Zadeh [12] put forth the idea of fuzzy set and the concept of fuzzy set brought about revolutionary changes in the area of interdisciplinary research. As Euler pioneered the concept of graph theory, Rosenfeld [8] developed fuzzy graph (FG) theory in 1975. Another innovative study on fuzzy set was made by Atanassov [1] who introduced intuitionistic fuzzy sets. Next breakthrough came when R.Parvathi [5] developed intutionistic fuzzy graph (IFG) and it was followed by T.Pathinathan [6] who developed the concept of hesitancy fuzzy graph (HFG). Later on, Ch.Chaitanya and T.V. Pradeep Kumar [2] introduced the idea of complete product of FGs. Many perspectives on hesitancy fuzzy sets and HFGs are discussed in [3, 4, 7, 9–11].

We define β-product of a pair of HFGs. In an HFG, the degree of hesitancy (ρ_1) of a vertex depends on the degree of membership (MS) λ_1 and non-membership (NMS) δ_1 of the vertex. An HFG is strong if it is λ-strong, δ-strong and ρ-strong. We establish that β-product of a pair of strong HFGs need not be a strong HFG because β-product of a pair of strong HFGs need not be ρ-strong. We introduce a new class of HFG, the intuitionistic HFG (IHFG) in which ρ_1 is independent of λ_1 and δ_1 and prove that β-product of a pair of strong IHFGs is a strong IHFG. For two complete IHFGs, their β-product is also a complete IHFG. If the β-product of a pair of IHFGs is strong, then at least one of the IHFG will be strong.

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2. Preliminaries

DEFINITION 2.1. [5] An IFG is $G = (V, E, \sigma, \mu)$, V is the vertex set, $\sigma = (\lambda_1, \delta_1), \mu = (\lambda_2, \delta_2)$ and $\lambda_1, \delta_1 : V \to [0, 1]$ represent the degree of MS, NMS of $v \in V$,

$$0 \leq \lambda_1(v) + \delta_1(v) \leq 1.$$

$\lambda_2, \delta_2 : V \times V \to [0, 1]$ represent the degree of MS, NMS of the edge $x = (u, v) \in V \times V$,

$$\lambda_2(x) \leq min\{\lambda_1(u), \lambda_1(v)\}$$
$$\delta_2(x) \leq max\{\delta_1(u), \delta_1(v)\}$$
$$0 \leq \lambda_2(x) + \delta_2(x) \leq 1, \forall x \in V \times V$$

DEFINITION 2.2. [6] An HFG is $G = (V, E, \sigma, \mu)$, V is the vertex set, $\sigma = (\lambda_1, \delta_1, \rho_1), \mu = (\lambda_2, \delta_2, \rho_2)$ and $\lambda_1, \delta_1, \rho_1 : V \to [0, 1]$ represent the degree of MS, NMS and hesitancy of $v \in V$,

$$\lambda_1(v) + \delta_1(v) + \rho_1(v) = 1$$
$$\text{where,} \quad \rho_1(v) = 1 - [\lambda_1(v) + \delta_1(v)].$$

$\lambda_2, \delta_2, \rho_2 : V \times V \to [0, 1]$ represent the degree of MS, NMS and hesitancy of $x = (u, v) \in V \times V$,

$$\lambda_2(x) \leq min\{\lambda_1(u), \lambda_1(v)\}$$
$$\delta_2(x) \leq max\{\delta_1(u), \delta_1(v)\}$$
$$\rho_2(x) \leq min\{\rho_1(u), \rho_1(v)\}$$
$$0 \leq \lambda_2(x) + \delta_2(x) + \rho_2(x) \leq 1, \forall x \in V \times V$$

3. Main Results

In an HFG, the degree of hesitancy (ρ_1) of a vertex v depends on the degree of MS (λ_1) and NMS (δ_1) of v. We define a new class of HFG namely, the intuitionistic HFG (IHFG) in which ρ_1 is independent of λ_1 and δ_1.

DEFINITION 3.1. An IHFG is $G = (V, E, \sigma, \mu)$, V is the vertex set, $\lambda_1, \delta_1, \rho_1 : V \to [0, 1]$ represent the degree of MS, NMS and hesitancy of $v \in V$,

$$0 \leq \lambda_1(v) + \delta_1(v) + \rho_1(v) \leq 1.$$

$\lambda_2, \delta_2, \rho_2 : V \times V \to [0, 1]$ represent the degree of MS, NMS and hesitancy of $x = (u, v) \in V \times V$,

$$\lambda_2(x) \leq min\{\lambda_1(u), \lambda_1(v)\}$$
$$\delta_2(x) \leq max\{\delta_1(u), \delta_1(v)\}$$
$$\rho_2(x) \leq min\{\rho_1(u), \rho_1(v)\}$$
$$0 \leq \lambda_2(x) + \delta_2(x) + \rho_2(x) \leq 1, \forall x$$

REMARK 3.2. All HFGs are IHFGs but all IHFGs need not be HFGs. In figure 1, G_1 is an HFG since $\lambda_1(u_i) + \delta_1(u_i) + \rho_1(u_i) = 1, \forall i$ where $\rho_1(u_i) = 1 - [\lambda_1(u_i) + \delta_1(u_i)]$. G_1 is also an IHFG. G_2 is an IHFG since $0 \leq \lambda_1(v_i) + \delta_1(v_i) + \rho_1(v_i) \leq 1, \forall i$, but G_2 is not a HFG.

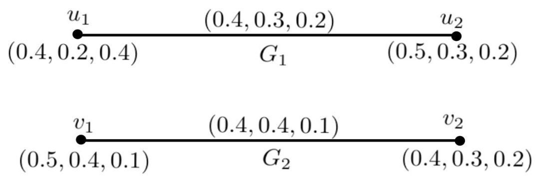

FIGURE 1. Example for HFG and IHFG

DEFINITION 3.3. An HFG G_1 or an IHFG G_2 is

$$\lambda\text{-strong if} \quad \lambda_2(x) = min\{\lambda_1(u), \lambda_1(v)\}, \forall x = (u, v) \in E$$

EXAMPLE 3.4. Consider the HFG G_1 with vertices u_1, u_2, u_3 and the IHFG G_2 with vertices v_1, v_2, v_3 in figure 2.
$\lambda_2(u_1, u_2) = 0.3, \lambda_1(u_1) \wedge \lambda_1(u_2) = 0.4 \wedge 0.3 = 0.3,$
$\lambda_2(u_2, u_3) = 0.3, \lambda_1(u_2) \wedge \lambda_1(u_3) = 0.3 \wedge 0.5 = 0.3$
$\lambda_2(u_1, u_2) = \lambda_1(u_1) \wedge \lambda_1(u_2)$
$\lambda_2(u_2, u_3) = \lambda_1(u_2) \wedge \lambda_1(u_3)$. Thus, G_1 is a λ-strong HFG.
$\lambda_2(v_1, v_2) = 0.4, \lambda_1(v_1) \wedge \lambda_1(v_2) = 0.4 \wedge 0.5 = 0.4,$
$\lambda_2(v_2, v_3) = 0.3, \lambda_1(v_2) \wedge \lambda_1(v_3) = 0.5 \wedge 0.3 = 0.3$
$\lambda_2(v_1, v_2) = \lambda_1(v_1) \wedge \lambda_1(v_2)$
$\lambda_2(v_2, v_3) = \lambda_1(v_2) \wedge \lambda_1(v_3)$. Thus, G_2 is a λ-strong IHFG.

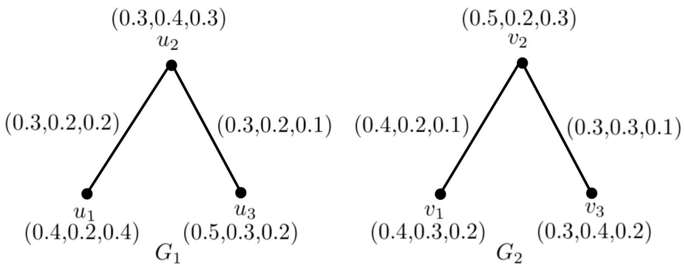

FIGURE 2. λ-strong HFG G_1 and λ-strong IHFG G_2

DEFINITION 3.5. An HFG G_1 or an IHFG G_2 is

$$\delta\text{-strong if} \quad \delta_2(x) = max\{\delta_1(u), \delta_1(v)\}, \forall x = (u, v) \in E$$

EXAMPLE 3.6. In figure 3,
$\delta_2(u_1, u_2) = 0.4, \delta_1(u_1) \vee \delta_1(u_2) = 0.2 \vee 0.4 = 0.4,$
$\delta_2(u_2, u_3) = 0.4, \delta_1(u_2) \vee \delta_1(u_3) = 0.4 \vee 0.3 = 0.4$
$\delta_2(u_1, u_2) = \delta_1(u_1) \vee \delta_1(u_2)$
$\delta_2(u_2, u_3) = \delta_1(u_2) \vee \delta_1(u_3)$. Thus, G_1 is a δ-strong HFG.
$\delta_2(v_1, v_2) = 0.3, \delta_1(v_1) \vee \delta_1(v_2) = 0.3 \vee 0.2 = 0.3,$
$\delta_2(v_2, v_3) = 0.4, \delta_1(v_2) \vee \delta_1(v_3) = 0.2 \vee 0.4 = 0.4$
$\delta_2(v_1, v_2) = \delta_1(v_1) \vee \delta_1(v_2)$
$\delta_2(v_2, v_3) = \delta_1(v_2) \vee \delta_1(v_3)$. Thus, G_2 is a δ-strong IHFG.

DEFINITION 3.7. An HFG G_1 or an IHFG G_2 is

$$\rho\text{-strong if} \quad \rho_2(x) = min\{\rho_1(u), \rho_1(v)\}, \forall x = (u, v) \in E$$

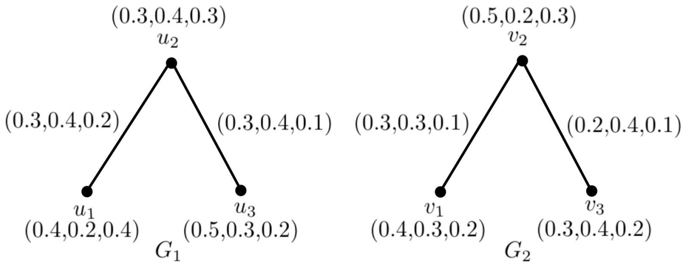

FIGURE 3. δ-strong HFG G_1 and δ-strong IHFG G_2

EXAMPLE 3.8. In figure 4, $\rho_2(u_1, u_2) = 0.3, \rho_1(u_1) \wedge \rho_1(u_2) = 0.4 \wedge 0.3 = 0.3,$
$\rho_2(u_2, u_3) = 0.2, \rho_1(u_2) \wedge \rho_1(u_3) = 0.3 \wedge 0.2 = 0.2$
$\rho_2(u_1, u_2) = \rho_1(u_1) \wedge \rho_1(u_2)$
$\rho_2(u_2, u_3) = \rho_1(u_2) \wedge \rho_1(u_3)$. Thus, G_1 is a ρ-strong HFG.
$\rho_2(v_1, v_2) = 0.2, \rho_1(v_1) \wedge \rho_1(v_2) = 0.2 \wedge 0.3 = 0.2,$
$\rho_2(v_2, v_3) = 0.2, \rho_1(v_2) \wedge \rho_1(v_3) = 0.3 \wedge 0.2 = 0.2$
$\rho_2(v_1, v_2) = \rho_1(v_1) \wedge \rho_1(v_2)$
$\rho_2(v_2, v_3) = \rho_1(v_2) \wedge \rho_1(v_3)$. Thus, G_2 is ρ-strong IHFG.

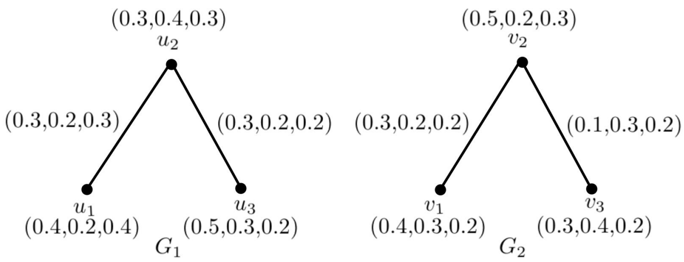

FIGURE 4. ρ-strong HFG G_1 and ρ-strong IHFG G_2

DEFINITION 3.9. An HFG G_1 or an IHFG G_2 is strong if it is λ-strong, δ-strong and ρ-strong.

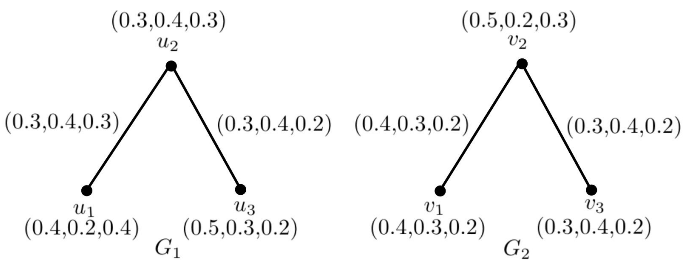

FIGURE 5. strong HFG G_1 and strong IHFG G_2

DEFINITION 3.10. A HFG G_1 or an IHFG G_2 is complete if

$$\lambda_2(x) = min\{\lambda_1(u), \lambda_1(v)\}$$
$$\delta_2(x) = max\{\delta_1(u), \delta_1(v)\}$$
$$\rho_2(x) = min\{\rho_1(u), \rho_1(v)\}, \forall u, v \in V.$$

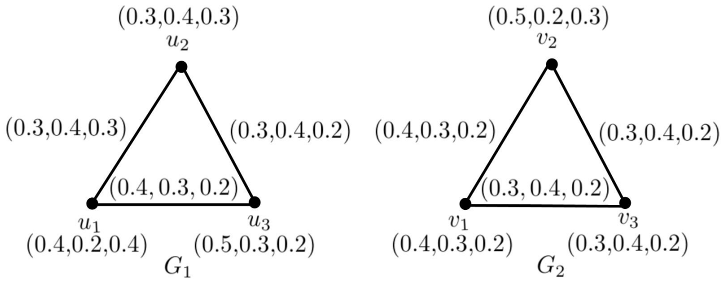

FIGURE 6. complete HFG G_1 and complete IHFG G_2

Now we discuss the β-product of HFG and IHFG.

DEFINITION 3.11. The β-product of two HFGs, $G_1 = (U, E_U, \sigma, \mu), G_2 = (V, E_V, \sigma', \mu')$ where $\sigma = (\lambda_1, \delta_1, \rho_1), \mu = (\lambda_2, \delta_2, \rho_2), \sigma' = (\lambda'_1, \delta'_1, \rho'_1)$ and $\mu' = (\lambda'_2, \delta'_2, \rho'_2)$ is the HFG $G = G_1 \times_\beta G_2 = (U \times V, E, \sigma \times_\beta \sigma', \mu \times_\beta \mu')$, $E = E_1 \cup E_2 \cup E_3$ where

$$\begin{aligned}
&E_1 = \{w : w_1 \in E_U, w_2 \in E_V\}\\
&E_2 = \{w : v_1 \neq v_2, w_1 \in E_U\}\\
&E_3 = \{w : u_1 \neq u_2, , w_2 \in E_V\},\\
&\; w = ((u_1, v_1), (u_2, v_2)), w_1 = (u_1, u_2), w_2 = (v_1, v_2).
\end{aligned}$$

$$\begin{aligned}
&(\lambda_1 \times_\beta \lambda'_1)(x) = \lambda_1(u) \wedge \lambda'_1(v)\\
&(\delta_1 \times_\beta \delta'_1)(x) = \delta_1(u) \vee \delta'_1(v)\\
&(\rho_1 \times_\beta \rho'_1)(x) = 1 - [\lambda_1(u) \wedge \lambda'_1(v) + \delta_1(u) \vee \delta'_1(v)]
\end{aligned}$$

$$(1) \qquad (\lambda_2 \times_\beta \lambda'_2)(w) = \begin{cases} \lambda_2(w_1) \wedge \lambda'_2(w_2), \text{if } w \in E_1 \\ \lambda'_1(v_1) \wedge \lambda'_1(v_2) \wedge \lambda_2(w_1), \text{if } w \in E_2 \\ \lambda_1(u_1) \wedge \lambda_1(u_2) \wedge \lambda'_2(w_2), \text{if } w \in E_3 \end{cases}$$

$$(2) \qquad (\delta_2 \times_\beta \delta'_2)(w) = \begin{cases} \delta_2(w_1) \vee \delta'_2(w_2), \text{if } w \in E_1 \\ \delta'_1(v_1) \vee \delta'_1(v_2) \vee \delta_2(w_1), \text{if } w \in E_2 \\ \delta_1(u_1) \vee \delta_1(u_2) \vee \delta'_2(w_2), \text{if } w \in E_3 \end{cases}$$

$$(3) \qquad (\rho_2 \times_\beta \rho'_2)(w) = \begin{cases} \rho_2(w_1) \wedge \rho'_2(w_2), \text{if } w \in E_1 \\ \rho'_1(v_1) \wedge \rho'_1(v_2) \wedge \rho_2(w_1), \text{if } w \in E_2 \\ \rho_1(u_1) \wedge \rho_1(u_2) \wedge \rho'_2(w_2), \text{if } w \in E_3 \end{cases}$$

REMARK 3.12. For two strong HFGs G_1, G_2, their β-product $G_1 \times_\beta G_2$ need not be ρ-strong and hence need not be a strong HFG. In figure 7, G_1 and G_2 are two strong HFGs and figure 8 is their β-product $G_1 \times_\beta G_2$.
$(\lambda_1 \times_\beta \lambda'_1)(u_1, v_2) = \lambda_1(u_1) \wedge \lambda'_1(v_2) = 0.3 \wedge 0.5 = 0.3$
$(\delta_1 \times_\beta \delta'_1)(u_1, v_2) = \delta_1(u_1) \vee \delta'_1(v_2) = 0.4 \vee 0.3 = 0.4$
$(\rho_1 \times_\beta \rho'_1)(u_1, v_2) = 1 - (0.3 + 0.4) = 0.3$
$(\lambda_1 \times_\beta \lambda'_1)(u_2, v_1) = \lambda_1(u_2) \wedge \lambda'_1(v_1) = 0.5 \wedge 0.4 = 0.4$
$(\delta_1 \times_\beta \delta'_1)(u_2, v_1) = \delta_1(u_2) \vee \delta'_1(v_1) = 0.3 \vee 0.2 = 0.3$
$(\rho_1 \times_\beta \rho'_1)(u_2, v_1) = 1 - (0.4 + 0.3) = 0.3$

Consider the edge $z = ((u_1, v_2), (u_2, v_1))$.
Since $z \in E_1$,
$(\lambda_2 \times_\beta \lambda_2')(z) = \lambda_2(w_1) \wedge \lambda_2'(w_2) = 0.3 \wedge 0.4 = 0.3$
$(\delta_2 \times_\beta \delta_2')(z) = \delta_2(w_1) \vee \delta_2'(w_2) = 0.4 \vee 0.3 = 0.4$
$(\rho_2 \times_\beta \rho_2')(z) = \rho_2(w_1) \wedge \rho_2'(w_2) = 0.2 \wedge 0.2 = 0.2$
i.e., $(\lambda_2 \times_\beta \lambda_2')(z) = (\lambda_1 \times_\beta \lambda_1')(u_1, v_2) \wedge (\lambda_1 \times_\beta \lambda_1')(u_2, v_1)$.
This is true for all the other edges in $G_1 \times_\beta G_2$ and hence $G_1 \times_\beta G_2$ is λ-strong.
$(\delta_2 \times_\beta \delta_2')(z) = (\delta_1 \times_\beta \delta_1')(u_1, v_2) \vee (\delta_1 \times_\beta \delta_1')(u_2, v_1)$, which is true for all the other edges and hence $G_1 \times_\beta G_2$ is δ-strong.
But, $(\rho_2 \times_\beta \rho_2')(z) \neq (\rho_1 \times_\beta \rho_1')(u_1, v_2) \wedge (\rho_1 \times_\beta \rho_1')(u_2, v_1)$. i.e., $G_1 \times_\beta G_2$ is not ρ-strong and hence not a strong HFG.

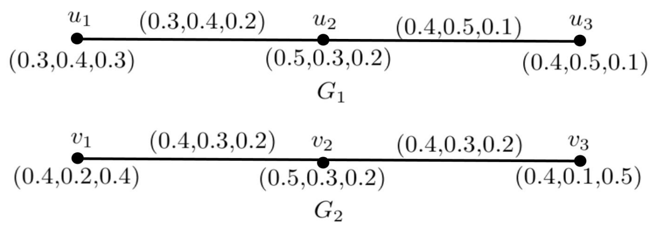

FIGURE 7. Strong HFGs G_1 and G_2

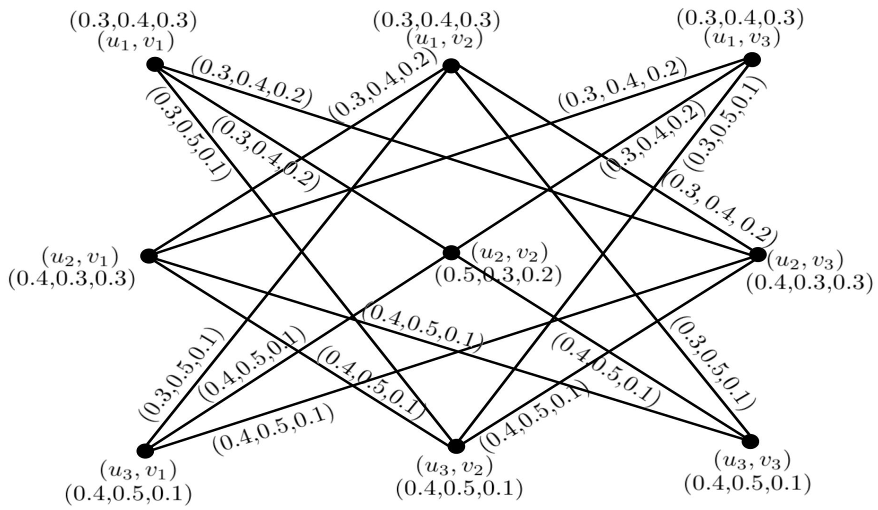

FIGURE 8. β-product of strong HFGs G_1 and G_2

DEFINITION 3.13. The β-product of two IHFGs $G_1 = (U, E_U, \sigma, \mu)$, $G_2 = (V, E_V, \sigma', \mu')$ where $\sigma = (\lambda_1, \delta_1, \rho_1), \mu = (\lambda_2, \delta_2, \rho_2), \sigma' = (\lambda_1', \delta_1', \rho_1')$ and $\mu' = (\lambda_2', \delta_2', \rho_2')$ is the IHFG $G = G_1 \times_\beta G_2 =(U \times V, E, \sigma \times_\beta \sigma', \mu \times_\beta \mu')$, $E = E_1 \cup E_2 \cup E_3$ where

$$E_1 = \{w : w_1 \in E_U, w_2 \in E_V\}$$
$$E_2 = \{w : v_1 \neq v_2, w_1 \in E_U\}$$
$$E_3 = \{w : u_1 \neq u_2, , w_2 \in E_V\}.$$

$$(\lambda_1 \times_\beta \lambda_1')(x) = \lambda_1(u) \wedge \lambda_1'(v)$$
$$(\delta_1 \times_\beta \delta_1')(x) = \delta_1(u) \vee \delta_1'(v)$$
$$(\rho_1 \times_\beta \rho_1')(x) = \rho_1(u) \wedge \rho_1'(v)$$

and equations (1), (2) and (3).

THEOREM 3.14. *If G_1, G_2 are two strong IHFGs, then their β-product $G_1 \times_\beta G_2$ is also a strong IHFG.*

Proof. Let G_1, G_2 be two strong IHFGs.
Then, for $w_1 \in E_U, w_2 \in E_V$,

$$\lambda_2(w_1) = \lambda_1(u_1) \wedge \lambda_1(u_2),$$
$$\lambda_2'(w_2) = \lambda_1'(v_1) \wedge \lambda_1'(v_2)$$
$$\delta_2(w_1) = \delta_1(u_1) \vee \delta_1(u_2),$$
$$\delta_2'(w_2) = \delta_1'(v_1) \vee \delta_1'(v_2)$$
$$\rho_2(w_1) = \rho_1(u_1) \wedge \rho_1(u_2),$$
$$\rho_2'(w_2) = \rho_1'(v_1) \wedge \rho_1'(v_2).$$

Case(i) When $w \in E_1$

$$\begin{aligned}(\lambda_2 \times_\beta \lambda_2')(w) &= \lambda_2(w_1) \wedge \lambda_2'(w_2)\\ &= \lambda_1(u_1) \wedge \lambda_1(u_2) \wedge \lambda_1'(v_1) \wedge \lambda_1'(v_2)\\ &= (\lambda_1 \times_\beta \lambda_1')(u_1, v_1) \wedge (\lambda_1 \times_\beta \lambda_1')(u_2, v_2)\end{aligned}$$

$$\begin{aligned}(\delta_2 \times_\beta \delta_2')(w) &= \delta_2(w_1) \vee \delta_2'(w_2)\\ &= \delta_1(u_1) \vee \delta_1(u_2) \vee \delta_1'(v_1) \vee \delta_1'(v_2)\\ &= (\delta_1 \times_\beta \delta_1')(u_1, v_1) \vee (\delta_1 \times_\beta \delta_1')(u_2, v_2)\end{aligned}$$

$$\begin{aligned}(\rho_2 \times_\beta \rho_2')(w) &= \rho_2(w_1) \wedge \rho_2'(w_2)\\ &= \rho_1(u_1) \wedge \rho_1(u_2) \wedge \rho_1'(v_1) \wedge \rho_1'(v_2)\\ &= (\rho_1 \times_\beta \rho_1')(u_1, v_1) \wedge (\rho_1 \times_\beta \rho_1')(u_2, v_2)\end{aligned}$$

Case(ii) When $w \in E_2$

$$\begin{aligned}(\lambda_2 \times_\beta \lambda_2')(w) &= \lambda_1'(v_1) \wedge \lambda_1'(v_2) \wedge \lambda_2(w_1)\\ &= \lambda_1(u_1) \wedge \lambda_1(u_2) \wedge \lambda_1'(v_1) \wedge \lambda_1'(v_2)\\ &= (\lambda_1 \times_\beta \lambda_1')(u_1, v_1) \wedge (\lambda_1 \times_\beta \lambda_1')(u_2, v_2)\end{aligned}$$

$$\begin{aligned}(\delta_2 \times_\beta \delta_2')(w) &= \delta_1'(v_1) \vee \delta_1'(v_2) \vee \delta_2(w_1)\\ &= \delta_1(u_1) \vee \delta_1(u_2) \vee \delta_1'(v_1) \vee \delta_1'(v_2)\\ &= (\delta_1 \times_\beta \delta_1')(u_1, v_1) \vee (\delta_1 \times_\beta \delta_1')(u_2, v_2)\end{aligned}$$

$$\begin{aligned}(\rho_2 \times_\beta \rho_2')(w) &= \rho_1'(v_1) \wedge \rho_1'(v_2) \wedge \rho_2(w_1)\\ &= \rho_1(u_1) \wedge \rho_1(u_2) \wedge \rho_1'(v_1) \wedge \rho_1'(v_2)\\ &= (\rho_1 \times_\beta \rho_1')(u_1, v_1) \wedge (\rho_1 \times_\beta \rho_1')(u_2, v_2)\end{aligned}$$

Case(iii) When $w \in E_3$

$$
\begin{aligned}
(\lambda_2 \times_\beta \lambda'_2)(w) &= \lambda_1(u_1) \wedge \lambda_1(u_2) \wedge \lambda'_2(w_2) \\
&= \lambda_1(u_1) \wedge \lambda_1(u_2) \wedge \lambda'_1(v_1) \wedge \lambda'_1(v_2) \\
&= (\lambda_1 \times_\beta \lambda'_1)(u_1, v_1) \wedge (\lambda_1 \times_\beta \lambda'_1)(u_2, v_2)
\end{aligned}
$$

$$
\begin{aligned}
(\delta_2 \times_\beta \delta'_2)(w) &= \delta_1(u_1) \vee \delta_1(u_2) \vee \delta'_2(w_2) \\
&= \delta_1(u_1) \vee \delta_1(u_2) \vee \delta'_1(v_1) \vee \delta'_1(v_2) \\
&= (\delta_1 \times_\beta \delta'_1)(u_1, v_1) \vee (\delta_1 \times_\beta \delta'_1)(u_2, v_2)
\end{aligned}
$$

$$
\begin{aligned}
(\rho_2 \times_\beta \rho'_2)(w) &= \rho_1(u_1) \wedge \rho_1(u_2) \wedge \rho'_2(w_2) \\
&= \rho_1(u_1) \wedge \rho_1(u_2) \wedge \rho'_1(v_1) \wedge \rho'_1(v_2) \\
&= (\rho_1 \times_\beta \rho'_1)(u_1, v_1) \wedge (\rho_1 \times_\beta \rho'_1)(u_2, v_2)
\end{aligned}
$$

Thus, $G = G_1 \times_\beta G_2$ is a strong IHFG. □

EXAMPLE 3.15. In figure 9, G_1 and G_2 are two strong IHFGs. Their β-product $G_1 \times_\beta G_2$ in figure 10 is a strong IHFG since all the edges are strong edges.

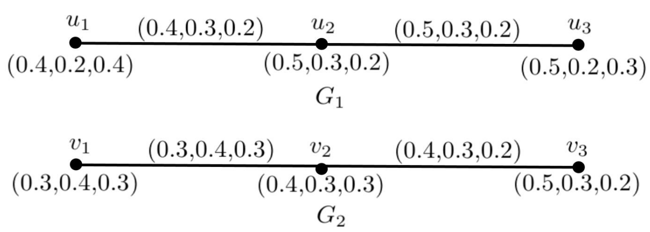

FIGURE 9. Strong IHFGs G_1 and G_2

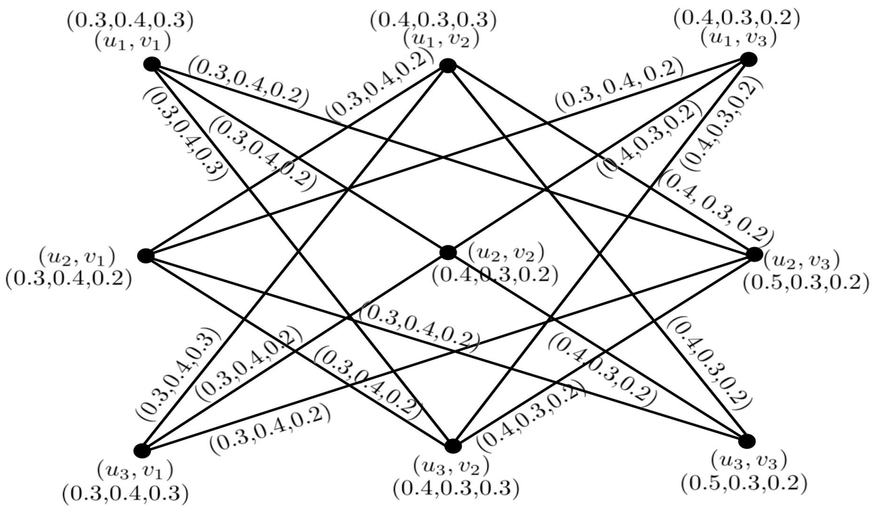

FIGURE 10. β-product of strong IHFGs G_1 and G_2

THEOREM 3.16. *If G_1 and G_2 are two complete IHFGs, then their β-product $G_1 \times_\beta G_2$ is also a complete IHFG .*

Proof. Similar to 3.14. □

THEOREM 3.17. *If G_1,G_2 are two IHFGs such that $G_1 \times_\beta G_2$ is strong, then at least one of G_1 or G_2 will be strong.*

Proof. Assume that the two IHFGs G_1,G_2 are not strong. Then there exists at least one $w_1 = (u_1, u_2) \in E_U, w_2 = (v_1, v_2) \in E_V$, with

$$\begin{aligned} \lambda_2(w_1) &< \lambda_1(u_1) \wedge \lambda_1(u_2), \quad \lambda_2'(w_2) < \lambda_1'(v_1) \wedge \lambda_1'(v_2), \\ \delta_2(w_1) &< \delta_1(u_1) \vee \delta_1(u_2), \quad \delta_2'(w_2) < \delta_1'(v_1) \vee \delta_1'(v_2), \\ \rho_2(w_1) &< \rho_1(u_1) \wedge \rho_1(u_2), \quad \rho_2'(w_2) < \rho_1'(v_1) \wedge \rho_1'(v_2). \end{aligned}$$

Let $w = ((u_1, v_1), (u_2, v_2)) \in E_1$. Then,

$$\begin{aligned} (\lambda_2 \times_\beta \lambda_2')(w) &= \lambda_2(w_1) \wedge \lambda_2'(w_2) \\ &< \lambda_1(u_1) \wedge \lambda_1(u_2) \wedge \lambda_1'(v_1) \wedge \lambda_1'(v_2) \\ i.e., (\lambda_2 \times_\beta \lambda_2')(w) &< (\lambda_1 \times_\beta \lambda_1')(u_1, v_1) \wedge (\lambda_1 \times_\beta \lambda_1')(u_2, v_2). \end{aligned}$$

$$\begin{aligned} (\delta_2 \times_\beta \delta_2')(w) &= \delta_2(w_1) \vee \delta_2'(w_2) \\ &< \delta_1(u_1) \vee \delta_1(u_2) \vee \delta_1'(v_1) \vee \delta_1'(v_2) \\ i.e., (\delta_2 \times_\beta \delta_2')(w) &< (\delta_1 \times_\beta \delta_1')(u_1, v_1) \vee (\delta_1 \times_\beta \delta_1')(u_2, v_2) \end{aligned}$$

$$\begin{aligned} (\rho_2 \times_\beta \rho_2')(w) &= \rho_2(w_1) \wedge \rho_2'(w_2) \\ &< \rho_1(u_1) \wedge \rho_1(u_2) \wedge \rho_1'(v_1) \wedge \rho_1'(v_2) \\ i.e., (\rho_2 \times_\beta \rho_2')(w) &< (\rho_1 \times_\beta \rho_1')(u_1, v_1) \wedge (\rho_1 \times_\beta \rho_1')(u_2, v_2) \end{aligned}$$

i.e., $G_1 \times_\beta G_2$ is not strong, a contradiction. So at least one of G_1 or G_2 will be strong. □

4. Application

IHFGs can be suitably used in real life problems. It can work as a good aid in solving companies' merger problems. Consider two strong networks of IHFGs G_1 and G_2 with vertices indicating distinct companies. The MS degree of the vertices and the edges indicates the market worth of the companies and the market worth of the companies' joint ventures respectively. Since the IHFGs G_1 and G_2 are strong, all the edges in G_1 and G_2 are strong and all the edges in the β-product $G_1 \times_\beta G_2$ are also strong. That means the joint venture of two strong networks will be strong and the production carried out by the joint venture will be surely successful. Thus, this product is stronger and more reliable and the decision on merger problems based on this result will be more accurate.

For example, consider two strong companies, one which is successful in the production of scooters and another company which is expert in the production of battery. A joint venture, if initiated, will benefit both the companies and expertise of both the companies in their respective production, will be a strong foundation to introduce a new production unit for manufacturing electric scooters and thus produce a new brand of electric scooters. Thus the production carried out by the joint venture will be surely successful and may result in making both the companies involved in the joint venture more stronger.

5. Conclusion

HFGs offers a wide range of uses in the fields of robotics, artificial intelligence and medical diagnosis. We proved that β-product of a pair of strong IHFGs is a strong IHFG and β-product of a pair of complete IHFGs is a complete IHFG. Also we proved that if β-product of a pair of IHFGs is strong, then at least one of the IHFG will be strong. IHFG models provide exact and accurate outcomes for making decisions and resolving merger related problems. Our future work is to broaden the scope of our investigation to study the complement of β-product of IHFG.

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