A STUDY OF GENERALIZED ADAMS-MOULTON METHOD FOR THE SATELLITE ORBIT DETERMINATION PROBLEM

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Abstract. In this paper, a generalized Adams-Moulton method that is a m-step method derived by using the Taylor's series is proposed to solve the satellite orbit determination problem. We show that our proposed method has produced much smaller error than the original Adams-Moulton method. Finally, the accuracy performance is demonstrated in the satellite orbit correction problem by giving a numerical example.

1. Introduction

Because a strongly stable multistep method in terms of round-off errors produces a relatively accurate approximation solution, the Adams-Bashforth-Moulton predictor-corrector method that is strongly stable is effectively used in many packages [1, 4]. In addition, since the multistep integrator needs relatively small number of function values [5], a multistep method for the orbit prediction and correction determination problem is preferable to a single step method. In [3], Hahm and Hong suggested a generalized Adams-Bashforth method that is explicit for the

satellite orbit determination problem. In this paper, we propose a generalized Adams-Moulton method for the same problem and our method is implicit.

Note that the original Adams-Moulton method is of the form

$$(1.1) \qquad y_{i+1} = y_i + h \sum_{l=-1}^{m-1} b_l f(t_{i-l}, y_{i-l})$$

where $b_{-1}, b_0, \ldots, b_{m-1}$ are constants to be determined. In this paper, by utilizing the Taylor's series, we formulate the generalized Adams-Moulton method associated with error control parameters

$$(1.2) \qquad y_{i+1} = \sum_{k=0}^{m-1} a_k y_{i-k} + h \sum_{l=-1}^{m-1} b_l f(t_{i-l}, y_{i-l})$$

where $\{a_0, a_1, \ldots, a_{m-1}\}$ and $\{b_{-1}, b_0, \ldots, b_{m-1}\}$ are constants to be determined. Then this method becomes a special case of the general multistep method and is only the method that can produce smaller local truncation error than the original Adams-Moulton method even if there are infinitely many strongly stable multistep methods.

For the comparison purposes, the two body problem of the Earth's satellite is integrated numerically. The two body problem is considered since it has an exact solution that can be used in the error quantification. The satellite for the numerical integration is a Geoscience Laser Altimeter System type low-altitude satellite [2].

2. Preliminaries

Let the first-order initial-value problem be of the form

$$(2.1) \qquad y' = f(t, y); \quad a \le t \le b, \ y(a) = y_0.$$

Suppose that $h = \frac{b-a}{N}$, $t_i = a + ih$ for $i = 0, 1, \ldots, N$, and that y_i is the approximation to $y(t_i)$ for each $i = 0, 1, \ldots, m-1$. Then the original m-step Adams-Moulton (AM) method is represented by

$$(2.2) \qquad y_{i+1} = y_i + h \sum_{l=-1}^{m-1} b_l f(t_{i-l}, y_{i-l})$$

for $i = m-1, m, \ldots, N-1$, where $b_{-1}, b_0, \ldots, b_{m-1}$ are constants satisfying the property

$$(2.3) \qquad 1 = \sum_{l=-1}^{m-1} b_l.$$

Moreover, this method is implicit since y_{i+1} occurs on both sides of (2.2). Note that the Taylor's series for y_{i+1} is given by

$$(2.4) \qquad y_{i+1} = y_i + \sum_{j=1}^{\infty} D_{i,j} h^j$$

where

$$(2.5) \qquad D_{i,j} = \frac{1}{j!} \frac{d^j y}{dt^j}(t_i).$$

In other word, $D_{i,j}$ is an operator representing the j-th derivative of y at t_i divided by $j!$. Therefore, we can easily show that the Taylor's series for y_{i-k} is

$$(2.6) \qquad y_{i-k} = y_i + \sum_{j=1}^{\infty} (-k)^j D_{i,j} h^j$$

since $t_{i-k} = t_i - kh$. As a result, if we use the Taylor's series of y_{i-k} (2.6) for $i = m-1, m, \ldots, N-1$, then we have the implicit m-step generalized Adams-Moulton (GAM) method for solving the problem (2.1) given by the difference equation

$$(2.7) \qquad y_{i+1} = \sum_{k=0}^{m-1} a_k y_{i-k} + h \sum_{l=-1}^{m-1} b_l f(t_{i-l}, y_{i-l})$$

where $\{a_0, a_1, \ldots, a_{m-1}\}$ and $\{b_{-1}, b_0, \ldots, b_{m-1}\}$ are constants to be determined.

REMARK 2.1. *In practice, each a_k should be constrained to provide the method with the roundoff stability. It is well known that the roots of the characteristic equation*

$$(2.8) \qquad \lambda^m - a_0 \lambda^{m-1} - a_1 \lambda^{m-2} - \cdots - a_{m-1} = 0$$

must satisfy the root conditions to be a strongly stable method:

Criterion 1 *$\lambda = 1$ is a simple root and is the only root of magnitude one.*

Criterion 2 *All roots except $\lambda = 1$ have absolute value less than 1.*

For all roots λ, $|\lambda| < 1$ except $\lambda = 1$.

REMARK 2.2. *If we set $a_0 = 1$ and $a_1 = \cdots = a_{m-1} = 0$ in the GAM method (2.7), then we simply obtain the original AM method (2.2).*

3. Main results

In this subsection, the generalized Adams-Moulton method that is a multistep method is derived by utilizing the Taylor's series. The coefficient matrix and the error vector of the generalized Adams-Moulton method are formulated. Strongly stable multistep methods can be obtained by choosing appropriate values of parameters associated with the local truncation error. The formula for the local truncation error gives an idea how to choose such values, however, it might not meet with good results because of the accumulative errors. It is known that those parameters should be non-negative.

3.1. The local truncation error of the GAM Method. In this subsection, we actually compute the local truncation error $\tau_{i+1}(h)$ at each step of the m-step GAM method which is smaller than that of the original m-step AM method.

THEOREM 3.1. *For $i = m-1, m, \ldots, N-1$, the local truncation error $\tau_{i+1}(h)$ for the GAM method (2.7) is*

(3.1)
$$\tau_{i+1}(h) = \left\{ 1 - \sum_{k=0}^{m-1}(-k)^{m+2}a_k - \sum_{l=-1}^{m-1}(m+2)(-l)^{m+1}b_l \right\} D_{i,m+2}h^{m+1}.$$

REMARK 3.2. *If we apply Criterion 1 to (2.8), then we can obtain a_0 by the equation*

(3.2)
$$a_0 = 1 - \sum_{k=1}^{m-1} a_k.$$

During the proof of Theorem 3.1, we will show that each b_l can be expressed as a linear combination of $\{a_0, a_1, \ldots, a_{m-1}\}$. However, due to (3.2), it is sufficient to determine $\{a_1, \ldots, a_{m-1}\}$ only.

Proof. To compute the local truncation error $\tau_{i+1}(h)$, let us apply the Taylor's series expansion to the equation $f = y'$ by considering that f is a function of t. Then the equation $f = y'$ is in the following form

$$(3.3) \qquad f(t_{i-l}, y_{i-l}) = \sum_{j=1}^{\infty} j(-l)^{j-1} D_{i,j} h^{j-1}.$$

Substituting (2.4), (2.6) and (3.3) into (2.7) and equating coefficients of the same powers of h give the system of equations,

$$(3.4) \qquad 1 = \sum_{k=0}^{m-1} (-k)^j a_k + \sum_{l=-1}^{m-1} j(-l)^{j-1} b_l \quad \text{for} \quad j = 1, 2, \ldots, m+1.$$

Therefore, a simple calculation gives that the local truncation error $\tau_{i+1}(h)$ at this step is

(3.5)

$$\begin{aligned}
\tau_{i+1}(h) &= \left(y_{i+1} - \sum_{k=0}^{m-1} a_k y_{i-k} \right)/h - \sum_{l=0}^{m-1} b_l f(t_{i-l}, y_{i-l}) \\
&= \left\{ 1 - \sum_{k=0}^{m-1} (-k)^{m+2} a_k - \sum_{l=-1}^{m-1} (m+2)(-l)^{m+1} b_l \right\} D_{i,m+2} h^{m+1}.
\end{aligned}$$

$\square$

REMARK 3.3. *From the equation (3.4), we have two things mentioned earlier. One is that, if we put $j = 1$ in the equation (3.4), then*

$$(3.6) \qquad 1 = \sum_{l=-1}^{m-1} b_l - \sum_{k=1}^{m-1} k a_k.$$

Therefore, since $a_0 = 1$ and $a_1 = \cdots = a_{m-1} = 0$ in the AM method, we have

$$1 = \sum_{l=-1}^{m-1} b_l$$

as shown in the equation (2.3). The other is that each b_l in the equation (3.4) is expressed in the linear combination of $\{a_1, \ldots, a_{m-1}\}$ only as mentioned in Remark 3.2.

3.2. The vector and matrix form of the GAM Method. In this subsection, we use vectors and matrices to represent the GAM method for a convenient computer computation. That is, we theoretically compute the coefficient matrix $\tilde{\mathbf{C}}$ and the error vector $\tilde{\mathbf{e}}$ of the GAM method.

First, we rewrite the system of equations in (3.4) as a vector-matrix form. For the first sum

$$(3.7) \qquad \sum_{k=0}^{m-1} (-k)^j a_k \ \text{ for } \ j = 1, 2, \ldots, m+1$$

of the equation (3.4), we let $\mathbf{A}$ be an $(m+1) \times m$ coefficient matrix of (3.7) such that

$$(3.8) \qquad \mathbf{A} = \begin{bmatrix} & & \overset{(k+1)\text{-th}}{\vdots} & \\ \cdots & -(-k)^j & \cdots \\ & & \vdots & \end{bmatrix} {}_{j\text{-th}}$$

for $j = 1, 2, \ldots, m+1$ and $k = 0, 1, \ldots, m-1$. Note that the first column of $\mathbf{A}$ is associated with the constant a_0 in (3.7). Therefore, we can easily see that the first column of $\mathbf{A}$ becomes a zero vector because $(-k)^j a_k = 0$ when $k = 0$.

Similarly, we define an $(m+1) \times (m+1)$ coefficient matrix $\mathbf{B}$ of the second sum

$$(3.9) \qquad \sum_{l=-1}^{m-1} j(-l)^{j-1} b_l \ \text{ for } \ j = 1, 2, \ldots, m+1$$

by

$$(3.10) \qquad \mathbf{B} = \begin{bmatrix} & & \overset{(l+2)\text{-th}}{\vdots} & \\ \cdots & j(-l)^{j-1} & \cdots \\ & & \vdots & \end{bmatrix} {}_{j\text{-th}}$$

for $j = 1, 2, \ldots, m+1$ and $l = -1, 0, \ldots, m-1$. But only in the case of $j = 1$ and $l = 0$, we set $\mathbf{B}_{12} = 1$.

Also, let $\tilde{\mathbf{a}}$ be an m-dimensional column vector and let $\mathbf{b}$ and $\mathbf{1}$ be $(m+1)$-dimensional vectors such that

$$(3.11) \qquad \tilde{\mathbf{a}} = \begin{bmatrix} 1 \\ a_1 \\ \vdots \\ a_{m-1} \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} b_{-1} \\ b_0 \\ b_1 \\ \vdots \\ b_{m-1} \end{bmatrix} \text{ and } \mathbf{1} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}.$$

Then the system of equations (3.4) is reduced to a vector-matrix form

$$(3.12) \qquad \mathbf{1} = -\mathbf{A}\tilde{\mathbf{a}} + \mathbf{B}\mathbf{b}.$$

From the equation (3.12), we end up with

$$(3.13) \qquad \mathbf{B}\mathbf{b} = \mathbf{1} + \mathbf{A}\tilde{\mathbf{a}}.$$

We now define a new matrix $\tilde{\mathbf{A}}$ by replacing the first column of the matrix $\mathbf{A}$ that is zero vector by a column vector $\mathbf{1}$. Then the new matrix $\tilde{\mathbf{A}}$ is of the form

$$(3.14) \qquad \tilde{\mathbf{A}} = \left[\mathbf{1} \,\middle|\, \mathbf{A}_{j,k+1} \right] \text{ for } j = 1, 2, \ldots, m+1; \; k = 1, \ldots, m-1.$$

As we can see that the matrix $\tilde{\mathbf{A}}$ is the same as the matrix $\mathbf{A}$ except for the first column. Therefore we have

$$(3.15) \qquad \tilde{\mathbf{A}}\tilde{\mathbf{a}} = \mathbf{1} + \mathbf{A}\tilde{\mathbf{a}}.$$

From (3.13) and (3.15), we have

$$(3.16) \qquad \mathbf{B}\mathbf{b} = \tilde{\mathbf{A}}\tilde{\mathbf{a}}.$$

Let

$$(3.17) \qquad \tilde{\mathbf{C}} = \mathbf{B}^{-1}\tilde{\mathbf{A}}.$$

Then, by (3.16) and (3.17), $\mathbf{b}$ can be expressed in a simple form

$$(3.18) \qquad \mathbf{b} = \tilde{\mathbf{C}}\tilde{\mathbf{a}}.$$

So, equation (3.18) shows that b_l for each $l = 1, 2, \ldots, m+1$ is a linear combination of $\{a_1, \ldots, a_{m-1}\}$.

If we set

$$(3.19) \qquad \mathbf{b}_0 = \mathbf{B}^{-1}\mathbf{1} \text{ and } \mathbf{C} = \mathbf{B}^{-1}\mathbf{A},$$

then, similar to $\tilde{\mathbf{A}}$, the matrix $\tilde{\mathbf{C}}$ is the same as the matrix $\mathbf{C}$ except for the first column. In fact, the first column of $\tilde{\mathbf{C}}$ is $\mathbf{b}_0$ so that $\tilde{\mathbf{C}}$ can be expressed as

$$(3.20) \quad \tilde{\mathbf{C}} = \left[\mathbf{b}_0 \Big| \mathbf{C}_{j,k+1} \right] \quad \text{for } j = 1, 2, \ldots, m+1; \ k = 1, \ldots, m-1.$$

Now, combining (3.15), (3.18) and (3.19) gives that $\mathbf{b}$ can be expressed in an additional form

$$(3.21) \qquad\qquad\qquad \mathbf{b} = \mathbf{b}_0 + \mathbf{C}\tilde{\mathbf{a}}.$$

Note that since the AM method has

$$(3.22) \qquad\qquad\qquad \tilde{\mathbf{a}}^T = \begin{bmatrix} 1 & 0 & \cdots & 0 \end{bmatrix},$$

equation (3.21) becomes $\mathbf{b} = \mathbf{b}_0$ as expected and $\mathbf{1}^T \mathbf{b}_0 = \mathbf{1}^T \mathbf{B}^{-1} \mathbf{1} = \sum_{l=-1}^{m-1} b_l = 1$ as shown in (2.3). Moreover, if we compare (3.6) to (3.21), an easy computation shows that

$$(3.23) \qquad\qquad\qquad \mathbf{1}^T \mathbf{C} = \begin{bmatrix} 0 & 1 & \cdots & m-1 \end{bmatrix}.$$

3.3. Error Analysis. In this paper, we theoretically find $\tilde{\mathbf{a}}$ that provides a smaller local truncation error $\tau_{i+1}(h)$ in (3.5)of the GAM method than that of the AM method.

For simplicity, let define ϵ by

$$(3.24) \qquad \epsilon = 1 - \sum_{k=0}^{m-1} (-k)^{m+2} a_k - \sum_{l=-1}^{m-1} (m+2)(-l)^{m+1} b_l.$$

Then local truncation error $\tau_{i+1}(h)$ in (3.5) is

$$(3.25) \qquad\qquad\qquad \tau_{i+1}(h) = \epsilon D_{i,m+2} h^{m+1}.$$

We now express ϵ in the equation (3.24) in terms of vectors and matrices. If we define an m-dimensional column vector $\mathbf{c}$ and an $m+1$-dimensional column vector $\mathbf{d}$ by

$$(3.26) \quad \mathbf{c} = \begin{bmatrix} \vdots \\ -(-k)^{m+2} \\ \vdots \end{bmatrix}_{(k+1)\text{-th}} \text{ and } \mathbf{d} = \begin{bmatrix} \vdots \\ -(m+2)(-l)^{m+1} \\ \vdots \end{bmatrix}_{(l+2)\text{-th}}$$

for $k = 0, 1, \ldots, m - 1$ and $l = -1, 0, \ldots, m - 1$, then by (3.21) and (3.26), ϵ in (3.24) becomes

$$(3.27) \qquad \epsilon = 1 + \mathbf{d}^T \mathbf{b} + \mathbf{c}^T \tilde{\mathbf{a}}$$

$$(3.28) \qquad = 1 + \mathbf{d}^T \mathbf{b}_0 + (\mathbf{d}^T \mathbf{C} + \mathbf{c}^T) \tilde{\mathbf{a}}$$

$$(3.29) \qquad = \epsilon_o + \mathbf{e}^T \tilde{\mathbf{a}}$$

where $\epsilon_o = 1 + \mathbf{d}^T \mathbf{b}_0$ and $\mathbf{e}^T = \mathbf{d}^T \mathbf{C} + \mathbf{c}^T$.

Note that the first entry of $\mathbf{e}$ is zero. Therefore if we replace the first entry of $\mathbf{e}$ by ϵ_o, we get a new column vector $\tilde{\mathbf{e}}$ such that

$$(3.30) \qquad \tilde{\mathbf{e}}^T = \begin{bmatrix} \epsilon_o & \Big| & \mathbf{e}_{k+1}^T \end{bmatrix} \quad \text{for } k = 1, 2, \ldots, m - 1.$$

As a result, this makes us have a compact form of ϵ

$$(3.31) \qquad \epsilon = \tilde{\mathbf{e}}^T \tilde{\mathbf{a}}.$$

Since $\mathbf{e}^T \tilde{\mathbf{a}} = 0$ in the AM method, it is easy to see that the local truncation error of the AM method is as follows,

$$(3.32) \qquad \epsilon = \epsilon_o \text{ and } \tau_{i+1}(h) = \epsilon_o D_{i,m+2} h^{m+1}.$$

3.4. Numerical values of coefficient matrix $\tilde{\mathbf{C}}$ and the error vector $\tilde{\mathbf{e}}$.

In this subsection, we provide numerical values of the coefficient matrix $\tilde{\mathbf{C}}$ and the error vector $\tilde{\mathbf{e}}$ of the GAM method through the C program for the numerical computation. As we can see in Subsections 3.2 and 3.3, the first column of $\tilde{\mathbf{C}}$ and the first entry of $\tilde{\mathbf{e}}$ represent the AM method.

The followings are numerical values of the coefficient matrix $\tilde{\mathbf{C}}$ and the error vector $\tilde{\mathbf{e}}$ of the GAM method. Because the error has been accumulated during the numerical computation, all numerical values of $\tilde{\mathbf{e}}$ below, except the first entry, are not non-negative always. But 2, 4 and 6-step methods do have non-negative values except the first entry.

2-step method:

$$(3.33) \qquad \tilde{\mathbf{C}} = \frac{1}{12} \begin{bmatrix} 5 & -1 \\ 8 & 8 \\ -1 & 5 \end{bmatrix} \text{ and } \tilde{\mathbf{e}}^T / 4! = \frac{1}{24} \begin{bmatrix} -1 & 1 \end{bmatrix}.$$

3-step method:

$$(3.34) \quad \tilde{\mathbf{C}} = \frac{1}{24} \begin{bmatrix} 9 & -1 & 0 \\ 19 & 13 & 8 \\ -5 & 13 & 32 \\ 1 & -1 & 8 \end{bmatrix} \text{ and } \tilde{\mathbf{e}}^T/5! = \frac{1}{720} \begin{bmatrix} -19 & 11 & -8 \end{bmatrix}.$$

4-step method:

$$\tilde{\mathbf{C}} = \frac{1}{720} \begin{bmatrix} 251 & -19 & -8 & -27 \\ 646 & 346 & 272 & 378 \\ -264 & 456 & 912 & 648 \\ 106 & -74 & 272 & 918 \\ -19 & 11 & -8 & 243 \end{bmatrix},$$

(3.35)

$$\text{and } \tilde{\mathbf{e}}^T/6! = \frac{1}{1440} \begin{bmatrix} -27 & 11 & 0 & 27 \end{bmatrix}.$$

5-step method:

$$\tilde{\mathbf{C}} = \frac{1}{1440} \begin{bmatrix} 475 & -27 & -16 & -27 & 0 \\ 1427 & 637 & 544 & 621 & 448 \\ -798 & 1022 & 1824 & 1566 & 2048 \\ 482 & -258 & 544 & 1566 & 768 \\ -173 & 77 & -16 & 621 & 2048 \\ 27 & -11 & 0 & -27 & 448 \end{bmatrix},$$

(3.36)

$$\text{and } \tilde{\mathbf{e}}^T/7! = \frac{1}{60480} \begin{bmatrix} -863 & 271 & 80 & 351 & -512 \end{bmatrix}.$$

6-step method:

(3.37)

$$\tilde{\mathbf{C}} = \frac{1}{60480} \begin{bmatrix} 19087 & -863 & -592 & -783 & -512 & -1375 \\ 65112 & 25128 & 22368 & 23976 & 21888 & 28200 \\ -46461 & 46989 & 77808 & 71037 & 78336 & 58125 \\ 37504 & -16256 & 21248 & 58752 & 42496 & 80000 \\ -20211 & 7299 & 528 & 31347 & 78336 & 31875 \\ 6312 & -2088 & -480 & -3240 & 21888 & 87000 \\ -863 & 271 & 80 & 351 & -512 & 18575 \end{bmatrix},$$

$$\tilde{\mathbf{e}}^T/8! = \frac{1}{120960} \begin{bmatrix} -1375 & 351 & 160 & 351 & 110 & 1375 \end{bmatrix}.$$

4. Numerical result

Since the two body problem of the Earth's satellite has an exact solution, we choose it as a numerical example for the error quantification of the GAM method. Also, this numerical integration is a Geoscience Laser Altimeter System type low-altitude satellite [2]. We consider the case that our satellite problem has a low altitude about 800 km.

The equation of a satellite orbit prediction problem is given by

$$(4.1) \qquad \begin{bmatrix} \dot{\mathbf{r}} \\ \dot{\mathbf{v}} \end{bmatrix} = \begin{bmatrix} \mathbf{v} \\ -(\mu/r^3)\mathbf{r} \end{bmatrix}$$

where $\mathbf{r}$ and $\mathbf{v}$ are position and velocity vectors, respectively; μ is a gravitational constant and r is the magnitude of $\mathbf{r}$ with the initial condition

$$(4.2) \qquad \mathbf{r}_0 = \begin{bmatrix} 7082414.740 \\ 3.957 \\ -56.618 \end{bmatrix} \text{ and } \mathbf{v}_0 = \begin{bmatrix} -9.567 \\ -1039.545 \\ 7485.424 \end{bmatrix}.$$

In practice, low-step methods give relatively low accuracy, we apply the 6-step method to this example.

As shown in (3.37), all entries of $\tilde{\mathbf{e}}$ are non-negative except for the first element. Therefore, the error of the GAM method should be less than or equal to the error of the AM method in each case of

$$(4.3) \quad \tilde{\mathbf{a}}_1 = \begin{bmatrix} 1 \\ a_1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \ \tilde{\mathbf{a}}_2 = \begin{bmatrix} 1 \\ 0 \\ a_2 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \ \tilde{\mathbf{a}}_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ a_3 \\ 0 \\ 0 \end{bmatrix}, \ \tilde{\mathbf{a}}_4 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ a_4 \\ 0 \end{bmatrix}, \ \tilde{\mathbf{a}}_5 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ a_5 \end{bmatrix}$$

for $a_k \geq 0$; $k = 1, 2, \ldots, 5$. Figure 1 shows the results of these when the a_k is increased by 0.1 from 0 to 1. The error is represented by the positional root mean squares (rms). As the a_k approaches to 1, it becomes unstable because of the Criterion 2. Since the unstable method is not necessary, unstable cases are omitted in the figure.

In fact, $\tilde{\mathbf{a}}_5$ reduces the error significantly. Since the error has been accumulated by its nature, this contradictable behavior can be explained in a way that the approximate solution y_{i-5} contains less error than $y_{i-4}, y_{i-3}, \ldots, y_i$. For the same reason, it is shown that $\tilde{\mathbf{a}}_5$ reduces the error much less than $\tilde{\mathbf{a}}_1$, $\tilde{\mathbf{a}}_2$, $\tilde{\mathbf{a}}_3$ and $\tilde{\mathbf{a}}_4$.
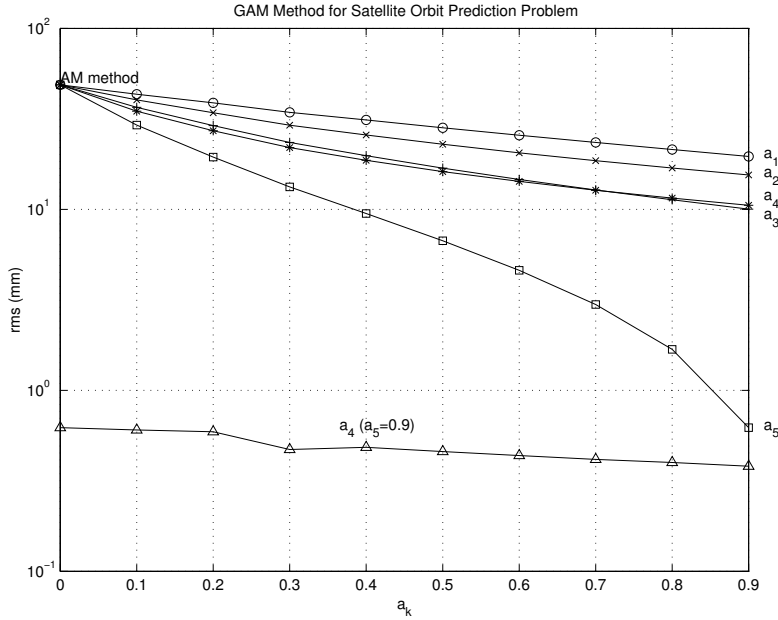
Figure 1. GAM Method for Satellite Orbit Prediction Problem

Among all cases of (4.3), the minimum error occurs when $a_5 = 0.9$ in $\tilde{\mathbf{a}}_5$. Moreover, we tried the following cases to find the better approximation: $\tilde{\mathbf{a}}_{1,5}^T = \begin{bmatrix} 1 & a_1 & 0 & 0 & 0 & 0.9 \end{bmatrix}$, $\tilde{\mathbf{a}}_{2,5}^T = \begin{bmatrix} 1 & 0 & a_2 & 0 & 0 & 0.9 \end{bmatrix}$, $\tilde{\mathbf{a}}_{3,5}^T = \begin{bmatrix} 1 & 0 & 0 & a_3 & 0 & 0.9 \end{bmatrix}$ and $\tilde{\mathbf{a}}_{4,5}^T = \begin{bmatrix} 1 & 0 & 0 & 0 & a_4 & 0.9 \end{bmatrix}$, and found that $\tilde{\mathbf{a}}_{i,5}$ for $i = 1, 2, 3$ are very unstable while $\tilde{\mathbf{a}}_{4,5}$ provides a smaller error as shown as the lowest curve in the figure. Consequently, a better empirical values for $\tilde{\mathbf{a}}$ is

Better 6-step GAM method (empirical): $\tilde{\mathbf{a}}^T = \begin{bmatrix} 1 & 0 & 0 & 0 & 0.9 & 0.9 \end{bmatrix}$.

Acknowledgements

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