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APPROXIMATION RESULTS OF A THREE STEP ITERATION METHOD IN BANACH SPACE

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Abstract. The purpose of this paper is to introduce a new three-step iterative process and show that our iteration scheme is faster than other existing iteration schemes in the literature. We provide a numerical example supported by graphs and tables to validate our proofs. We also prove convergence and stability results for the approximation of fixed points of the contractive-like mapping in the framework of uniformly convex Banach space. In addition, we have established some weak and strong convergence theorems for nonexpansive mappings.

1. Introduction

Let K be a Banach space and M be any nonempty set. Consider the self map $T : M \to M$. A fixed point of T is a point $x \in M$ which is mapped onto itself, that is

$$Tx = x.$$

The set of all fixed points of M is denoted by $F(T)$.

Fixed point theory provides very useful tools to solve most of the nonlinear problems, that have application in different fields, as they can be easily transformed into a fixed point problem. Fixed point problems possess either existing results or approximate solutions. Iterative methods are popular tools to approximate fixed points of nonlinear mappings. When studying an iterative procedure, it should be considered two criteria which are faster and simpler. Till now many iterative processes have been developed, all of which can not be covered.

In 1953, Mann [25], introduces an iterative scheme. Later in 1974, Ishikawa [18], introduced an iterative scheme which was a two-step iterative scheme. In 2000, Noor [12], introduced three three-step iterative schemes for approximating fixed points of asymptotically nonexpansive mappings. Later several researchers modified Mann, Ishikawa, and Noor iterations, etc.

After some years Kadioglu et al. [15] introduced the Picard Normal S-iteration process. The Picard Normal S-iteration process is defined as follows: Let K be a

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normed linear space and M be a nonempty convex subset of K. Let $T : M \to M$ be any nonlinear mapping and for each $x_0 \in M$, the sequence $\{x_n\}$ in M is defined by

$$\begin{cases} z_n = (1-\beta_n)x_n + \beta_n T x_n, \\ y_n = (1-\alpha_n)z_n + \alpha_n T z_n, \\ x_{n+1} = T y_n, \quad n \geq 1, \end{cases} \tag{1.1}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $(0,1)$. They proved that the Picard Normal S-iteration process is faster than the Normal S-iteration process.

In 2016, Thakur et al. [2] introduced the following iteration scheme in the framework of Banach space. Let K be a Banach space and M be any nonempty subset of K and for each $x_0 \in M$, the sequence $\{x_n\}$ in M is defined by

$$\begin{cases} z_n = (1-\beta_n)x_n + \beta_n T x_n, \\ y_n = T((1-\alpha_n)x_n + \alpha_n z_n), \\ x_{n+1} = T y_n, \quad n \geq 1, \end{cases} \tag{1.2}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $(0,1)$. They proved that their iterative scheme is faster than Picard, Mann, Ishikawa, Agrawal [16], Noor and Abbas et al. [10] iteration process

In 2017, Karakaya et al. [24] introduced the following iteration scheme. Let K be a normed linear space and M be any nonempty convex subset of K. Let $T : M \to M$ be any nonlinear mapping and for each $x_0 \in M$, the sequence $\{x_n\}$ in M is defined by

$$\begin{cases} z_n = T x_n, \\ y_n = (1-\alpha_n)z_n + \alpha_n T z_n, \\ x_{n+1} = T y_n, \quad n \geq 1, \end{cases} \tag{1.3}$$

where $\{\alpha_n\}$ is a sequences in $(0,1)$. They proved that their iterative process converges faster than all of Picard's, Mann, Ishikawa, Noor, Abass et al. process.

In 2018, Ullah et al. [9] introduced a new iterative scheme. This iteration scheme is called the M-iteration process which follows as. Let K be a normed linear space and M be any nonempty convex subset of K. Let $T : M \to M$ be any nonlinear mapping and for each $x_0 \in M$, the sequence $\{x_n\}$ in M is defined by

$$\begin{cases} z_n = (1-\alpha_n)x_n + \alpha_n T x_n, \\ y_n = T z_n, \\ x_{n+1} = T y_n, \quad n \geq 1, \end{cases} \tag{1.4}$$

where $\{\alpha_n\}$ is a sequence in $(0,1)$. They proved that their iterative process converges faster than all of Picard, Mann, Ishikawa, Noor, Abass et al., the above-listed iterative process, and some existing ones in the literature.

In 2018, Abass et al. [5] introduced the following iterative process. Let K be a normed linear space and M be any nonempty convex subset of K. Let $T : M \to M$ be any nonlinear mapping and for each $x_0 \in M$, the sequence $\{x_n\}$ in M defined by

$$\begin{cases} z_n = T x_n, \\ y_n = T z_n, \\ x_{n+1} = (1-\alpha_n)y_n + \alpha_n T y_n, \quad n \geq 1. \end{cases} \tag{1.5}$$

where $\{\alpha_n\}$ is a sequence in $(0,1)$. They showed that this process converges at a rate the same as the (1.3) and (1.4).

In 2021, Akutsah et al. [4] introduced the following iteration process. Let K be a uniformly Banach space and M be any nonempty closed convex subset of K. Let $T : M \to M$ be a mapping and for each $x_0 \in M$, the sequence $\{x_n\}$ in M is defined by

$$\begin{cases} z_n = (1-\beta_n)x_n + \beta_n T x_n, \\ y_n = T z_n, \\ x_{n+1} = T((1-\alpha_n)y_n + \alpha_n T y_n), \quad n \geq 1, \end{cases} \tag{1.6}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $(0,1)$. They proved that this iteration process (1.6) converges faster than (1.1), (1.2), (1.3), (1.4) and (1.5).

Motivated by the previous iteration process, we introduce a new iteration process for approximating fixed points of contractive-like mapping in the framework of uniformly Banach space. Let K be a uniformly Banach space and M be any nonempty closed convex subset of K. Let $T : M \to M$ be a mapping and for each $x_0 \in M$, the sequence $\{x_n\}$ in M is defined by

$$\begin{cases} z_n = T((1-\alpha_n)x_n + \alpha_n T x_n), \\ y_n = T((1-\beta_n)z_n + \beta_n T z_n) \\ x_{n+1} = T y_n, \quad n \geq 1, \end{cases} \tag{1.7}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $(0,1)$.

In this paper, we have to show that our iteration process (1.7) converges faster than (1.6) and some other existing iterative process in literature, for contractive-like mapping with the help of numerical examples. Also, we investigate the convergence and stability results of the proposed iteration scheme.

2. Preliminaries

In this section, we recall some definitions and results to be used in establishing the main results.

Definition 2.1. [8] A Banach space K is said to be uniformly convex if for each $\epsilon \in (0,2]$ there is a $\delta > 0$ such that $\forall\, x, y \in K$

$$||(x+y)/2|| \leq 1 - \delta \textit{ whenever } \quad ||x-y|| \geq \epsilon \textit{ and } ||x|| = ||y|| = 1.$$

Definition 2.2. [26]A Banach space K is said to satisfy Opal's property if for each sequence $\{x_n\}$ in K converging weakly to $x \in K$, we have

$$\limsup_{n\to\infty} ||x_n - x|| < \limsup_{n\to\infty} ||x_n - y||$$

for all $y \in K$ s.t. $x \neq y$.

Definition 2.3. [3] Let K be a Banach space and let $T : K \to K$ be a self map. The mapping T is called contractive like mapping if there exist a constant $\delta \in [0,1)$ and a strictly increasing and continuous function $\xi : [0,\infty) \to [0,\infty)$ with $\xi(0) = 0$ such that for all $x, y \in K$,

$$||Tx - Ty|| \leq \delta ||x-y|| + \xi(||x - Tx||). \tag{2.1}$$

DEFINITION 2.4. Let M be a nonempty closed convex subset of a Banach space K. A mapping $T : M \to M$ be said to be nonexpansive if

$$||Tx - Ty|| \leq ||x - y||$$

$\forall\ x, y \in M$.

Senter and Dotson [6] introduced a class of mappings satisfying condition (I)

DEFINITION 2.5. [6] A mapping $T : K \to K$ is said to satisfy condition I, if $\exists$ a non decreasing function $f : [0, \infty) \to [0, \infty)$ with $f(0) = 0$ and $f(c) > 0$ for all $c > 0$ s.t. $||x - Tx|| \geq f(d(x_n, F(T)))$, for all $x \in K$, where $d(x, F(T)) = \inf\{||x - x^*|| : x^* \in F(T)\}$.

The following definition is about the rate of convergence due to Berinde [22] which is used to verify that our iteration process (1.7) convergence is faster than the other existing iteration process.

DEFINITION 2.6. [22] Let $\{a_n\}_{n=0}^{\infty}$ and $\{b_n\}_{n=0}^{\infty}$ be two sequences of positive numbers such that both converge to a and b respectively. Assume that there exists

$$\lim_{n\to\infty} \frac{|a_n - a|}{|b_n - b|} = l.$$

(i) If $l = 0$, then the sequence $\{a_n\}$ converges faster than the sequence $\{b_n\}$.
(ii) If $0 < l < \infty$, then we say that the sequence $\{a_n\}$ and $\{b_n\}$ have the same rate of convergence.
(iii) If $l = \infty$, then the sequence $\{b_n\}$ converges faster than sequence $\{a_n\}$.

We next recall the definition of T-stable, equivalence and w^2- stable for an iteration process.

DEFINITION 2.7. [1] Let $\{t_n\}$ be any arbitrary sequence in K. Then an iteration procedure $x_{n+1} = f(T, x_n)$, converging to fixed point p, is said to T-stable, if for $\epsilon_n = ||t_{n+1} - f(T, t_n)||$, $\forall n \in N$, we have $\lim_{n\to\infty} \epsilon_n = 0$ if and only if $\lim_{n\to\infty} t_n = p$.

DEFINITION 2.8. [19] Two sequences say $\{x_n\}$ and $\{y_n\}$ are said to be equivalence if $\lim_{n\to\infty} ||x_n - y_n|| = 0$.

DEFINITION 2.9. [19] Let $\{t_n\}$ be an equivalent sequence of $\{x_n\}$. Then an iteration procedure $x_{n+1} = f(T, x_n)$, converging to fixed point p, is said to be weak w^2-stable with respect to T, if and only if $\lim_{n\to\infty} ||t_{n+1} - f(T, t_n)|| = 0$, implies that $\lim_{n\to\infty} t_n = p$.

LEMMA 2.10. [22] *Suppose that for two fixed point iteration processes $\{u_n\}$ and $\{v_n\}$ both converging to the same fixed point x^*, the error estimates*

$$||u_n - x^*|| \leq a_n \quad n \geq 1,$$
$$||v_n - x^*|| \leq b_n \quad n \geq 1.$$

are available where $\{a_n\}$ and $\{b_n\}$ are two sequences of positive numbers converging to zero. If $\{a_n\}$ converges faster than $\{b_n\}$, then $\{u_n\}$ converges faster than $\{v_n\}$ to x^*.

LEMMA 2.11. [20] *If λ is a real number such that $0 \leq \lambda < 1$ and $\{\epsilon_n\}$ is the sequence of positive numbers such that*

$$\lim_{n\to\infty} \epsilon_n = 0$$

then for a sequence of positive numbers v_n satisfying

$$v_{n+1} \leq \lambda v_n + \epsilon_n, \quad for\ n = 1, 2, ...,$$

we have

$$\lim_{n\to\infty} v_n = 0.$$

LEMMA 2.12. [7] *Let K be a uniformly convex Banach space and $0 < p \leq t_n \leq q < 1$ $\forall n \in N$. Let $\{x_n\}$ and $\{y_n\}$ be two sequences of K s.t. $\limsup_{n\to\infty} ||x_n|| \leq a$, $\limsup_{n\to\infty} ||y_n|| \leq a$ and $\limsup_{n\to\infty} ||t_n x_n + (1-t_n)y_n|| = a$ hold for some $a \geq 0$. Then $\lim\limits_{n\to\infty} ||x_n - y_n|| = 0$.*

LEMMA 2.13. [8] *Let K be a uniformly convex Banach space and M be any nonempty closed convex subset of K. Let T be a nonexpansive mapping on K. Then, $I - T$ is demi closed at zero.*

3. Rate of Convergence

In this section, we prove that our iteration process (1.7) converges faster than (1.6). To support our results, we provide a numerical example using MATLAB software. Before going to the main results, first, we now establish the following useful results:

LEMMA 3.1. *Let K be a uniformly convex Banach space and M be any nonempty closed convex subset of K. Let $T : M \to M$ be a mapping satisfying equation (2.1) and x^* be a fixed point of T. Suppose that $\{x_n\}$ is generated by the iterative process (1.6). Consider the following cases of (1.6):*

$$(3.1) \qquad \begin{cases} z_n &= \beta_n x_n + (1-\beta_n)Tx_n, \\ y_n &= Tz_n \\ x_{n+1} &= T((1-\alpha_n)y_n + \alpha_n Ty_n), \ n \geq 1, \end{cases}$$

$$(3.2) \qquad \begin{cases} z_n = (1-\beta_n)x_n + \beta_n Tx_n, \\ y_n = Tz_n \\ x_{n+1} = T(\alpha_n y_n + (1-\alpha_n)Ty_n), \quad n \geq 1, \end{cases}$$

and

$$(3.3) \qquad \begin{cases} z_n = \beta_n x_n + (1-\beta_n)Tx_n, \\ y_n = Tz_n \\ x_{n+1} = T(\alpha_n y_n + (1-\alpha_n)Ty_n), \ n \geq 1, \end{cases}$$

If $1-\alpha_n < \alpha_n$ and $1-\beta_n < \beta_n$, then the iteration (1.6) converges faster than (3.1), (3.2) and (3.3). Also (3.1) and (3.2) have the same rate of convergence. Both (3.1) and (3.2) converges faster than (3.3).

Proof. Using (1.6) and (2.1), we have

$$\begin{aligned}||z_n - x^*|| &\leq (1-\beta_n)||x_n - x^*|| + \beta_n||Tx_n - x^*||\\ &= (1-\beta_n)||x_n - x^*|| + \beta_n||Tx^* - Tx_n||\\ &\leq (1-\beta_n)||x_n - x^*|| + \beta_n\delta||x_n - x^*|| + \xi(||x^* - Tx^*||)\\ &= (1-(1-\delta)\beta_n)||x_n - x^*||.\end{aligned}$$

So that

$$\begin{aligned}||y_n - x^*|| &= ||Tz_n - x^*||\\ &\leq \delta||z_n - x^*||\\ &\leq \delta(1-(1-\delta)\beta_n)||x_n - x^*||.\end{aligned}$$

Also, we have

$$\begin{aligned}||x_{n+1} - x^*|| &= ||Tx^* - T((1-\alpha_n)y_n + \alpha_n Ty_n)||\\ &\leq \delta||x^* - ((1-\alpha_n)y_n + \alpha_n Ty_n)|| + \xi(||x^* - Tx^*||)\\ &\leq \delta[(1-\alpha_n)||y_n - x^*|| + \alpha_n||Ty_n - x^*||]\\ &\leq \delta[(1-\alpha_n)||y_n - x^*|| + \alpha_n\delta||y_n - x^*||]\\ &= \delta(1-(1-\delta)\alpha_n)||y_n - x^*||\\ &\leq \delta^2(1-(1-\delta)\alpha_n)(1-(1-\delta)\beta_n)||x_n - x^*||. \qquad (3.4)\end{aligned}$$

Since $\{\alpha_n\}$, $\{\beta_n\}$ are in $\left(\frac{1}{2}, 1\right)$, so

$$1-(1-\delta)\alpha_n < \frac{1+\delta}{2}. \qquad (3.5)$$

$$1-(1-\delta)\beta_n < \frac{1+\delta}{2}. \qquad (3.6)$$

Using (3.4), (3.5) and (3.6), we have

$$\begin{aligned}||x_{n+1} - x^*|| &\leq \delta^2\frac{(1+\delta)^2}{2^2}||x_n - x^*||\\ &\vdots\\ &\leq \delta^{2(n+1)}\frac{(1+\delta)^{2(n+1)}}{2^{2(n+1)}}||x_0 - x^*||. \qquad (3.7)\end{aligned}$$

Let $a_n = \delta^{2(n+1)}\frac{(1+\delta)^{2(n+1)}}{2^{2(n+1)}}||x_0 - x^*||$. Now from the given the case (3.1) and (2.1)

$$\begin{aligned}||z_n - x^*|| &= ||\beta_n x_n + (1-\beta_n)Tx_n - x^*||\\ &\leq \beta_n||x_n - x^*|| + (1-\beta_n)||Tx_n - x^*||\\ &= \beta_n||x_n - x^*|| + (1-\beta_n)||Tx_n - Tx^*||\\ &\leq \beta_n||x_n - x^*|| + (1-\beta_n)(\delta||x_n - x^*|| + \xi(||x^* - Tx^*||))\\ &= (\beta_n + (1-\beta_n)\delta)||x_n - x^*||\\ &= (\delta + (1-\delta)\beta_n)||x_n - x^*||\\ &< (1+(1-\delta)\beta_n)||x_n - x^*||. \qquad (3.8)\end{aligned}$$

So that

$$||y_n - x^*|| = ||Tz_n - x^*|| \tag{3.9}$$
$$\leq \delta||z_n - x^*||.$$

From (3.8) and (3.9), we have

$$||y_n - x^*|| \leq \delta(1 + (1 - \delta)\beta_n)||x_n - x^*||. \tag{3.10}$$

Also we have

$$\begin{aligned} ||x_{n+1} - x^*|| &= ||Tx^* - T((1 - \alpha_n)y_n + \alpha_n Ty_n)|| \\ &\leq \delta||x^* - ((1 - \alpha_n)y_n + \alpha_n Ty_n)|| + \xi(||x^* - Tx^*||) \\ &\leq \delta[(1 - \alpha_n)||y_n - x^*|| + \alpha_n||Ty_n - x^*||] \\ &\leq \delta[(1 - \alpha_n)||y_n - x^*|| + \alpha_n\delta||y_n - x^*||] \\ &= \delta(1 - (1 - \delta)\alpha_n)||y_n - x^*||. \end{aligned} \tag{3.11}$$

Thus from (3.10) and (3.11), we have

$$||x_{n+1} - x^*|| \leq \delta^2(1 - (1 - \delta)\alpha_n)(1 + (1 - \delta)\beta_n)||x_n - x^*||. \tag{3.12}$$

Since $\{\alpha_n\}$, $\{\beta_n\}$ are in $\left(\frac{1}{2}, 1\right)$, so

$$\begin{aligned} 1 + (1 - \delta)\beta_n &= 1 - (1 - \delta)\beta_n + 2(1 - \delta_n)\beta_n \\ &\leq 1 - \frac{(1 - \delta)}{2} + 2(1 - \delta) \\ &= \frac{(5 - 3\delta)}{2} \end{aligned}$$

and

$$1 - (1 - \delta)\alpha_n < \frac{1 + \delta}{2}.$$

From (3.12), we have

$$\begin{aligned} ||x_{n+1} - x^*|| &\leq \delta^2 \frac{(1 + \delta)}{2} \frac{(5 - 3\delta)}{2} ||x_n - x^*|| \\ &\vdots \\ &\leq \delta^{2(n+1)} \frac{(1 + \delta)^{(n+1)}}{2^{(n+1)}} \frac{(5 - 3\delta)^{(n+1)}}{2^{(n+1)}} ||x_0 - x^*||. \end{aligned} \tag{3.13}$$

Let $b_n = \delta^{2(n+1)} \frac{(1+\delta)^{(n+1)}}{2^{(n+1)}} \frac{(5-3\delta)^{(n+1)}}{2^{(n+1)}} ||x_0 - x^*||$. Thus we conclude that

$$\begin{aligned} \lim_{n\to\infty} \frac{a_n}{b_n} &= \lim_{n\to\infty} \frac{\delta^{2(n+1)} \frac{(1+\delta)^{2(n+1)}}{2^{2(n+1)}} ||x_0 - x^*||}{\delta^{2(n+1)} \frac{(1+\delta)^{(n+1)}}{2^{(n+1)}} \frac{(5-3\delta)^{(n+1)}}{2^{(n+1)}} ||x_0 - x^*||} \\ &= \lim_{n\to\infty} \frac{(1 + \delta)^{(n+1)}}{(5 - 3\delta)^{(n+1)}} \\ &= 0. \end{aligned}$$

And so the iteration (1.6) converges faster than case (3.1) by Lemma 2.10. Now from the given case (3.2) and equation (2.1), we get

$$\begin{aligned} ||z_n - x^*|| &\leq (1-\beta_n)||x_n - x^*|| + \beta_n||Tx_n - x^*|| \\ &= (1-\beta_n)||x_n - x^*|| + \beta_n||Tx^* - Tx_n|| \\ &\leq (1-\beta_n)||x_n - x^*|| + \beta_n\delta||x_n - x^*|| + \xi(||x^* - Tx^*||) \\ &= (1-(1-\delta)\beta_n)||x_n - x^*||. \end{aligned}$$

So that

$$\begin{aligned} ||y_n - x^*|| &= ||Tz_n - x^*|| \\ &\leq \delta||z_n - x^*|| \\ &\leq \delta(1-(1-\delta)\beta_n)||x_n - x^*||. \end{aligned} \tag{3.14}$$

Also, we have

$$||x_{n+1} - x^*|| = ||T(\alpha_n y_n + (1-\alpha_n)Ty_n) - x^*||. \tag{3.15}$$

Using equation (2.1) in equation (3.15), we obtain

$$\begin{aligned} ||x_{n+1} - x^*|| &\leq \delta||\alpha_n y_n + (1-\alpha_n)Ty_n - x^*|| + \xi(||x^* - Tx^*||) \\ &\leq \delta[\alpha_n||y_n - x^*|| + (1-\alpha_n)||Ty_n - x^*||] \\ &\leq \delta[\alpha_n||y_n - x^*|| + (1-\alpha_n)\delta||y_n - x^*||] \\ &= \delta[(\alpha_n + (1-\alpha_n)\delta)||y_n - x^*||] \\ &= \delta(\delta + (1-\delta)\alpha_n)||y_n - x^*|| \\ &\leq \delta(1 + (1-\delta)\alpha_n)||y_n - x^*||. \end{aligned}$$

From (3.14), we have

$$||x_{n+1} - x^*|| \leq \delta^2(1 + (1-\delta)\alpha_n)(1-(1-\delta)\beta_n)||x_n - x^*||. \tag{3.16}$$

Since $\{\alpha_n\}$, $\{\beta_n\}$ are in $\left(\frac{1}{2}, 1\right)$. So

$$\begin{aligned} 1 + (1-\delta)\alpha_n &= 1 - (1-\delta)\alpha_n + 2(1-\delta_n)\alpha_n \\ &\leq 1 - \frac{(1-\delta)}{2} + 2(1-\delta) \\ &= \frac{(5-3\delta)}{2} \end{aligned}$$

and

$$1 - (1-\delta)\beta_n < \frac{1+\delta}{2}.$$

So (3.15) becomes

$$||x_{n+1} - x^*|| \leq \delta^2 \frac{(1+\delta)}{2}\frac{(5-3\delta)}{2}||x_n - x^*|| \tag{3.17}$$

$$\vdots$$

$$\leq \delta^{2(n+1)} \frac{(1+\delta)^{(n+1)}}{2^{(n+1)}} \frac{(5-3\delta)^{(n+1)}}{2^{(n+1)}}||x_0 - x^*||. \tag{3.18}$$

Let $c_n = \delta^{2(n+1)}\frac{(1+\delta)^{(n+1)}}{2^{(n+1)}}\frac{(5-3\delta)^{(n+1)}}{2^{(n+1)}}||x_0 - x^*||$. Thus we conclude that

$$\lim_{n\to\infty}\frac{a_n}{c_n} = \lim_{n\to\infty}\frac{\delta^{2(n+1)}\frac{(1+\delta)^{2(n+1)}}{2^{2(n+1)}}||x_0 - x^*||}{\delta^{2(n+1)}\frac{(1+\delta)^{(n+1)}}{2^{(n+1)}}\frac{(5-3\delta)^{(n+1)}}{2^{(n+1)}}||x_0 - x^*|} = \lim_{n\to\infty}\frac{(1+\delta)^{(n+1)}}{(5-3\delta)^{(n+1)}} = 0. \tag{3.19}$$

Hence by Lemma 2.10, we have iteration process (1.6) converges faster than case (3.2). Now from the given case (3.3) and equation (2.1), we get

$$\begin{aligned}||z_n - x^*|| &= ||\beta_n x_n + (1-\beta_n)Tx_n - x^*||\\ &\le \beta_n||x_n - x^*|| + (1-\beta_n)||Tx_n - x^*||\\ &= \beta_n||x_n - x^*|| + (1-\beta_n)||Tx_n - Tx^*||\\ &\le \beta_n||x_n - x^*|| + (1-\beta_n)(\delta||x_n - x^*|| + \xi(||x^* - Tx^*||))\\ &= (\beta_n + (1-\beta_n)\delta)||x_n - x^*||\\ &= (\delta + (1-\delta)\beta_n)||x_n - x^*||\\ &< (1 + (1-\delta)\beta_n)||x_n - x^*||. \end{aligned} \tag{3.20}$$

So that

$$\begin{aligned}||y_n - x^*|| &= ||Tz_n - x^*||\\ &\le \delta||z_n - x^*||.\end{aligned} \tag{3.21}$$

Also, we have

$$||x_{n+1} - x^*|| = ||T(\alpha_n y_n + (1-\alpha_n)Ty_n) - x^*||. \tag{3.22}$$

Using (2.1) in equation(3.22)

$$\begin{aligned}||x_{n+1} - x^*|| &\le \delta||\alpha_n y_n + (1-\alpha_n)Ty_n - x^*|| + \xi(||x^* - Tx^*||)\\ &\le \delta[\alpha_n||y_n - x^*|| + (1-\alpha_n)||Ty_n - x^*||]\\ &\le \delta[\alpha_n||y_n - x^*|| + (1-\alpha_n)\delta||y_n - x^*||]\\ &= \delta[(\alpha_n + (1-\alpha_n)\delta)||y_n - x^*||]\\ &= \delta(\delta + (1-\delta)\alpha_n)||y_n - x^*||\\ &\le \delta(1 + (1-\delta)\alpha_n)||y_n - x^*||.\end{aligned} \tag{3.23}$$

From equation (3.20), (3.21) and (3.23), we have

$$||x_{n+1} - x^*|| \le \delta^2(1 + (1-\delta)\alpha)(1 + (1-\delta)\beta)||x_n - x^*||. \tag{3.24}$$

Since $\{\alpha_n\}$, $\{\beta_n\}$ are in $\left(\frac{1}{2}, 1\right)$, so

$$\begin{aligned}1 + (1-\delta)\alpha_n &= 1 - (1-\delta)\alpha_n + 2(1-\delta_n)\alpha_n\\ &\le 1 - \frac{(1-\delta)}{2} + 2(1-\delta)\\ &= \frac{(5-3\delta)}{2},\end{aligned}$$

and

$$\begin{aligned}1+(1-\delta)\beta_n &= 1-(1-\delta)\beta_n+2(1-\delta_n)\beta_n\\ &\le 1-\frac{(1-\delta)}{2}+2(1-\delta)\\ &= \frac{(5-3\delta)}{2}.\end{aligned}$$

From equation(3.24), we have

$$\begin{aligned}||x_{n+1}-x^*|| &\le \delta^2\frac{(5-3\delta)^2}{2^2}||x_n-x^*||\\ &\ \vdots\\ &\le \delta^{2(n+1)}\frac{(5-3\delta)^{2(n+1)}}{2^{2(n+1)}}||x_0-x^*||. \qquad (3.25)\end{aligned}$$

Let $d_n=\delta^{2(n+1)}\frac{(5-3\delta)^{2(n+1)}}{2^{2(n+1)}}||x_0-x^*||$. Thus we conclude that

$$\begin{aligned}\lim_{n\to\infty}\frac{a_n}{d_n} &= \lim_{n\to\infty}\frac{\delta^{2(n+1)}\frac{(1+\delta)^{2(n+1)}}{2^{2(n+1)}}||x_0-x^*|}{\delta^{2(n+1)}\frac{(5-3\delta)^{2(n+1)}}{2^{2(n+1)}}||x_0-x^*||}\\ &= \lim_{n\to\infty}\frac{\frac{(1+\delta)^{2(n+1)}}{2^{2(n+1)}}}{\frac{(5-3\delta)^{2(n+1)}}{2^{2(n+1)}}}\\ &= 0. \qquad (3.26)\end{aligned}$$

Hence by Lemma 2.10, iteration secheme (1.6) converges faster than case (3.3). Now

$$\begin{aligned}\lim_{n\to\infty}\frac{b_n}{c_n} &= \lim_{n\to\infty}\frac{\delta^{2(n+1)}\frac{(1+\delta)^{(n+1)}}{2^{(n+1)}}\frac{(5-3\delta)^{(n+1)}}{2^{(n+1)}}||x_0-x^*||}{\delta^{2(n+1)}\frac{(1+\delta)^{(n+1)}}{2^{(n+1)}}\frac{(5-3\delta)^{(n+1)}}{2^{(n+1)}}||x_0-x^*||} \qquad (3.27)\\ &= 1.\end{aligned}$$

$$\begin{aligned}\lim_{n\to\infty}\frac{b_n}{d_n} &= \lim_{n\to\infty}\frac{\delta^{2(n+1)}\frac{(1+\delta)^{(n+1)}}{2^{(n+1)}}\frac{(5-3\delta)^{(n+1)}}{2^{(n+1)}}||x_0-x^*||}{\delta^{2(n+1)}\frac{(5-3\delta)^{2(n+1)}}{2^{2(n+1)}}||x_0-x^*||},\\ &= 0.\end{aligned}$$

$$\begin{aligned}\lim_{n\to\infty}\frac{c_n}{d_n} &= \lim_{n\to\infty}\frac{\delta^{2(n+1)}\frac{(1+\delta)^{(n+1)}}{2^{(n+1)}}\frac{(5-3\delta)^{(n+1)}}{2^{(n+1)}}||x_0-x^*||}{\delta^{2(n+1)}\frac{(5-3\delta)^{2(n+1)}}{2^{2(n+1)}}||x_0-x^*||}\\ &= 0. \qquad (3.28)\end{aligned}$$

By Definition (2.6), cases (3.1) and (3.2) have same rate of convergence and cases (3.1) and (3.2) converges faster than (3.3). □

Example 3.2. Let $K=R$ and $M\subset K$, where $M=[8,100]$. Let $T:M\to M$ be mapping defined by $T(x)=x/2+4$, for all $x\in M$. Clearly, the unique fixed point of T is 8. We have to show that T satisfies (2.1). Now for $\delta=\frac{1}{2}$ and an increasing function $\xi : [0, \infty) \to [0, \infty)$ defined by $\xi(x) = x$ with $\xi(0) = 0$, for all $x \in [0, \infty)$, we have

$$\begin{aligned}||Tx - Ty|| - \delta||x - y|| - \xi(||x - Tx||) &= |(\frac{x}{2} + 4) - (\frac{y}{2} + 4)| - \frac{1}{2}|x - y| - \xi(|x - (\frac{x}{2} + 4)|) \\ &= \frac{1}{2}|x - y| - \frac{1}{2}|x - y| - \xi(|\frac{x}{2} - 4|) \\ &= -(\frac{x}{2} - 4) \\ &\leq 0, \quad \forall x, y \in M.\end{aligned}$$

We take $\alpha_n = \beta_n = \frac{2}{3}$ and initial value $x_0 = 20$.Table 1 shows that the iteration process (1.6) converges to $x^* = 8$ faster than iteration process (3.1), (3.2) and (3.3) and also iteration process (3.1) and (3.2) converges faster than iteration (3.3). The convergence behavior of the iteration process is represented in figure 1.

Iteration	(1.6)	(3.1)	(3.2)	(3.3)
0	20.00000000	20.00000000	20.00000000	20.00000000
1	9.33333333	9.66666667	9.66666667	10.08333333
2	8.14814815	8.23148148	8.23148148	8.36168981
3	8.01646091	8.03215021	8.03215021	8.06279337
4	8.00182899	8.00446531	8.00446531	8.01090163
5	8.00020322	8.00062018	8.00062018	8.00189264
6	8.00002258	8.00008614	8.00008614	8.00032858
7	8.00000251	8.00001196	8.00001196	8.00005705
8	8.00000028	8.00000166	8.00000166	8.00000990
9	8.00000003	8.00000023	8.00000023	8.00000172
10	8.00000000	8.00000003	8.00000003	8.00000030
11	8.00000000	8.00000000	8.00000000	8.00000005
12	8.00000000	8.00000000	8.00000000	8.00000001
13	8.00000000	8.00000000	8.00000000	8.00000000
14	8.00000000	8.00000000	8.00000000	8.00000000
15	8.00000000	8.00000000	8.00000000	8.00000000
16	8.00000000	8.00000000	8.00000000	8.00000000
17	8.00000000	8.00000000	8.00000000	8.00000000
18	8.00000000	8.00000000	8.00000000	8.00000000
19	8.00000000	8.00000000	8.00000000	8.00000000
20	8.00000000	8.00000000	8.00000000	8.00000000

TABLE 1. Comparison Table

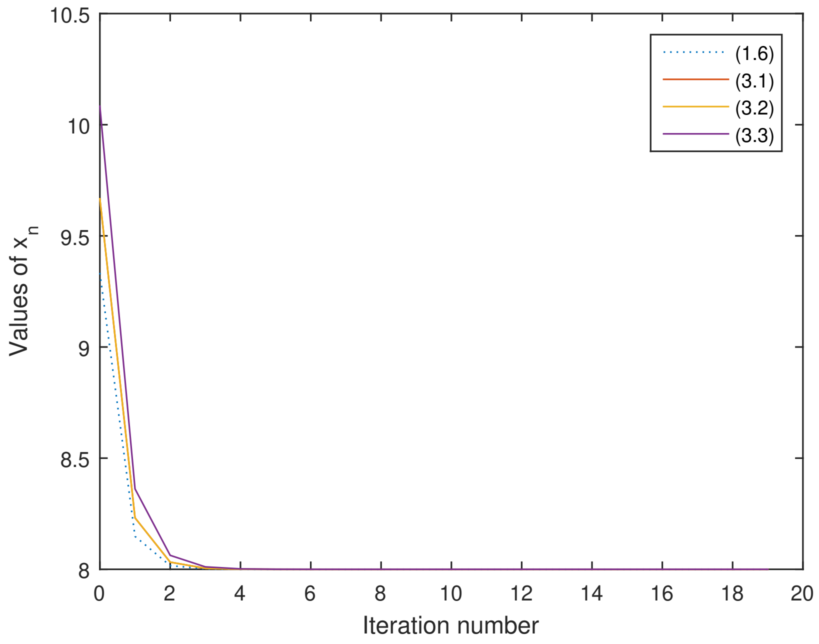

FIGURE 1. Comparison Plot

LEMMA 3.3. *Let K be a uniformly convex Banach space and M be any nonempty closed convex subset of K. Let $T : M \to M$ be a mapping satisfying* (2.1) *and x^* be a fixed point of T. Suppose that $\{x_n\}$ is generated by iteration process* (1.7)*. Consider the following cases of* (1.7)*:*

$$\begin{cases} z_n = T(\alpha_n x_n + (1-\alpha_n)Tx_n), \\ y_n = T((1-\beta_n)z_n + \beta_n Tz_n) \\ x_{n+1} = Ty_n, \ \forall n \geq 1, \end{cases} \tag{3.29}$$

$$\begin{cases} z_n = T((1-\alpha_n)x_n + \alpha_n Tx_n), \\ y_n = T(\beta_n z_n + (1-\beta_n)Tz_n), \\ x_{n+1} = Ty_n, \forall n \geq 1, \end{cases} \tag{3.30}$$

and

$$\begin{cases} z_n = T(\alpha_n x_n + (1-\alpha_n)Tx_n), \\ y_n = T(\beta_n z_n + (1-\beta_n)Tz_n) \\ x_{n+1} = Ty_n, \forall n \geq 1, \end{cases} \tag{3.31}$$

If $1-\alpha_n < \alpha_n$ and $1-\beta_n < \beta_n$, then the iteration process (1.7) *converges faster than* (3.29), (3.30) *and* (3.31) *and also* (3.29) *and* (3.30) *converges faster than* (3.31).

Proof. Using (1.7) and (2.1), we have

$$\begin{aligned} ||z_n - x^*|| &= ||T((1-\alpha_n)x_n + \alpha_n T x_n) - x^*|| \\ &= ||Tx^* - T((1-\alpha_n)x_n + \alpha_n T x_n)|| \\ &\leq \delta||x^* - ((1-\alpha_n)x_n + \alpha_n T x_n|| + \xi(||x^* - Tx^*||) \\ &\leq \delta[(1-\alpha_n)||x_n - x^*|| + \alpha_n||Tx_n - x^*||] \\ &\leq \delta[((1-\alpha_n)||x_n - x^*|| + \alpha_n \delta||x_n - x^*||] \\ &= \delta(1-(1-\delta))\alpha_n)||x_n - x^*||. \end{aligned} \tag{3.32}$$

$$\begin{aligned} ||y_n - x^*|| &= ||T((1-\beta_n)z_n + \beta_n T z_n) - x^*|| \\ &= ||Tx^* - T((1-\beta_n)z_n + \beta_n T z_n)|| \\ &\leq \delta[||x^* - ((1-\beta_n)z_n + \beta_n T z_n)||] \\ &= \delta[(1-\beta_n)||z_n - x^*|| + \beta_n||Tz_n - x^*||] \\ &\leq \delta[(1-\beta_n)||z_n - x^*|| + \beta_n \delta||z_n - x^*||] \\ &= \delta(1-(1-\delta)\beta_n)||z_n - x^*||. \end{aligned}$$

From eq. (3.32), we have

$$||y_n - x^*|| \leq \delta^2((1-(1-\delta)\alpha_n)((1-(1-\delta)\beta_n)||x_n - x^*||. \tag{3.33}$$

And so

$$\begin{aligned} ||x_{n+1} - x^*|| &= ||Ty_n - x^*|| \\ &\leq \delta||y_n - x^*|| + \xi(||x^* - Tx^*||). \end{aligned}$$

From (3.33), we have

$$||x_{n+1} - x^*|| \leq \delta^3((1-(1-\delta)\alpha_n)((1-(1-\delta)\beta_n)||x_n - x^*||. \tag{3.34}$$

Since $\{\alpha_n\}$, $\{\beta_n\}$ are in $\left(\frac{1}{2}, 1\right)$, so

$$\begin{aligned} 1-(1-\delta)\alpha_n &< \frac{1+\delta}{2}. \\ 1-(1-\delta)\beta_n &< \frac{1+\delta}{2}. \end{aligned}$$

From (3.34)

$$\begin{aligned} ||x_{n+1} - x^*|| &\leq \frac{\delta^3(1+\delta)^2}{2^2}||x_n - x^*|| \\ &\vdots \\ &\leq \frac{\delta^{3(n+1)}(1+\delta)^{2(n+1)}}{2^{2(n+1)}}||x_0 - x^*||. \end{aligned} \tag{3.35}$$

Let $a_n = \frac{\delta^{3(n+1)}(1+\delta)^{2(n+1)}}{2^{2(n+1)}}||x_0 - x^*||$.
By using iteration (3.29), we have

$$\begin{aligned} ||z_n - x^*|| &= ||T(\alpha_n x_n + (1-\alpha_n)Tx_n) - x^*|| \\ &\leq \delta[||x^* - (\alpha_n x_n + (1-\alpha_n)Tx_n)||] \\ &\leq \delta[\alpha_n||x_n - x^*|| + (1-\alpha_n)||Tx_n - x^*||] \\ &\leq \delta[\alpha_n||x_n - x^*|| + (1-\alpha_n)\delta||x_n - x^*||] \\ &= \delta(\delta + (1-\delta)\alpha_n)||x_n - x^*|| \\ &\leq \delta(1 + (1-\delta)\alpha_n)||x_n - x^*||. \end{aligned} \tag{3.36}$$

$$\begin{aligned} ||y_n - x^*|| &= ||T((1-\beta_n)z_n + \beta_n Tz_n) - x^*|| \\ &= ||Tx^* - T((1-\beta_n)z_n + \beta_n Tz_n)|| \\ &\leq \delta[||x^* - ((1-\beta_n)z_n + \beta_n Tz_n)||] \\ &= \delta[(1-\beta_n)||z_n - x^*|| + \beta_n||Tz_n - x^*||] \\ &\leq \delta[(1-\beta_n)||z_n - x^*|| + \beta_n\delta||z_n - x^*||] \\ &= \delta(1 - (1-\delta)\beta_n)||z_n - x^*||. \end{aligned}$$

From eq. (3.36) we have

$$||y_n - x^*|| \leq \delta^2(1 + (1-\delta)\alpha_n)(1 - (1-\delta)\beta_n)||x_n - x^*||. \tag{3.37}$$

And so

$$\begin{aligned} ||x_{n+1} - x^*|| &= ||Ty_n - x^*|| \\ &\leq \delta||y_n - x^*|| + \xi(||x^* - Tx^*||). \end{aligned}$$

From eq. (3.37) we have

$$||x_{n+1} - x^*|| \leq \delta^3(1 + (1-\delta)\alpha_n)(1 - (1-\delta)\beta_n)||x_n - x^*||. \tag{3.38}$$

Since $\{\alpha_n\}$, $\{\beta_n\}$ are in $\left(\frac{1}{2}, 1\right)$, so

$$\begin{aligned} 1 + (1-\delta)\alpha_n &= 1 - (1-\delta)\alpha_n + 2(1-\delta_n)\alpha_n \\ &\leq 1 - \frac{(1-\delta)}{2} + 2(1-\delta) \\ &= \frac{(5-3\delta)}{2}. \end{aligned}$$

and

$$1 - (1-\delta)\beta_n < \frac{1+\delta}{2}.$$

From equation (3.38) we have

$$\begin{aligned} ||x_{n+1} - x^*|| &\leq \frac{\delta^3(5-3\delta)(1+\delta)}{2^2}||x_n - x^*|| \\ &\vdots \\ &\leq \frac{\delta^{3(n+1)}(1+\delta)^{n+1}(5-3\delta)^{n+1}}{2^{2(n+1)}}||x_0 - x^*||. \end{aligned} \tag{3.39}$$

Let $b_n = \frac{\delta^{3(n+1)}(1+\delta)^{n+1}(5-3\delta)^{n+1}}{2^{2(n+1)}}||x_0 - x^*||$.

$$\lim_{n\to\infty}\frac{a_n}{b_n} = \lim_{n\to\infty}\frac{\frac{\delta^{3(n+1)}(1+\delta)^{2(n+1)}}{2^{2(n+1)}}||x_0 - x^*||}{\frac{\delta^{3(n+1)}(1+\delta)^{n+1}(5-3\delta)^{n+1}}{2^{2(n+1)}}||x_0 - x^*||}$$

$$= \lim_{n\to\infty}\frac{(1+\delta)^{n+1}}{(5-3\delta)^{n+1}} \tag{3.40}$$

$$= 0.$$

Using Definition (2.6) and Lemma (2.10), we have iteration (1.7) converges faster than (3.29). Similarly we conclude same argument, we have iteration (1.7) converges faster than cases (3.30) and (3.31). Also (3.29) and (3.30) converges faster than (3.31). □

EXAMPLE 3.4. Let $K = R$ and $M \subset K$, where $M = [0, 40]$. Let $T : M \to M$ be a mapping defined by $T(x) = \frac{3x}{4}$. Clearly, the unique fixed point of T is 0. We have to prove that T satisfies equation (2.1). Now for $\delta = \frac{3}{4}$ and for any increasing function ξ with $\xi(0) = 0$, $\forall x, y \in M$;

$$||Tx - Ty|| - \delta||x - y|| - \xi(||x - Tx||) = \frac{3}{4}|x - y| - \frac{3}{4}|x - y| - \xi(|x - \frac{3x}{4}|)$$
$$= -\xi(|\frac{x}{4}|).$$
$$\leq 0.$$

We take $\alpha_n = \beta_n = \frac{3}{4}$ and initial value $x_0 = 15$.Table 2 shows that the iteration process (1.7) converges to $x^* = 0$ faster than iteration process (3.29), (3.30) and (3.31) and also shows that iteration process (3.29) and (3.30) converges faster than (3.31). The convergence behavior of the iteration process is represented in the figure 2.

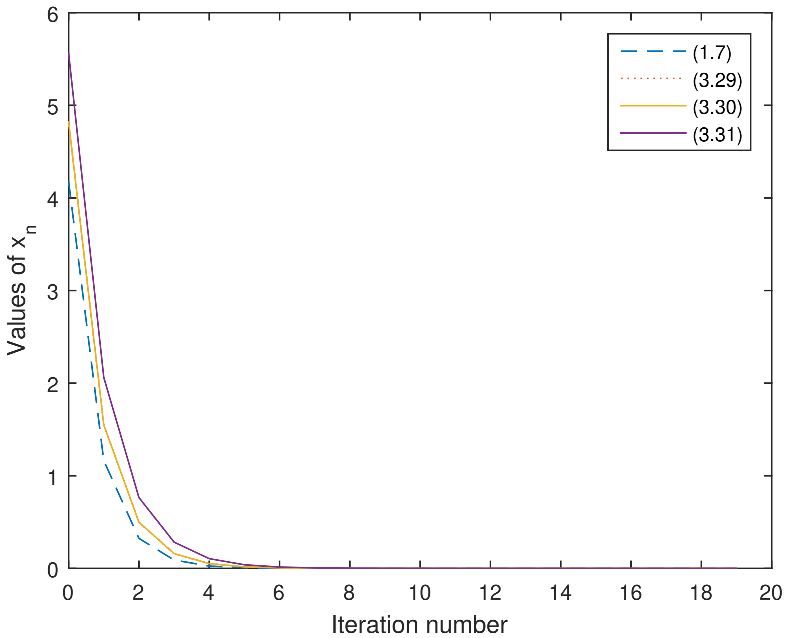

FIGURE 2. Comparison Plot

Iteration	(1.7)	(3.29)	(3.30)	(3.31)
0	15.00000000	15.00000000	15.00000000	15.00000000
1	4.17755127	4.82025146	4.82025146	5.56182861
2	1.16346231	1.54898828	1.54898828	2.06226250
3	0.32402823	0.49776753	0.49776753	0.76466337
4	0.09024297	0.15995765	0.15995765	0.28352844
5	0.02513298	0.05140241	0.05140241	0.10512911
6	0.00699962	0.01651817	0.01651817	0.03898067
7	0.00194942	0.00530811	0.00530811	0.01445359
8	0.00054292	0.00170576	0.00170576	0.00535923
9	0.00015120	0.00054815	0.00054815	0.00198714
10	0.00004211	0.00017615	0.00017615	0.00073681
11	0.00001173	0.00005660	0.00005660	0.00027320
12	0.00000327	0.00001819	0.00001819	0.00010130
13	0.00000091	0.00000585	0.00000585	0.00003756
14	0.00000025	0.00000188	0.00000188	0.00001393
15	0.00000007	0.00000060	0.00000060	0.00000516
16	0.00000002	0.00000019	0.00000019	0.00000191
17	0.00000001	0.00000006	0.00000006	0.00000071
18	0.00000000	0.00000002	0.00000002	0.00000026
19	0.00000000	0.00000001	0.00000001	0.00000010
20	0.00000000	0.00000000	0.00000000	0.00000004

TABLE 2. Comparison Table

THEOREM 3.5. *Let K be a uniformly convex Banach space and M be any nonempty closed convex subset of K. Let $T : M \to M$ be a mapping satisfying* (2.1) *with constant $\delta \in [0,1)$ and x^* be a fixed point of T. Suppose $\{u_n\}$ is defined by the iteration process* (1.6) *and $\{x_n\}$ defined by* (1.7)*, where $\{\alpha_n\}$, $\{\beta_n\}$ are in* $(0,1)$*. Then $\{x_n\}$ convergences faster than $\{u_n\}$, i.e. our process* (1.7) *converges faster than* (1.6).

Proof. By using Lemma 3.1, in iteration (1.6) and in equation (3.4), we have

$$
\begin{aligned}
||u_{n+1} - x^*|| &\leq \delta^2(1-(1-\delta)\alpha_n)(1+(1-\delta)\beta_n)||u_n - x^*|| \\
&\leq \delta^{2\times 2}\prod_{k=n-1}^{n}(1-(1-\delta)\alpha_k)(1+(1-\delta)\beta_k)||u_{n-1} - x^*|| \\
&\vdots \\
&\leq \delta^{2(n+1)}\prod_{k=0}^{n}(1-(1-\delta)\alpha_k)(1-(1-\delta)\beta_k)||u_0 - x^*||. \qquad (3.41)
\end{aligned}
$$

Now , by using Lemma 3.3, in iteration (1.7) and in equation (3.34) , we have

$$\begin{aligned} ||x_{n+1} - x^*|| &\leq \delta^3(1 + (1-\delta)\alpha_n)(1 - (1-\delta)\beta_n)||x_n - x^*|| \\ &\leq \delta^{3\times 2} \prod_{k=(n-1)}^{n} (1 - (1-\delta)\alpha_k)(1 - (1-\delta)\beta_k)||x_{n-1} - x^*|| \\ &\vdots \\ &\leq \delta^{3(n+1)} \prod_{k=0}^{n}(1 - (1-\delta)\alpha_k)(1 - (1-\delta)\beta_k)||x_0 - x^*||. \end{aligned} \tag{3.42}$$

so

$$\begin{aligned} \lim_{n\to\infty} \frac{||x_{n+1} - x^*||}{||u_{n+1} - x^*||} &= \lim_{n\to\infty} \frac{\delta^{3(n+1)} \prod_{k=0}^{n}(1 - (1-\delta)\alpha_k)(1 - (1-\delta)\beta_k)||x_0 - x^*||}{\delta^{2(n+1)} \prod_{k=0}^{n}(1 - (1-\delta)\alpha_k)(1 - (1-\delta)\beta_k)||u_0 - x^*||} \\ &= \lim_{n\to\infty} \delta^{n+1} \frac{||x_0 - x^*||}{||u_0 - x^*||}. \end{aligned} \tag{3.43}$$

Since $\delta \in [0, 1)$, we have

$$\lim_{n\to\infty} \frac{||x_{n+1} - x^*||}{||u_{n+1} - x^*||} = 0. \tag{3.44}$$

Hence by Definition (2.6) and Lemma 2.10, we have $\{x_n\}$ converges faster than $\{u_n\}$. So our iteration process (1.7) converges faster than (1.6). □

Example 3.6. Let K be a set of real number and M be a subset of K, where $M = [0, 60]$. Let $T : M \to M$ be a mapping defined by $T(x) = \frac{2x}{3}$. Clearly, T has a unique fixed point is 0. We have to show that T satisfies equation (2.1). Now for $\delta = \frac{2}{3}$ and for any increasing function ξ with $\xi(0) = 0$, for all $x, y \in M$, we have

$$\begin{aligned} ||Tx - Ty|| - \delta||x - y|| - \xi(||x - Tx|| &= \frac{2}{3}|x - y| - \frac{2}{3}|x - y| - \xi(|x - \frac{2x}{3}|) \\ &= -\xi(|\frac{x}{3}|) \\ &\leq 0. \end{aligned}$$

We take $\alpha_n = \beta_n = \frac{2}{3}$ and initial value $x_0 = 25$. The comparison table 3 shows that the iteration process(1.7) converges to $x^* = 0$ faster than Akutsah et al. [4], Abbas et al. [5], Ullah et al. [9], Karakaya et al. [24], Thakur at el. [2] and Kadiogul et al. [15]. The convergence behavior of these iteration process are represented in the figure 3.

4. Convergence and Stability Theorems

In this section, we establish convergence, T-stable and w^2-stable theorems using our iteration process (1.7).

Theorem 4.1. *Let K be a uniformly convex Banach space and M be any nonempty closed convex subset of K. Let $T : M \to M$ be a mapping satisfying (2.1) with constant $\delta \in [0, 1)$ and x^* be a fixed point of T. Suppose $\{x_n\}$ is defined by the iteration process (1.7) with sequences $\{\alpha_n\}$ and $\{\beta_n\}$ in $(0, 1)$ and $\sum_{n=0}^{\infty} \alpha_n = \infty$ or $\sum_{n=0}^{\infty} \beta_n = \infty$. Then $\{x_n\}$ converges strongly to a unique fixed point of T.*

It.No.	New	Akutsah	Abbas	Ullah	Karakaya	Thakur	Kadioglu
0	25.00000000	25.00000000	25.00000000	25.00000000	25.00000000	25.00000000	25.00000000
1	4.48102423	6.72153635	8.64197531	8.64197531	8.64197531	9.46502058	10.08230453
2	0.80318313	1.80716204	2.98734949	2.98734949	2.98734949	3.58346458	4.06611458
3	0.14396332	0.48587621	1.03266402	1.03266402	1.03266402	1.35670264	1.63983222
4	0.02580413	0.13063339	0.35697028	0.35697028	0.35697028	0.51364874	0.66133151
5	0.00462516	0.03512228	0.12339713	0.12339713	0.12339713	0.19446783	0.26670983
6	0.00082902	0.00944303	0.04265580	0.04265580	0.04265580	0.07362568	0.10756199
7	0.00014859	0.00253887	0.01474521	0.01474521	0.01474521	0.02787474	0.04337891
8	0.00002663	0.00068260	0.00509711	0.00509711	0.00509711	0.01055340	0.01749437
9	0.00000477	0.00018353	0.00176196	0.00176196	0.00176196	0.00399553	0.00705534
10	0.00000086	0.00004934	0.00060907	0.00060907	0.00060907	0.00151271	0.00284537
11	0.00000015	0.00001327	0.00021054	0.00021054	0.00021054	0.00057271	0.00114751
12	0.00000003	0.00000357	0.00007278	0.00007278	0.00007278	0.00021683	0.00046278
13	0.00000000	0.00000096	0.00002516	0.00002516	0.00002516	0.00008209	0.00018664
14	0.00000000	0.00000026	0.00000870	0.00000870	0.00000870	0.00003108	0.00007527
15	0.00000000	0.00000007	0.00000301	0.00000301	0.00000301	0.00001177	0.00003036
16	0.00000000	0.00000002	0.00000104	0.00000104	0.00000104	0.00000445	0.00001224
17	0.00000000	0.00000001	0.00000036	0.00000036	0.00000036	0.00000169	0.00000494
18	0.00000000	0.00000000	0.00000012	0.00000012	0.00000012	0.00000064	0.00000199
19	0.00000000	0.00000000	0.00000004	0.00000004	0.00000004	0.00000024	0.00000080
20	0.00000000	0.00000000	0.00000001	0.00000001	0.00000001	0.00000009	0.00000032
21	0.00000000	0.00000000	0.00000001	0.00000001	0.00000001	0.00000003	0.00000013
22	0.00000000	0.00000000	0.00000000	0.00000000	0.00000000	0.00000001	0.00000005
23	0.00000000	0.00000000	0.00000000	0.00000000	0.00000000	0.00000000	0.00000002
24	0.00000000	0.00000000	0.00000000	0.00000000	0.00000000	0.00000000	0.00000001
25	0.00000000	0.00000000	0.00000000	0.00000000	0.00000000	0.00000000	0.00000000
26	0.00000000	0.00000000	0.00000000	0.00000000	0.00000000	0.00000000	0.00000000
27	0.00000000	0.00000000	0.00000000	0.00000000	0.00000000	0.00000000	0.00000000
28	0.00000000	0.00000000	0.00000000	0.00000000	0.00000000	0.00000000	0.00000000
29	0.00000000	0.00000000	0.00000000	0.00000000	0.00000000	0.00000000	0.00000000
30	0.00000000	0.00000000	0.00000000	0.00000000	0.00000000	0.00000000	0.00000000
31	0.00000000	0.00000000	0.00000000	0.00000000	0.00000000	0.00000000	0.00000000
32	0.00000000	0.00000000	0.00000000	0.00000000	0.00000000	0.00000000	0.00000000
33	0.00000000	0.00000000	0.00000000	0.00000000	0.00000000	0.00000000	0.00000000
34	0.00000000	0.00000000	0.00000000	0.00000000	0.00000000	0.00000000	0.00000000
35	0.00000000	0.00000000	0.00000000	0.00000000	0.00000000	0.00000000	0.00000000

TABLE 3. Comparison table

Proof. By the proof of Lemma 3.3, for iteration (1.7) we have

$$\begin{aligned} ||x_{n+1} - x^*|| &\leq \delta^3(1 + (1-\delta)\alpha_n)(1 - (1-\delta)\beta_n)||x_n - x^*|| \\ &\leq \delta^{3\times 2} \prod_{k=(n-1)}^{n} (1 - (1-\delta)\alpha_k)(1 - (1-\delta)\beta_k)||x_{n-1} - x^*|| \\ &\vdots \\ &\leq \delta^{3(n+1)} \prod_{k=0}^{n}(1 - (1-\delta)\alpha_k)(1 - (1-\delta)\beta_k)||x_0 - x^*||. \qquad (4.1) \end{aligned}$$

Since $\{\alpha_n\}$ and $\{\beta_n\}$ are in $(0,1)$, we have $(1-(1-\delta)\alpha_n)(1-(1-\delta)\beta_n) \in (0,1)$. For inequality $1 - x \leq e^{-x}\ \forall\ x \in [0,1]$, thus from (4.1), we have

$$||x_{n+1} - x^*|| \leq \frac{\delta^{3(n+1)}||x_0 - x^*||}{e^{(1-\delta)\sum_{n=0}^{\infty}\alpha_n + \sum_{n=0}^{\infty}\beta_n}}.$$

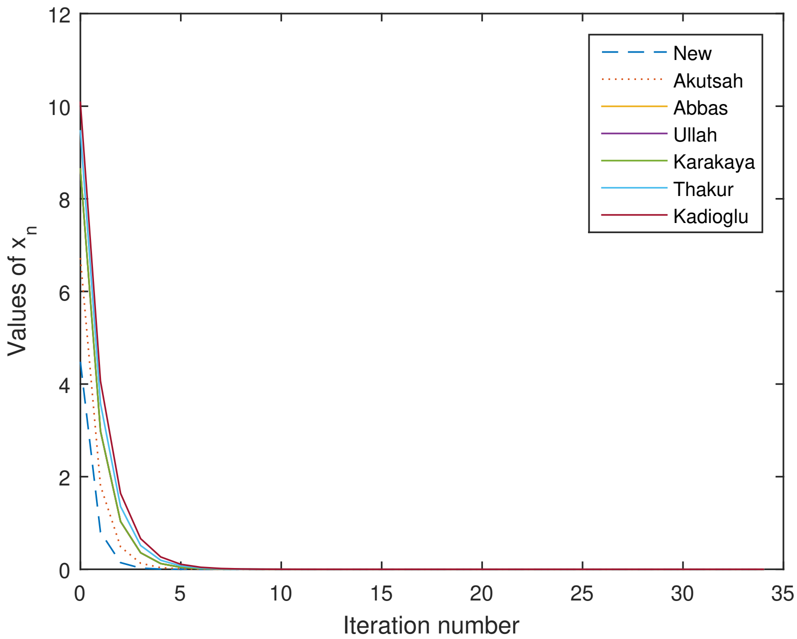

FIGURE 3. Comparison Plot

So

$$\lim_{n\to\infty} ||x_{n+1} - x^*|| \leq \lim_{n\to\infty} \frac{\delta^{3(n+1)}||x_0 - x^*||}{e^{(1-\delta)\sum_{n=0}^{\infty}\alpha_n + \sum_{n=0}^{\infty}\beta_n}}$$
$$= 0. \tag{4.2}$$

We now prove that x^* is unique. Let x^*, $x^{**} \in F(T)$, such that $x^* \neq x^{**}$. Now

$$||x^* - x^{**}|| = ||Tx^* - Tx^{**}|| \tag{4.3}$$

using equation (2.1), we have

$$||Tx^* - Tx^{**}|| \leq \delta||x^* - x^{**}|| + \xi(||x^* - Tx^*||)$$
$$\leq ||x^* - x^{**}|| \tag{4.4}$$

From equation (4.3) and (4.4), we have

$$||x^* - x^{**}|| \leq ||x^* - x^{**}||.$$

Clearly we have that $||x^* - x^{**}|| = ||x^* - x^{**}||$. Hence $x^* = x^{**}$. □

THEOREM 4.2. *Let K be a uniformly convex Banach space and M be any nonempty closed convex subset of K. Let $T : M \to M$ be a mapping satisfying* (2.1) *with constant $\delta \in [0, 1)$ and x^* be a fixed point of T. suppose $\{x_n\}$ is generated by the iteration process* (1.7) *with sequences $\{\alpha_n\}$ and $\{\beta_n\}$ in* $(0, 1)$*.Then the iteration process* (1.7) *is $T-$ stable.*

Proof. Let $\{t_n\}$ be an arbitrary sequence in M and the sequence generated by (1.7) $x_{n+1} = f(T, x_n)$ converges to a unique fixed point x^* and $\epsilon = ||t_{n+1} - f(T, t_n)||$. We shall prove that $\lim_{n\to\infty} \epsilon = 0$ if and only if $\lim_{n\to\infty} t_n = x^*$. Suppose $\lim_{n\to\infty} \epsilon = 0$

and

$$\begin{aligned}
||t_{n+1} - x^*|| &= ||t_{n+1} - f(T,t_n) + f(T,t_n) - x^*|| \\
&\le ||t_{n+1} - f(T,t_n)|| + ||f(T,t_n) - x^*|| \\
&= \epsilon + ||Ts_n - x^*|| \\
&\le \epsilon + \delta||s_n - x^*|| \\
&= \epsilon + \delta||T((1-\beta_n)r_n + \beta_n Tr_n) - x^*|| \\
&\le \epsilon + \delta^2||x^* - ((1-\beta_n)r_n + \beta_n Tr_n)|| \\
&\le \epsilon + \delta^2[(1-\beta_n)||r_n - x^*|| + \beta_n||Tr_n - x^*||] \\
&\le \epsilon + \delta^2[(1-\beta_n)||r_n - x^*|| + \beta_n\delta||r_n - x^*||] \\
&= \epsilon + \delta^2[(1-(1-\delta)\beta_n)||r_n - x^*||] \\
&\le \epsilon + \delta^2(1-(1-\delta)\beta_n)||T((1-\alpha_n)t_n + \alpha_n Tt_n) - x^*|| \\
&\le \epsilon + \delta^3(1-(1-\delta)\beta_n)||x^* - ((1-\alpha_n)t_n + \alpha_n Tt_n)|| \\
&\le \epsilon + \delta^3(1-(1-\delta)\beta_n)[(1-\alpha_n)||t_n - x^*|| + \alpha_n||Tt_n - x^*||] \\
&\le \epsilon + \delta^3(1-(1-\delta)\beta_n)[(1-\alpha_n)||t_n - x^*|| + \alpha_n\delta||t_n - x^*||] \\
&= \epsilon + \delta^3(1-(1-\delta)\beta_n)(1-(1-\delta)\alpha_n)||t_n - x^*||.
\end{aligned}$$

Since $\delta \in [0,1)$ and $\{\alpha_n\}$, $\{\beta_n\} \in (0,1)$, so $\delta^3(1-(1-\delta)\beta_n)(1-(1-\delta)\alpha_n) < 1$. Using Lemma 2.11, we have

$$\lim_{n\to\infty} ||t_n - x^*|| = 0$$
$$\lim_{n\to\infty} t_n = x^*.$$

Converse part: Suppose $\lim_{n\to\infty} t_n = x^*$ and

$$\begin{aligned}
\epsilon &= ||t_{n+1} - f(T,t_n)|| \\
&\le ||t_{n+1} - x^*|| + ||x^* - f(T,t_n)|| \\
&= ||t_{n+1} - x^*|| + ||Ts_n - x^*|| \\
&\le ||t_{n+1} - x^*|| + \delta||s_n - x^*|| \\
&= ||t_{n+1} - x^*|| + \delta||T((1-\beta_n)r_n + \beta_n Tr_n) - x^*|| \\
&\le ||t_{n+1} - x^*|| + \delta^2||x^* - ((1-\beta_n)r_n + \beta_n Tr_n)|| \\
&\le ||t_{n+1} - x^*|| + \delta^2[(1-\beta_n)||r_n - x^*|| + \beta_n||Tr_n - x^*||] \\
&\le ||t_{n+1} - x^*|| + \delta^2[(1-\beta_n)||r_n - x^*|| + \beta_n\delta||r_n - x^*||] \\
&= ||t_{n+1} - x^*|| + \delta^2[(1-(1-\delta)\beta_n)||r_n - x^*||] \\
&\le ||t_{n+1} - x^*|| + \delta^2(1-(1-\delta)\beta_n)||T((1-\alpha_n)t_n + \alpha_n Tt_n) - x^*|| \\
&\le ||t_{n+1} - x^*|| + \delta^3(1-(1-\delta)\beta_n)||x^* - ((1-\alpha_n)t_n + \alpha_n Tt_n)|| \\
&\le ||t_{n+1} - x^*|| + \delta^3(1-(1-\delta)\beta_n)[(1-\alpha_n)||t_n - x^*|| + \alpha_n||Tt_n - x^*||] \\
&\le ||t_{n+1} - x^*|| + \delta^3(1-(1-\delta)\beta_n)[(1-\alpha_n)||t_n - x^*|| + \alpha_n\delta||t_n - x^*||] \\
&= ||t_{n+1} - x^*|| + \delta^3(1-(1-\delta)\beta_n)(1-(1-\delta)\alpha_n)||t_n - x^*||.
\end{aligned}$$

Since $\lim_{n\to\infty} t_n = x^*$, we have

$$\lim_{n\to\infty} \epsilon = 0.$$

Hence iteration process (1.7) is $T - stable$. □

THEOREM 4.3. *Let K be a uniformly convex Banach space and M be any nonempty closed convex subset of K. Let $T : M \to M$ be a mapping satisfying* (2.1) *with constant $\delta \in [0, 1)$ and x^* be a fixed point of T. Suppose $\{x_n\}$ is defined by iteration process* (1.7) *with sequences $\{\alpha_n\}$ and $\{\beta_n\}$ in* $(0, 1)$. *Then the iteration process* (1.7) *is weak $w^2 - stable$ with respect to T.*

Proof. Let $\{p_n\} \subset M$ be an equivalent sequence of $\{x_n\}$ and suppose that $\epsilon_n = ||p_{n+1} - Tr_n||$, where $r_n = T((1 - \beta_n)q_n + \beta_n Tq_n)$ and $q_n = T((1 - \alpha_n)p_n + \alpha_n Tp_n)$. Let $\lim_{n\to\infty} \epsilon_n = 0$ and

$$\begin{aligned}
||p_{n+1} - x^*|| &= ||p_{n+1} - x_{n+1} + x_{n+1} - x^*|| \\
&\le ||p_{n+1} - x_{n+1}|| + ||x_{n+1} - x^*|| \\
&\le ||p_{n+1} - Tr_n|| + ||Tr_n - x_{n+1}|| + ||x_{n+1} - x^*|| \\
&\le ||p_{n+1} - Tr_n|| + ||Tr_n - Ty_n|| + ||x_{n+1} - x^*|| \\
&= \epsilon_n + ||Ty_n - Tr_n|| + ||x_{n+1} - x^*|| \\
&\le \epsilon_n + \delta||y_n - r_n|| + \xi(||y_n - Ty_n||) + ||x_{n+1} - x^*||. \qquad (4.5)
\end{aligned}$$

Now

$$\begin{aligned}
||y_n - r_n|| &\le (1 - \beta_n)||z_n - q_n|| + \beta_n||Tz_n - Tq_n|| \\
&\le (1 - \beta_n)||z_n - q_n|| + \beta_n\delta||z_n - q_n|| + \beta_n\xi(||z_n - Tz_n||) \\
&= (1 - (1 - \delta)\beta_n)||z_n - q_n|| + \beta_n\xi(||z_n - Tz_n||). \qquad (4.6)
\end{aligned}$$

Also

$$\begin{aligned}
||z_n - q_n|| &\le (1 - \alpha_n)||x_n - p_n|| + \alpha_n||Tx_n - Tp_n|| \\
&\le (1 - \alpha_n)||x_n - p_n|| + \alpha_n\delta||x_n - p_n|| + \alpha_n\xi(||x_n - Tx_n||) \\
&= (1 - (1 - \delta)\alpha_n)||x_n - p_n|| + \alpha_n\xi(||x_n - Tx_n||). \qquad (4.7)
\end{aligned}$$

From eq. (4.5),(4.6) and (4.7), we have

$$\begin{aligned}
||p_{n+1} - x^*|| \le \epsilon_n + \delta[(1 - (1 - \delta)\beta_n)(1 - (1 - \delta)\alpha_n)||x_n - p_n|| + \alpha_n\xi(||x_n - Tx_n||) \\
+ \beta_n\xi(||z_n - Tz_n||) + \xi(||y_n - Ty_n||)] + ||x_{n+1} - x^*||. \qquad (4.8)
\end{aligned}$$

Using Theorem 4.1, we have $\lim_{n\to\infty} x_n = x^*$ and $\{x_n\}$ and $\{p_n\}$ are equivalent, we have $\lim_{n\to\infty} ||x_n - p_n|| = 0$. It also follows that

$$\begin{aligned}
||x_n - Tx_n|| &\le ||x_n - x^*|| + ||Tx^* - Tx_n|| \\
&\le ||x_n - x^*|| + \delta||x_n - x^*|| + \xi(||x^* - Tx^*||) \\
&\le (1 + \delta)||x_n - x^*|| \to 0 \quad as\ n \to \infty.
\end{aligned}$$

Applying same argument, we have $\lim_{n\to\infty} ||z_n - Tz_n|| = 0$ and $\lim_{n\to\infty} ||y_n - Ty_n|| = 0$, since $\lim_{n\to\infty} ||y_n - Ty_n|| = \xi(\lim_{n\to\infty} ||y_n - Ty_n||) = 0$.
Similarly we apply same argument we have $\xi(\lim_{n\to\infty} ||z_n - Tz_n||) = 0$ and $\xi(\lim_{n\to\infty} ||x_n - Tx_n|| = 0$. From Equation (4.8) we have

$$\lim_{n\to\infty} ||p_{n+1} - x^*|| = 0.$$

Consequently

$$\lim_{n\to\infty} ||p_n - x^*|| = 0.$$

Thus $\{x_n\}$ is weak $w^2 - stable$ with respect to T. □

5. Convergence Results for Nonexpansive mapping

In this section, we establish some convergence results using our iteration scheme (1.7) for nonexpansive mapping:

LEMMA 5.1. *Let K be a uniformly convex Banach space and M be any nonempty closed convex subset of K. Let $T : M \to M$ be a nonexpansive self mapping with $F(T) \neq \phi$. Suppose that $\{x_n\}$ is generated by (1.7), then $\lim_{n\to\infty} ||x_n - x^*||$ exists $\forall x^* \in F(T)$.*

Proof. Let $x^* \in F(T)$, $\forall n \in N$. From iteration process (1.7), we have

$$\begin{aligned} ||z_n - x^*|| &= ||T((1-\alpha)x_n + \alpha T x_n) - x^*|| \\ &\leq ||(1-\alpha_n)x_n + \alpha_n T x_n - x^*|| \\ &\leq (1-\alpha_n)||x_n - x^*|| + \alpha_n ||T x_n - x^*|| \\ &= ||x_n - x^*||. \end{aligned} \tag{5.1}$$

$$\begin{aligned} ||y_n - x^*|| &= ||T((1-\beta_n)z_n + \beta_n T z_n) - x^*|| \\ &\leq ||(1-\beta_n)z_n + \beta_n T z_n - x^*|| \\ &\leq (1-\beta_n)||z_n - x^*|| + \beta_n ||z_n - x^*|| \\ &= ||z_n - x^*||. \end{aligned} \tag{5.2}$$

Thus from (5.1) and (5.2), we have

$$||y_n - x^*|| \leq ||x_n - x^*||. \tag{5.3}$$

Also

$$\begin{aligned} ||x_{n+1} - x^*|| &= ||T y_n - x^*|| \\ &\leq ||y_n - x^*||. \end{aligned} \tag{5.4}$$

From (5.3) and (5.4), we have

$$||x_{n+1} - x^*|| \leq ||x_n - x^*||.$$

Hence $\lim_{n\to\infty} ||x_n - x^*||$ exists for all $x^* \in F(T)$. □

LEMMA 5.2. *Let K be a uniformly convex Banach space and let M be any nonempty closed convex subset of K. Let $T : M \to M$ be a nonexpansive self mapping with $F(T) \neq \phi$. Suppose $\{x_n\}$ is generated by iteration process (1.7). Then $\lim_{n\to\infty} ||x_n - T x_n|| = 0$.*

Proof. By using Lemma 5.1, we have

$$\lim_{n\to\infty} ||x_n - x^*|| = exists.$$

Suppose that $\lim_{n\to\infty} ||x_n - x^*|| = a$. If $a = 0$, then we are done. Thus we consider $a > 0$.

$$||Tx_n - x^*|| \leq ||x_n - x^*||.$$

Thus taking $\limsup_{n\to\infty}$, we have

$$\limsup_{n\to\infty} ||Tx_n - x^*|| \leq a.$$

From (5.1), we have

$$||z_n - x^*|| \leq ||x_n - x^*||,$$

which implies that

$$\limsup_{n\to\infty} ||z_n - x^*|| \leq a. \tag{5.5}$$

From (5.2) and (5.4), we have

$$\begin{aligned} ||x_{n+1} - x^*|| &\leq ||y_n - x^*|| \\ &\leq ||z_n - x^*||. \end{aligned}$$

Taking $\liminf_{n\to\infty}$, we have

$$a \leq \liminf_{n\to\infty} ||z_n - x^*||. \tag{5.6}$$

From (5.5) and (5.6), we have

$$\lim_{n\to\infty} ||z_n - x^*|| = a.$$

That is

$$\lim_{n\to\infty} ||T((1-\alpha_n)x_n + \alpha_n Tx_n) - x^*|| = a.$$

Thus by Lemma 2.12, we have

$$\lim_{n\to\infty} ||x_n - Tx_n|| = 0.$$

□

THEOREM 5.3. *Let K be a uniformly convex Banach space and M be any nonempty closed convex subset of K with satisfy Opial's property. Let $T : M \to M$ be a nonexpansive mapping with $F(T) \neq \phi$. Let $\{x_n\}$ is generated by iteration process (1.7). Then $\{x_n\}$ converges weakly to a fixed point of T.*

Proof. Let $x^* \in F(T)$. By Lemma 5.1, then $\lim_{n\to\infty} ||x_n - x^*||$ exists. We prove that $\{x_n\}$ has a unique weak subsequential limit in $F(T)$. Let u and v be weak limits of the subsequences $\{x_m\}$ and $\{x_k\}$ of $\{x_n\}$ respectively. From Lemma 5.2, we have $\lim_{n\to\infty} ||x_n - Tx_n|| = 0$ and $I - T$ is demiclosed with respect to zero by Lemma 2.13, we have that $Tu = u$. Same approach we have to prove that $v \in F(T)$.We have to show that uniqueness.From Lemma 5.1, we have $\lim_{n\to\infty} ||x_n - v||$ exist. Now suppose that $u \neq v$, then by opial's condition,

$$\begin{aligned} \lim_{n\to\infty} ||x_n - u|| &= \lim_{m\to\infty} ||x_m - u|| \\ &< \lim_{m\to\infty} ||x_m - v|| \\ &= \lim_{n\to\infty} ||x_n - v|| \\ &= \lim_{k\to\infty} ||x_k - v|| \\ &\leq \lim_{k\to\infty} ||x_k - u|| \\ &= \lim_{n\to\infty} ||x_n - u||. \end{aligned}$$

Which is contradiction, so $u = v$. Hence $\{x_n\}$ converges weakly to a fixed point of $F(T)$. □

Theorem 5.4. *Let K be a uniformly convex Banach space and M be any nonempty closed convex subset of K. Let $T : M \to M$ be a nonexpansive mapping with $F(T) \neq \phi$. Suppose that $\{x_n\}$ is defined by the iteration process* (1.7). *Then the sequence $\{x_n\}$ converges to a fixed point of T iff* $\liminf_{n\to\infty} d(x_n, F(T)) = 0$, *where* $d(x, F(T)) = \inf\{||x - x^*|| : x^* \in F(T)\}$.

Proof. Let $\{x_n\}$ converges to x^*, then $\lim_{n\to\infty} d(x_n, x^*) = 0$.It follows that $\lim_{n\to\infty} d(x_n, F(T)) = 0$. Therefore, $\liminf_{n\to\infty} d(x_n, F(T)) = 0$.
Conversely: Suppose that $\lim_{n\to\infty} d(x_n, F(T)) = 0$. It follows that from Lemma 5.1, we have $\lim\limits_{n\to\infty} ||x_n - x^*||$ exists and that $\liminf_{n\to\infty} d(x_n, F(T))$ exists for all $x^* \in F(T)$. Our assumption $\liminf_{n\to\infty} d(x_n, F(T)) = 0$, therefore we have $\lim\limits_{n\to\infty} d(x_n, F(T)) = 0$. We will show that $\{x_n\}$ is a cauchy sequence in M. Since $\lim\limits_{n\to\infty} d(x_n, F(T)) = 0$ for given $\epsilon > 0$, $\exists\ n_0 \in N$ s.t. $\forall n \geq n_0$,

$$d(x_n, F(T)) < \frac{\epsilon}{2}.$$

In particular, $\inf\{||x_{n_0} - x^*|| : x_1^* \in F(T)\} < \frac{\epsilon}{2}$. Hence $\exists\ x_1^* \in F(T)$ s.t.$||x_{n_0} - x_1^*|| < \frac{\epsilon}{2}$. Now for $m, n \geq n_0$,

$$\begin{aligned} ||x_{n+m} - x_n|| &\leq ||x_{n+m} - x_1^*|| + ||x_n - x_1^*|| \\ &\leq 2||x_{n_0} - x_1^*|| \\ &< \epsilon. \end{aligned}$$

Hence $\{x_n\}$ is a cauchy sequence in M. Since M is closed in a uniformly Banach space K, thus $\exists$ a point $x^* \in M$ s.t. $\lim\limits_{n\to\infty} x_n = x^*$. Now $\lim\limits_{n\to\infty} d(x_n, F(T)) = 0$, gives that $d(x^*, F(T)) = 0$ which implies that $x^* \in F(T)$. □

Theorem 5.5. *Let K be a uniformly convex Banach space and M be any nonempty closed convex subset of K. Let $T : M \to M$ be a nonexpansive self mapping with $F(T) \neq \phi$. Suppose that $\{x_n\}$ is defined by iteration process* (1.7). *Let T satisfy condition (I), then $\{x_n\}$ strongly converges to a fixed point of T.*

Proof. By using Lemma 5.2, we have

$$\lim_{n\to\infty} ||x_n - Tx_n|| = 0. \tag{5.7}$$

From condition (I) and (5.7), we have

$$\lim_{n\to\infty} f(d(x_n, F(T))) \leq \lim_{n\to\infty} ||x_n - Tx_n||$$
$$\Rightarrow \lim_{n\to\infty} f(d(x_n, F(T))) = 0.$$

Since $f : [0, \infty) \to [0, \infty)$ is a non decreasing function satisfying $f(0) = 0$ and $f(c) > 0$ for all $c \in (0, \infty)$, therefore, we have

$$\lim_{n\to\infty} d(x_n, F(T)) = 0.$$

By Theorem 5.4, the sequence $\{x_n\}$ strongly converges to a fixed point of $F(T)$. □

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