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APPROXIMATE BEST PROXIMITY PAIR RESULTS ON METRIC SPACES USING CONTRACTION OPERATORS

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Abstract. The aim of this paper is to prove some new approximate best proximity pair theorems on metric spaces using contraction mappings such as P-Bianchini contraction, $P-B$ contraction and so on. A few examples are provided to exemplify our findings. Finally, we discuss some applications that are related to the main results.

1. Introduction

Best proximity point theory and fixed point theory are now crucial in many mathematics-related fields and its applications, notably in economics, astronomy, dynamical systems, decision theory, and parameter estimation. In 1922 [2], Banach proposed the Banach fixed point results. After that, various authors extended these principle and gave many results using contractive mappings on metric spaces (refer, [8], [9], [15], [19], [20], [21], [30] & [37]). In addition to that, many researchers found new approximate fixed point theorems on metric spaces that do not require completeness in both contractive and rational contractive operators (refer, [4], [5], [6], [10], [11], [18], [27] & [32]). On the other hand, the best proximity point theory also has the same importance as fixed point theory. In the absence of exact proximity points, approximate best proximity points may be used because the best proximity point results have overly strict limitations. There seem to be numerous problems in applied mathematics that can be handled using the concept of best proximity pair theory. Nonetheless, experience demonstrates that for many instances, an approximate computation is more than acceptable; hence, having the best proximity pair is not always necessary, but having an almost-best proximity pair is essential. Another type of growing challenge that leads to this approximate occurs when the requirements that must be enforced to ensure the presence of the best proximity pairings for the major challenge at hand are much more stringent. In [24], the authors achieved some results on the optimum proximity pairs. In the same way, the authors Antony

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Eldred. A. et al [12], proved many results on proximity pairs. One can also refer to many results, which is also recent, about proximity point of the pairs and their theorems in [1], [3], [13], [14], [16], [17], [23], [29], [31], [33], [34], [35], [36]. Additionally, B-contraction and Bianchini contraction definitions are located in [7] & [25], and using these, we define $P-B$ contraction and P-Bianchini contraction.

This manuscript is laid out as follows: In Section 2, we recall the basics from the previous literature. In Section 3, we present the main results, which include the approximate best proximity pairs in contraction operators such as $P-B$ contraction, P-Bianchini contraction and so on. Also, we discuss diameter of an approximate best proximity point for the pair (W, V) by using various operators based on the results of [26] and [28]. In Section 4, we provide some applications of our main results in the field of applied mathematics. Finally, in Section 5, we reach a conclusion.

2. Preliminaries

In this section, some definitions and lemmas from earlier research are recalled. These are then employed throughout the remainder of the main findings of this manuscript.

Definition 2.1. [26], [28] Let W and V be two nonempty subsets of a metric space M and $B : W\cup V \to W\cup V$ such that $B(W) \subseteq V$ and $B(V) \subseteq W$. Then w is said to be an approximate best proximity point of the pair (W, V), if $d_b(w, Bw) \leq d_b(W, V) + \epsilon$.

Remark 2.2. [26], [28] Let $P_{B\epsilon}(W, V) = \{w \in (W, V) : d_b(w, Bw) < d_b(W, V) + \epsilon, \text{for some}, \epsilon > 0\}$ be denotes the set of all approximate best proximity pairs of pair (W, V) for a given $\epsilon > 0$. Also the pair (W, V) is said to be an approximate best proximity pair property if $d_b(w, Bw) \leq d_b(W, V) \neq 0$.

Example 2.3. Let us take $M = \mathbb{R}^2$ and $W = \{(w, v) \in M : (w - v)^2 + v^2 \leq 1\}$ and $V = \{(-w, v) \in M : (w + v)^2 + v^2 \leq 1\}$ with $B(w, v) = (-w, v)$ for $(w, v) \in M$. Then $d_b(w, v), B(w, v) \leq d_b(W, V) + \epsilon$, for some $\epsilon > 0$. Hence $P_{B\epsilon}(W, V) \neq \emptyset$.

Theorem 2.4. [26], [28] *Let W and V be two nonempty subsets of a metric space M. Suppose that the mapping $B : W \cup V \to W \cup V$ satisfying $B(W) \subseteq V$ and $B(V) \subseteq W$ and $\lim_{n\to\infty} d_b(B^n w, B^{n+1}w) = d_b(W, V)$, for some $w \in (W \cup V)$. Then the pair (W, V) is called an approximate best proximity pair.*

Definition 2.5. [26], [28] Let $B : W \cup V \to W \cup V$ be a continuous map such that $B(W) \subseteq V, B(V) \subseteq W$ and $\epsilon > 0$. We define the diameter $Dtr(P_{B\epsilon}(W, V))$, i.e., $Dtr(P_{B\epsilon}(W, V)) = sup\{d_b(w, v) : w, v \in P_{B\epsilon}(W, V)\}$.

Theorem 2.6. [26], [28] *Let W and V be two non-empty subsets of a metric space M. Suppose that a mapping $B : W \cup V \to W \cup V$ satisfying $B(W) \subseteq V, B(V) \subseteq W$ is a $P - \alpha$ contraction and $\epsilon > 0$. Suppose that:*

(i) $P_{B\epsilon}(W, V) \neq \emptyset$;
(ii) for every $\varphi > 0, \exists\psi(\varphi) > 0$ such that
$d_b(w, v) - d_b(Bw, Bv) \leq \varphi \Rightarrow d_b(w, v) \leq \psi(\varphi)$, *for evry $w, v \in P_{B\epsilon}(W, V) \neq \emptyset$.*

Then, $Dtr(P_{B\epsilon}(W, V)) \leq \psi(2d_b(W, V) + \epsilon)$.

Definition 2.7. A mapping $B : W \cup V \to W \cup V$ satisfying $B(W) \subseteq V$ and $B(V) \subseteq W$ is a $P - B$ contraction operator if there exists $b_1, b_2, b_3 \in (0,1)$ with $b_1 + b_2 + b_3 < 1$ such that

$$\begin{aligned} d_b(Bw, Bv) &\leq b_1[d_b(w, Bw) + d_b(v, Bv)] + b_2[d_b(w, v)] \\ &\quad + b_3[d_b(w, Bv) + d_b(v, Bw)], \text{ for all } w, v \in W \cup V. \end{aligned} \tag{1}$$

Definition 2.8. A mapping $B : W \cup V \to W \cup V$ satisfying $B(W) \subseteq V$ and $B(V) \subseteq W$ is a P-Bianchini contraction operator if there exists $b_1 \in (0,1)$ such that

$$\begin{aligned} &d_b(Bw, Bv) \leq b_1 B_{ia}(w, v), \\ &\text{where } B_{ia}(w, v) = max\{d_b(w, Bw), d_b(v, Bv)\}, \text{ for all } w, v \in W \cup V. \end{aligned} \tag{2}$$

Definition 2.9. A mapping $B : W \cup V \to W \cup V$ satisfying $B(W) \subseteq V$ and $B(V) \subseteq W$ is a P- Hardy and Rogers contraction operator if there exists $b_1, b_2, b_3, b_4, b_5 \in (0,1)$ with $b_1 + b_2 + b_3 + b_4 + b_5 < 1$ such that

$$\begin{aligned} d_b(Bw, Bv) &\leq b_1 d_b(w, v) + b_2 d_b(w, Bw) + b_3 d_b(v, Bv) \\ &\quad + b_4 d_b(w, Bv) + b_5 d_b(v, Bw), \text{ for all } w, v \in W \cup V. \end{aligned} \tag{3}$$

3. Main Results

This section is divided into two parts. The first one deals with qualitative results, and the other one deals with quantitative results; both are related to the approximate best proximity points for the pairs (V, W) on metric spaces.

3.1. Qualitative theorems for P-contraction operators: In this subsection, we prove some qualitative theorems about the approximate best proximity point for the pair (V, W) by using contraction operators such as the $P - B$ contraction operator, the P-Bianchini contraction operator, and the P-Hardy Rogers contraction operator on a metric space.

Theorem 3.1. *Let W and V be two non-empty subsets of a metric space M. Suppose that a mapping $B : W \cup V \to W \cup V$ satisfying $B(W) \subseteq V$ and $B(V) \subseteq W$ is a $P - B$ contraction operator then for every $\epsilon > 0, P_{B\epsilon}(W, V) \neq \emptyset$.*

Proof. Let $\epsilon > 0$ and $w \in W \cup V$. Consider,

$$\begin{aligned} d_b(B^n w, B^{n+1} w) &= d_b(B(B^{n-1} w), B(B^n w)) \\ &\leq b_1[d_b(B^{n-1} w, B^n w) + d_b(B^n w, B^{n+1} w)] + b_2[d_b(B^{n-1} w, B^n w)] \\ &\quad + b_3[d_b(B^{n-1} w, B^{n+1} w) + d_b(B^n w, B^n w)] [\text{ By equation (1)}] \\ &\leq b_1 d_b(B^{n-1} w, B^n w) + b_1 d_b(B^n w, B^{n+1} w) + b_2 d_b(B^{n-1} w, B^n w) \\ &\quad + b_3 d_b(B^{n-1} w, B^n w) + b_3 d_b(B^n w, B^{n+1} w) \\ &= \left(\frac{b_1 + b_2 + b_3}{1 - b_2 - b_3}\right) d_b(B^{n-1} w, B^n w) \\ &= \lambda d_b(B^{n-1} w, B^n w), \text{ where } \lambda = \frac{b_1 + b_2 + b_3}{1 - b_2 - b_3}. \end{aligned}$$

But b_1, b_2 and $b_3 \in (0,1)$ implies that $\lambda \in (0,1)$. Therefore,

$$\lim_{n\to\infty} d_b(B^n w, B^{n+1} w) = 0, \text{ for all } w \in W \cup V.$$

Hence, by Theorem 2.4, it follows that

$$P_{B\epsilon}(W,V) \neq \emptyset, \text{ for all } \epsilon > 0.$$

□

THEOREM 3.2. *Let W and V be two non-empty subsets of a metric space M. Suppose that a mapping $B : W \cup V \to W \cup V$ satisfying $B(W) \subseteq V$ and $B(V) \subseteq W$ is a P-Bianchini contraction operator then for every $\epsilon > 0, P_{B\epsilon}(W,V) \neq \emptyset$.*

Proof. Let $\epsilon > 0$ and $w \in W \cup V$. Consider,

CASE 1. Suppose $B_{ia}(w,v) = d_b(w, Bw)$. Then the Definition 2.8 becomes

$$d_b(Bw, Bv) \leq b_1 d_b(w, Bw)$$

Substitute $v = Bw$ we get,

$$d_b(Bw, B^2 w) \leq b_1 d_b(w, Bw)$$

Again substituting $w = Bw$ implies,

$$\begin{aligned} d_b(B^2 w, B^3 w) &\leq b_1 d_b(Bw, B^2 w) \\ &\leq (b_1)^2 d_b(w, Bw) \end{aligned}$$

Continuing this process we have,

$$d_b(B^n w, B^{n+1} w) \leq (b_1)^n d_b(w, Bw)$$

CASE 2. Suppose $B_{ia}(w,v) = d_b(v, Bv)$. Then the Definition 2.8 becomes

$$d_b(Bw, Bv) \leq b_1 d_b(v, Bv)$$

Substitute $v = Bw$ we get,

$$d_b(Bw, B^2 w) \leq b_1 d_b(w, B^2 w)$$

This is impossible because $b_1 \in (0,1)$. Therefore, CASE 2 does not exists. Now using CASE 1 and Theorem 2.4, we have

$$\lim_{n\to\infty} d_b(B^n w, B^{n+1} w) = 0, \text{ for all } w \in W \cup V.$$

And it follows that

$$P_{B\epsilon}(W,V) \neq \emptyset, \text{ for all } \epsilon > 0.$$

□

COROLLARY 3.3. *Let W and V be two non-empty subsets of a metric space M. Suppose that a mapping $B : W \cup V \to W \cup V$ satisfying $B(W) \subseteq V$ and $B(V) \subseteq W$ and defined by $d_b(Bw, Bv) \leq b_1 d_b(w, Bw)$ operator then for every $\epsilon > 0, P_{B\epsilon}(W,V) \neq \emptyset$.*

Proof. It is a direct consequence of Theorem 3.2. □

THEOREM 3.4. *Let W and V be two non-empty subsets of a metric space M. Suppose that a mapping $B : W \cup V \to W \cup V$ satisfying $B(W) \subseteq V$ and $B(V) \subseteq W$ is a P-Hardy Rogers operator then for every $\epsilon > 0, P_{B\epsilon}(W,V) \neq \emptyset$.*

Proof. Let $\epsilon > 0$ and $w \in W \cup V$. Consider,

$$\begin{aligned} d_b(B^n w, B^{n+1} w) &= d_b(B(B^{n-1}w), B(B^n w)) \\ &\leq b_1 d_b(B^{n-1}w, B^n w) + b_2 d_b(B^{n-1}w, B^n w) + b_3 d_b(B^n w, B^{n+1} w) \\ &\quad + b_4 d_b(B^{n-1}w, B^{n+1}w) + b_5 d_b(B^n w, B^n w) [\text{ By equation (3)}] \\ &\leq b_1 d_b(B^{n-1}w, B^n w) + b_2 d_b(B^{n-1}w, B^n w) + b_3 d_b(B^n w, B^{n+1} w) \\ &\quad + b_4 d_b(B^{n-1}w, B^n w) + b_4 d_b(B^n w, B^{n+1} w) \\ &= \left(\frac{b_1 + b_2 + b_4}{1 - b_3 - b_4}\right) d_b(B^{n-1}w, B^n w) \\ &= \lambda d_b(B^{n-1}w, B^n w), \text{ where } \lambda = \frac{b_1 + b_2 + b_4}{1 - b_3 - b_4}. \end{aligned}$$

But b_1, b_2, b_3, b_4 and $b_5 \in (0,1)$ implies that $\lambda \in (0,1)$. Therefore,

$$\lim_{n\to\infty} d_b(B^n w, B^{n+1} w) = 0, \text{ for all } w \in W \cup V.$$

Hence, by Theorem 2.4, it follows that

$$P_{B\epsilon}(W,V) \neq \emptyset, \text{ for all } \epsilon > 0.$$

□

Remark 3.5. 1. In Definition 2.7, substitute $b_2 = \alpha$ and $b_1 = b_3 = 0$, then it becomes $P - \alpha$ contraction operator and for every $\epsilon > 0, P_{B\epsilon}(W,V) \neq \emptyset$.
2. In Definition 2.7, substitute $b_2 = b_3 = 0$, then it becomes P-Kannan operator and for every $\epsilon > 0, P_{B\epsilon}(W,V) \neq \emptyset$.
3. In Definition 2.7, substitute $b_1 = b_2 = 0$, then it becomes P-Chatterjea operator and for every $\epsilon > 0, P_{B\epsilon}(W,V) \neq \emptyset$.
4. In Definition 2.8, substitute $b_4 = b_5 = 0$, then it becomes P-Reich operator and for every $\epsilon > 0, P_{B\epsilon}(W,V) \neq \emptyset$.
5. In Definition 2.8, substitute $b_4 = b_5$, then it becomes P-Ciric operator and for every $\epsilon > 0, P_{B\epsilon}(W,V) \neq \emptyset$.

3.2. Quantitative results for P-contraction operators: In this subsection, we prove some quantitative results of approximate best proximity point of the pairs(V,W) by using contraction operators such as the $P-B$ contraction operator, the P-Bianchini contraction operator, and the P-Hardy Rogers contraction operator on a metric space.

Theorem 3.6. *Let (M, d_b) be a metric space and $B : W \cup V \to W \cup V$ satisfies the conditions of Theorem 3.1. Then,*

$$Dtr(P_{B\epsilon}(W,V)) \leq \frac{2(b_1 + b_3) d_b(W,V) + 2\epsilon(b_1 + b_3 + 1)}{1 - b_2 - 2b_3}, \text{ for all } \epsilon > 0.$$

Proof. Let $\epsilon > 0$. Also, condition(i) of Theorem 2.6 is proved by using Theorem 3.1. To show, condition (ii) of Theorem 2.6 holds. For that, take $\varphi > 0$ and $w, v \in P_{B\epsilon}(W,V)$. Also $d_b(w,v) - d_b(Bw,Bv) \leq \varphi$. Then $d_b(w,v) \leq d_b(Bw,Bv) + \varphi$. since $w, v \in P_{B\epsilon}(W,V)$ implies that $d_b(w,Bw) \leq d_b(W,V) + \epsilon_1$ and $d_b(v,Bv) \leq$

$d_b(W,V)+\epsilon_2$. And choose $\epsilon=\{\epsilon_1,\epsilon_2\}$. Now,

$$\begin{aligned} d_b(w,v) &\le d_b(Bw,Bv)+\varphi \\ &\le b_1[d_b(W,V)+\epsilon+d_b(W,V)+\epsilon]+b_2[d_b(w,v)] \\ &\quad + b_3[d_b(w,v)+d_b(v,Bv)+d_b(v,w)+d_b(w,Bw)]+\varphi \\ &= b_1[2d_b(W,V)+2\epsilon]+b_2 d_b(w,v)+b_3[2d_b(w,v)+2d_b(W,V)+2\epsilon]+\varphi \\ &= \frac{2(b_1+b_3)d_b(W,V)+2\epsilon(b_1+b_3)+\varphi}{1-b_2-2b_3} \\ &= \psi(\varphi) \end{aligned}$$

Thus, for every $\varphi>0$, there exists $\psi(\varphi)>0$ such that $d_b(w,v)-d_b(Bw,Bv)\le\varphi$ implies $d_b(w,v)=\psi(\varphi)$. Then by Theorem 2.6 ,

$$Dtr(P_{B\epsilon}(W,V))\le\psi(2\epsilon),\ \text{for all } \epsilon>0.$$

This means exactly

$$Dtr(P_{B\epsilon}(W,V))\le\frac{2(b_1+b_3)d_b(W,V)+2\epsilon(b_1+b_3+1)}{1-b_2-2b_3},\ \text{for all } \epsilon>0.$$

□

THEOREM 3.7. *Let (M,d_b) be a metric space and $B:W\cup V\to W\cup V$ satisfies the conditions of Theorem 3.2. Then,*

$$Dtr(P_{B\epsilon}(W,V))\le b_1 d_b(W,V)+\epsilon(b_1+2),\ \textit{for all } \epsilon>0.$$

Proof. Let $\epsilon>0$. Also, condition (i) of Theorem 2.6 is proved by using Theorem 3.2. To show, condition (ii) of Theorem 2.6 holds. For that, take $\varphi>0$ and $w,v\in P_{B\epsilon}(W,V)$. Also $d_b(w,v)-d_b(Bw,Bv)\le\varphi$. Then $d_b(w,v)\le d_b(Bw,Bv)+\varphi$. since $w,v\in P_{B\epsilon}(W,V)$ implies that $d_b(w,Bw)\le d_b(W,V)+\epsilon_1$ and $d_b(v,Bv)\le d_b(W,V)+\epsilon_2$. And choose $\epsilon=\{\epsilon_1,\epsilon_2\}$. Now,

$$\begin{aligned} d_b(w,v) &\le d_b(Bw,Bv)+\varphi \\ &\le b_1 d_b(w,Bw)+\varphi \\ &\le b_1(d_b(W,V)+\epsilon)+\varphi \\ &= \psi(\varphi) \end{aligned}$$

Thus, for every $\varphi>0$, there exists $\psi(\varphi)>0$ such that $d_b(w,v)-d_b(Bw,Bv)\le\varphi$ implies $d_b(w,v)=\psi(\varphi)$. Then the Theorem 2.6 gives,

$$Dtr(P_{B\epsilon}(W,V))\le\psi(2\epsilon),\ \text{for all } \epsilon>0.$$

This means exactly

$$Dtr(P_{B\epsilon}(W,V))\le b_1 d_b(W,V)+\epsilon(b_1+2),\ \text{for all } \epsilon>0.$$

□

THEOREM 3.8. *Let (M,d_b) be a metric space and $B:W\cup V\to W\cup V$ satisfies the conditions of Theorem 3.4. Then,*

$$Dtr(P_{B\epsilon}(W,V))\le\frac{(1-b_1)d_b(W,V)+\epsilon(3-b_1)}{b_2+b_3},\ \textit{for all } \epsilon>0.$$

Proof. $\epsilon > 0$. Also, condition (i) of Theorem 2.6 is proved by using Theorem 3.4. To show, condition (ii) of Theorem 2.6 holds. For that, take $\varphi > 0$ and $w, v \in P_{B\epsilon}(W, V)$. Also $d_b(w, v) - d_b(Bw, Bv) \leq \varphi$. Then $d_b(w, v) \leq d_b(Bw, Bv) + \varphi$. since $w, v \in P_{B\epsilon}(W, V)$ implies that $d_b(w, Bw) \leq d_b(W, V) + \epsilon_1$ and $d_b(v, Bv) \leq d_b(W, V) + \epsilon_2$. And choose $\epsilon = \{\epsilon_1, \epsilon_2\}$. Now,

$$\begin{aligned}
d_b(w, v) &\leq d_b(Bw, Bv) + \varphi \\
&\leq b_1 d_b(w, v) + b_2[d_b(W, V) + \epsilon] + b_3[d_b(W, V) + \epsilon] + b_4 d_b(w, v) \\
&\quad + b_4[d_b(W, V) + \epsilon] + b_5 d_b(w, v) + b_5[d_b(W, V) + \epsilon] + \varphi \\
&= (b_1 + b_4 + b_5) d_b(w, v) + (b_2 + b_3 + b_4 + b_5)[d_b(W, V) + \epsilon] + \varphi \\
&= \frac{(b_2 + b_3 + b_4 + b_5)[d_b(W, V) + \epsilon] + \varphi}{1 - (b_1 + b_4 + b_5)} \\
&= \psi(\varphi)
\end{aligned}$$

Thus, for every $\varphi > 0$, there exists $\psi(\varphi) > 0$ such that $d_b(w, v) - d_b(Bw, Bv) \leq \varphi$ implies $d_b(w, v) = \psi(\varphi)$. Then, the Theorem 2.6 gives

$$Dtr(P_{B\epsilon}(W, V)) \leq \psi(2\epsilon), \text{ for all } \epsilon > 0.$$

That is,

$$Dtr(P_{B\epsilon}(W, V)) \leq \frac{(b_2 + b_3 + b_4 + b_5) d_b(W, V) + \epsilon(b_2 + b_3 + b_4 + b_5 + 2)}{1 - (b_1 + b + 4 + b_5)}, \text{ for all } \epsilon > 0.$$

This means exactly

$$Dtr(P_{B\epsilon}(W, V)) \leq \frac{(1 - b_1) d_b(W, V) + \epsilon(3 - b_1)}{b_2 + b_3}, \text{ for all } \epsilon > 0.$$

□

COROLLARY 3.9. *Let (M, d_b) be a metric space and $B : W \cup V \to W \cup V$ satisfies the conditions of Corollary 3.3. Then,*

$$Dtr(P_{B\epsilon}(W, V)) \leq b_1 d_b(W, V) + \epsilon(b_1 + 2), \text{ for all } \epsilon > 0.$$

Proof. It is a direct consequence of Theorem 3.7. □

REMARK 3.10.

1. In $P - \alpha$ contraction operator,
$$Dtr(P_{B\epsilon}(W, V)) \leq \frac{2(\epsilon + d_b(W, V))}{b_1}, \text{ for all } \epsilon > 0.$$
2. In P-Kannan operator, $Dtr(P_{B\epsilon}(W, V)) \leq 2\epsilon(1+b_1)+2b_1 d_b(W, V)$, for all $\epsilon > 0$.
3. In P-Chatterjea operator, $Dtr(P_{B\epsilon}(W, V)) \leq \dfrac{2[\epsilon(1 + b_1) + b_1 d_b(W, V)]}{1 - 2b_1}$, for all $\epsilon > 0$.
4. In P-Reich operator, $Dtr(P_{B\epsilon}(W, V)) \leq \dfrac{(1 - b_1)(d_b(W, V)) + (3 - b_1)\epsilon}{1 - b_1}$, for all $\epsilon > 0$.
5. In P-Ciric operator, $Dtr(P_{B\epsilon}(W, V)) \leq \dfrac{(1 - b_1) d_b(W, V)) + (3 - b_1)\epsilon}{b_2 + b_3}$, for all $\epsilon > 0$.

4. Applications

Approximate best proximity point theory covers a wide range of applications in mathematics, particularly differential geometry, numerical analysis, and so on. By reading [22] and the references there in, one can find a variety of applications involving approximate best proximity point results in the field of mathematics. The two examples below demonstrate how to apply approximate best proximity point findings in differential equations.

Example 4.1. Consider $z''(w) = 6z^2(w), 0 \leq w \leq 1$ subect to $z(0) = \frac{1}{4}, z(1) = \frac{1}{9}$. Exact solution is $z_0(w) = \frac{-5w}{36} + \frac{1}{4}$. Consider a mapping $T : [0,1] \to [0,1]$ defined by

$$\begin{aligned} T(z) &= z + \int_0^1 G(w,v)[z''(v) - \phi(v, z(v), z'(v))]dv \\ &= \frac{-5w}{36} + \frac{1}{4} - \int_0^1 G(w,v)\phi(v, z(v), z'(v))dv \\ &= \frac{-5w}{36} + \frac{1}{4} - \int_0^1 G(w,v)6z''(v)dv \end{aligned}$$

Consider,

$$\begin{aligned} |T(z_1) - T(z_2)| &= 6\left|-\int_0^1 G(w,v)z_1^2(v)dv + \int_0^1 G(w,v)z_2^2(v)dv\right| \\ &= 6\left(\int_0^1 |G(w,v)|^2 dv\right)^{\frac{1}{2}} \left(\int_0^1 |z_2^2(v) - z_1^2(v)|^2 dv\right)^{\frac{1}{2}} \\ &\leq \frac{1}{4\sqrt{3}}\left(\int_0^1 |z_2^2(v) - z_1^2(v)|^2 dv\right)^{\frac{1}{2}} \\ &< \sup_{[0,1]} |z_1(v) - z_2(v)| \end{aligned}$$

Hence, T is a contraction and it has approximate best proximity point.

Example 4.2. Consider $z''(w) = \frac{3v^2(w)}{2}, 0 \leq v \leq 1$ subect to $z(0) = 4, z(1) = 1$. Exact solution is $z(w) = \frac{4}{(1+w)^2}$. Consider a mapping $W : [0,1] \to [0,1]$ by

$$W(z) = w_2 + \int_0^1 G(v,w)[z''(w) - \phi(w, z(w))]dw \tag{4}$$

Consider, $z''(v) = 0$ which implies

$$z(v) = c_1 v + c_2 \tag{5}$$

By initial condition we have $c_2 = 4$ and $c_1 = -3$. Then (5) becomes $z(v) = -3w_1 + 4$.

$$\begin{aligned} W(z) &= -3v + 4 + \int_0^1 G(v,w)[z''(w) - \phi(w, z(w))]dw \\ &= -3v + 4 + \int_0^1 G(v,w)z''(w)dw - \int_0^1 G(v,w)\phi(w, z(w))dw \\ &= -3v + 4 + \int_0^1 G(v,w)\frac{3}{2}z^2(w)dw \end{aligned}$$

Consider,

$$
\begin{aligned}
|W(z_1) - W(z_2)| &= \left| -\int_0^1 G(v,w)\frac{3}{2}z_2^2(w)dw + \int_0^1 G(v,w)\frac{3}{2}z_2^2(w)dw \right| \\
&= \frac{3}{2}\left| \int_0^1 G(v,w)[z_2^2(w) - z_1^2(w)]dw \right| \\
&\leq \frac{3}{2}\left(\int_0^1 |G(v,w)|^2 dw \right)^{\frac{1}{2}} \left[\int_0^1 |z_2^2(w) - z_1^2(w)|^2 dw \right]^{\frac{1}{2}} \\
&\leq \frac{3}{2}\left(\int_0^w w^2(1-v)^2 dv + \int_v^1 v^2(1-w)^2 dw \right)^{\frac{1}{2}} \left[\int_0^1 |z_2^2(w) - z_1^2(w)|^2 dw \right]^{\frac{1}{2}} \\
&\leq \frac{3}{2}\left\{ \frac{(1-v)^2 v^3}{3} + \frac{v^2(1-v)^3}{3} \right\}^{\frac{1}{2}} \left[\int_0^1 |z_2^2(w) - z_1^2(w)|^2 dw \right]^{\frac{1}{2}} \\
&\leq \frac{3}{2}\left\{ \frac{(1-v)^2}{3}[v^3 + v^2(1-v)] \right\}^{\frac{1}{2}} \left[\int_0^1 |z_2^2(w) - z_1^2(w)|^2 dw \right]^{\frac{1}{2}} \\
&\leq \frac{3}{2}\left\{ \frac{(1-v)^2 v^2}{3} \right\}^{\frac{1}{2}} \left[\int_0^1 |z_2^2(w) - z_1^2(w)|^2 dw \right]^{\frac{1}{2}} \\
&\leq \frac{3}{8\sqrt{3}} \left[\int_0^1 |z_2^2(w) - z_1^2(w)|^2 dw \right]^{\frac{1}{2}} \\
&\leq \frac{\sqrt{3}}{8} \left[\int_0^1 |z_2^2(w) - z_1^2(w)|^2 dw \right]^{\frac{1}{2}} \\
&\leq \frac{\sqrt{3}}{8} \sup_{[0,1]} |z_2(w) - z_1(w)| \\
&\leq \sup_{[0,1]} |z_2(w) - z_1(w)|
\end{aligned}
$$

Hence, W is contraction, it has approximate best proximity point.

EXAMPLE 4.3. Let us consider a numerical problem $\int_0^\pi sinpdp$. To solve this, by using Simpson's rule, we get $p = 2.0008$. Similarly, to solve this, by using Trapezoidal rule, we get $p = 1.955$. But the actual solution is $p = 2$. Therefore, in both methods, we get only an approximate solution.

5. Conclusion

This work provides a series of contraction mappings to demonstrate several approximate best proximity point theorems on metric spaces. It is essential to note that all of the conclusions made in the current paper generate better constrained approximations of best proximity points, mostly in minimising condition $\epsilon \longrightarrow 0$. In order to confirm the presence of an approximate fixed points, alternative discoveries presented in the later can be demonstrated in a lower environment. Thus, the concept of an approximate best proximity point of the pair (W, V) is just as significant as the concept of best proximity point of the pair (W, V).

Author Contribution Statements

For making this article, all the author's contributed equally.

Competing Interests

All the author's said that they have no competing interests.

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