Korean J. Math. **31** (2023), No. 4, pp. 445–463
https://dx.doi.org/10.11568/kjm.2023.31.4.445

DIRECT PRODUCT, SUBDIRECT PRODUCT, AND REPRESENTABILITY IN AUTOMETRIZED ALGEBRAS

Gebrie Yeshiwas Tilahun *, Radhakrishna Kishore Parimi, and Mulugeta Habte Melesse

Abstract. The paper introduces the concept of direct product and discusses some basic facts about distant ideals. We also introduce the definition of directly indecomposable in an autometrized algebra. Furthermore, we present the concept of a subdirect product and simple autometrized algebra and its behavior. We also introduce the definition of subdirectly irreducible in an autometrized algebras. In particular, we prove that every subdirectly irreducible monoid autometrized algebra is directly indecomposable. Finally, we discuss different properties of chain autometrized algebras and introduce the representability in the autometrized algebra. We also prove that if a weak chain monoid normal autometrized l-algebra is nilradical, then it is representable.

1. Introduction

Swamy in [1] proposed the concept of autometrized algebra to create a comprehensive theory that encompasses the known autometrized algebras at the time: Boolean algebras (Blumenthal [2] and Ellis [3]), Brouwerian algebras (Nordhaus and Lapidus [4]), Newman algebras (Kamala Ranjan [5]), autometrized lattices (Nordhaus and Lapidus [4]) and commutative lattice ordered groups or l-groups (Swamy [6]). The theory of autometrized algebra was further developed by Swamy and Rao [7], Rachunek [8–11], Hansen [12], Kovář [13], and Chajda and Rachŭnek [14].

Furthermore, the notion of representable autometrized algebras was examined by Subba Rao and Yedlapalli in [15], as well as by Subba Rao, Kanakam, and Yedlapalli in [16–18]. The theory of strong ideals and monoid autometrized algebras was developed by Tilahun, Parimi, and Melesse in [19,20], who also explored the relationships among normal autometrized semialgebras, normal autometrized l-algebras, and representable autometrized algebras.

The main objective of this paper is to introduce and examine a direct product, subdirect product, and representability in autometrized algebra. That can be viewed as an extension of the work done by Tilahun, Parimi, and Melesse in [19]. Additionally,

---

Received June 29, 2023. Revised October 17, 2023. Accepted October 18, 2023.
2010 Mathematics Subject Classification: 20M14, 20M32, 06A06, 06B05, 06B15.
Key words and phrases: autometrized algebra, direct product, subdirect product, chain, representable.
∗ Corresponding author.

we investigate and describe the connections between subdirect products, chains, and representability in autometrized algebras.

This paper shall be arranged in the following way. In Section 2, we recall some definitions and terminologies that are essential. Section 3 introduces the concept of direct product and discusses some basic facts about distant ideals in an autometrized algebra. In Section 4, we present the definition of subdirect product and simple autometrized algebras and their behavior. Section 5 discusses the different properties of chain autometrized algebra and introduces the representability in autometrized algebra. Lastly, in Section 6, we will provide a conclusion to the paper.

2. Preliminaries

This section reviews some basic concepts, definitions, and terms.

Definition 2.1 ([1]). A system A= $(A,\ +,\ 0,\ \leq,\ *)$ is called an autometrized algebra if

(i)**:** $(A,\ +,\ 0)$ is a commutative monoid.
(ii)**:** $(A,\ \leq)$ is a partial ordered set, and $\leq$ is translation invariant, that is, $\forall a,\ b,\ c \in A;\ a \leq b \Rightarrow a+c \leq b+c$.
(iii)**:** $* : A \times A \to A$ is autometric on A, that is, $*$ satisfies metric operation axioms:
(M_1)**:** $\forall a,\ b \in A;\ a*b \geq 0$ and, $a*b = 0 \Leftrightarrow a = b$,
(M_2)**:** $\forall a,\ b \in A;\ a*b = b*a$,
(M_3)**:** $\forall a,\ b,\ c \in A;\ a*c \leq a*b + b*c$.

Definition 2.2 ([7]). An autometrized algebra A $= (A,\ +,\ 0,\ \leq,\ *)$ is called normal if and only if

(i)**:** $a \leq a*0\ \forall a \in A$.
(ii)**:** $(a+c)*(b+d) \leq (a*b)+(c*d)\ \forall a,\ b,\ c,\ d \in A$.
(iii)**:** $(a*c)*(b*d) \leq (a*b)+(c*d)\ \forall a,\ b,\ c,\ d \in A$.
(iv)**:** For any a and b in A, $a \leq b \Rightarrow \exists x \geq 0$ such that $a+x = b$.

Definition 2.3 ([7]). Let $A = (A,\ +,\ 0,\ \leq,\ *)$ be a system. Then A is said to be a lattice ordered autometrized algebra (or) autometrized l-algebra if

(i)**:** $(A,\ +,\ 0)$ is a commutative semigroup with 0.
(ii)**:** $(A,\ \leq)$ is a lattice, and $\leq$ is translation invariant, that is, $\forall a,\ b,\ c \in A$;

$$a+(b \vee c) = (a+b) \vee (a+c)$$
$$a+(b \wedge c) = (a+b) \wedge (a+c)$$

(iii)**:** $* : A \times A \to A$ is autometric on A, that is, $*$ satisfies metric operation axioms: M_1, M_2 and M_3.

Definition 2.4 ([7]). Let A be an autometrized algebra. Then A is said to be semiregular if for any $a \in A$, $a \geq 0 \Rightarrow a*0 = a$.

Definition 2.5 ([19]). A nonempty subset I of an autometrized algebra A $= (A,\ +,\ 0,\ \leq,\ *)$ is called an ideal if and only if

(i)**:** $a,\ b \in I$ imply $a+b \in I$.
(ii)**:** $a \in I,\ b \in A$ *and* $b*0 \leq a*0$ imply $b \in I$.

Definition 2.6 ([19]). Let A be an autometrized algebra. Then radical of A is the set $Rad(A) = \bigcap\{M \mid M\ is\ a\ maximal\ ideal\ of\ A\}$.

Definition 2.7 ([19]). Let A be an autometrized algebra. An ideal I of A is called a strong ideal if

(i): $a \in I \Leftrightarrow a * I = I$ and
(ii): $a * I = b * I \Leftrightarrow a * b \in I$ for $a,\ b \in A$.

Definition 2.8 ([19]). Let A $= (A,\ +,\ 0,\ \leq,\ *)$ and B $= (B,\ +,\ 0,\ \leq,\ *)$ be autometrized algebras. Let $f : A \to B$ be a map. Then f is said to be a homomorphism from A to B if and only if

(i): $f(a+b) = f(a) + f(b) \forall a,\ b \in A$,
(ii): $f(a*b) = f(a) * f(b) \forall a,\ b \in A$ and
(iii): $a \leq b \Rightarrow f(a) \leq f(b) \forall a,\ b \in A$.

A homomorphism $f : A \to B$ is called

(i): an epimorphism if and only if f is onto.
(ii): a monomorphism(embedding) if and only if f is one-to-one.
(iii): an isomorphism if and only if f is a bijection.

Definition 2.9 ([19]). Let A and B be autometrized algebras. Let $f : A \to B$ be a map. If $a \leq b \Leftrightarrow f(a) \leq f(b) \forall a,\ b \in A$, then f is said to be an order-embedding of A into B. That is; f is both order-preserving and order-reversing.

Definition 2.10 ([19]). Let A $= (A,\ +,\ 0,\ \leq,\ *)$ and B $= (B,\ +,\ 0,\ \leq,\ *)$ be autometrized algebras. Let $f : A \to B$ be a homomorphism. Then $\ker f = \{x \in A \mid f(x) = \overline{0}\}$ where $\overline{0}$ is the zero element of B.

Clearly, f is one-to-one if and only if $\ker f = \{0\}$.

Theorem 2.11 ([19]). *Let $A,\ B$ be autometrized l-algebras. Let $f : A \to B$ be an epimorphism and order-reversing. Let I be a prime ideal of A. Then, $L = f(I) = \{f(a) \in B \mid a \in I\}$ is a prime ideal of B.*

Definition 2.12 ([19]). An autometrized algebra $(A,\ +,\ 0,\ \leq,\ *)$ is called monoid if and only if

(i): $a * (b * c) = (a * b) * c\ \forall a,\ b,\ c \in A$.[Associative]
(ii): $a * 0 = a\ \forall a \in A$.[Identity]

Then we say that A is a monoid autometrized algebra.

Theorem 2.13 ([19]). *Let A be a monoid autometrized algebra. Then every ideal of A is strong.*

Theorem 2.14 ([19]). *Let A be an autometrized algebra. Let M is an ideal of A. Let $A/M = \{a * M | a \in A\}$. For any $a * M,\ b * M \in A/M$, define the operations:*

$$(a*M) + (b*M) = (a+b)*M.$$
$$(a*M)*(b*M) = (a*b)*M.$$
$$a*M \leq b*M \Leftrightarrow a \leq b.$$

Then $(A/M,\ +,\ \leq,\ *)$ is an autometrized algebra is called the quotient algebra of A by ideal M.

Theorem 2.15 ([19]). *Let A be autometrized algebra. Let M be a strong ideal of A. Define a map $\phi : A \to A/M$ by $\phi(a) = a * M$. Then ϕ is an epimorphism and* $\ker \phi = M$.

Theorem 2.16 ([19]). [First Isomorphism Theorem] *Let A be a monoid autometrized algebra. Let B be an autometrized algebra. Let $f : A \to B$ be a homomorphism. Then $A/\ker f \cong Imf$.*

In particular, if f is onto, then $A/\ker f \cong B$.

3. Direct Products and Distant Ideals

This section introduces direct products and distant ideals in an autometrized algebra. We also prove that a monoid autometrized algebra A is directly indecomposable if and only if the only distant ideals on A are $\{0\}$, A. Now, we shall begin with the definition of direct product.

Definition 3.1. Let $\{A_i\}_{i\in I}$ be a family of autometrized algebras. Let $A = \prod_{i\in I} A_i = \{a = (a(1),\ a(2),\ \ldots) \mid a(i) \in A_i\}$. Define for any $a = (a_i)_{i\in I},\ b = (b_i)_{i\in I}$:

$$a + b = (a_i + b_i)_{i\in I}.$$
$$a * b = (a_i * b_i)_{i\in I}.$$
$$a \leq b \Leftrightarrow a_i \leq b_i \forall i \in I.$$

Then $A = \prod_{i\in I} A_i$ is an autometrized algebra under these operations. This is called the direct product of $\{A_i\}_{i\in I}$.

Theorem 3.2. *Let $\{A_i\}_{i\in I}$ be a family of monoid autometrized algebras. Let $A = A_1 \times \ \ldots \times \ A_k$. Then A is a monoid autometrized algebra.*

Proof. To show that A is a monoid autometrized algebra. Let $a,\ b,\ c \in A$. That is; $a = (a_i)_{i\in I},\ b = (b_i)_{i\in I}$ and $c = (c_i)_{i\in I}$.

(i)**:** Consider,

$$\begin{aligned}(a * b) * c &= [(a_i)_{i\in I} * (b_i)_{i\in I}] * (c_i)_{i\in I}.\\ &= (a_i * b_i)_{i\in I} * (c_i)_{i\in I}.\\ &= [(a_i * b_i) * c_i]_{i\in I}.\\ &= [a_i * (b_i * c_i)]_{i\in I}.[Since\ A_i\ is\ associative]\\ &= (a_i)_{i\in I} * (b_i * c_i)_{i\in I}.\\ &= (a_i)_{i\in I} * [(b_i)_{i\in I} * (c_i)_{i\in I}].\\ &= a * (b * c).\end{aligned}$$

Hence, $*$ is associative.

(ii)**:** Consider,

$$\begin{aligned}a * 0 &= (a_i)_{i\in I} * (0_i)_{i\in I}.\\ &= (a_i * 0_i)_{i\in I}.\\ &= (a_i)_{i\in I}.[Since\ 0\ is\ identity\ for\ *]\\ &= a.\end{aligned}$$

Hence, 0 is the identity element for $*$. Therefore, A is monoid autometrized algebra.

$\square$

Definition 3.3. Let A be an autometrized algebra. Let $\{A_i\}_{i\in I}$ be a family of autometrized algebras. We know that $\prod_{i\in I} A_i = \{a = (a(1),\ a(2),\ \ldots) \mid a(i) \in A_i\}$ is an autometrized algebra.

Let $\alpha_i : A \to A_i$ be a map for $i \in I$. Define a map $\alpha : A \to \prod_{i\in I} A_i$ by $\alpha(a) = (\alpha_1(a),\ \alpha_2(a),\ \ldots)$. That is $\alpha(a)(i) = \alpha_i(a)$ for $i \in I$.

Theorem 3.4. *Let A be an autometrized algebra. Let $\{A_i\}_{i\in I}$ be a family of autometrized algebras. If each $\alpha_i : A \to A_i$ is a homomorphism, then the map $\alpha : A \to \prod_{i\in I} A_i$ is also a homomorphism and* $\ker\alpha = \cap_{i\in I} \ker\alpha_i$.

Proof. Suppose each $\alpha_i : A \to A_i$ is a homomorphism. To show that $\alpha : A \to \prod_{i\in I} A_i$ is a homomorphism. Let $a_1,\ a_2 \in A$. Now consider;

(i)**:**

$$\begin{aligned}\alpha(a_1 + a_2)(i) &= \alpha_i(a_1 + a_2).\\ &= \alpha_i(a_1) + \alpha_i(a_2).\\ &= \alpha(a_1)(i) + \alpha(a_2)(i).\\ &= (\alpha(a_1) + \alpha(a_2))(i).\end{aligned}$$

Therefore, $\alpha(a_1 + a_2) = \alpha(a_1) + \alpha(a_2)$.

(ii)**:**

$$\begin{aligned}\alpha(a_1 * a_2)(i) &= \alpha_i(a_1 * a_2).\\ &= \alpha_i(a_1) * \alpha_i(a_2).\\ &= \alpha(a_1)(i) * \alpha(a_2)(i).\\ &= (\alpha(a_1) * \alpha(a_2))(i).\end{aligned}$$

Therefore, $\alpha(a_1 * a_2) = \alpha(a_1) * \alpha(a_2)$.

(iii)**:** Suppose $a \le b$. Since α_i are homomorphisms;

$$\begin{aligned}&\Rightarrow \alpha_i(a) \le \alpha_i(b).\\ &\Rightarrow \alpha(a)(i) \le \alpha(b)(i).\\ &\Rightarrow \alpha(a) \le \alpha(b).\end{aligned}$$

Hence, α is a homomorphism.

Now, we shall prove that $\ker\alpha = \cap_{i\in I} \ker\alpha_i$.

$$\begin{aligned}\ker\alpha &= \{a \in A \mid \alpha(a) = 0\}.\\ &= \{a \in A \mid \alpha(a)(i) = 0(i)\}.\\ &= \{a \in A \mid \alpha_i(a) = 0_i\}.\\ &= \{a \in A \mid a \in \ker\alpha_i \forall i \in I\}.\\ &= \{a \in A \mid a \in \cap_{i\in I} \ker\alpha_i\}.\\ &= \cap_{i\in I} \ker\alpha_i.\end{aligned}$$

$\square$

Definition 3.5. Let $\{A_i\}_{i\in I}$ and $\{B_i\}_{i\in I}$ be families of autometrized algebras. Let $\alpha_i : A_i \to B_i$ be a map for $i \in I$. Define a map $\alpha : \prod_{i\in I} A_i \to \prod_{i\in I} B_i$. For any $a = (a(1),\ a(2),\ \ldots) \in \prod_{i\in I} A_i$; $\alpha(a) = (\alpha_1(a(1)),\ \alpha_2(a(2)),\ \ldots)$. That is $\alpha(a)(i) = \alpha_i(a(i))$; for $i \in I$.

Theorem 3.6. *Let $\{A_i\}_{i\in I}$ and $\{B_i\}_{i\in I}$ be families of autometrized algebras. If each $\alpha_i : A_i \to B_i$ is a homomorphism, then the map $\alpha : A = \prod_{i\in I} A_i \to \prod_{i\in I} B_i$ is also a homomorphism.*

Proof. Suppose $\alpha_i : A_i \to B_i$ is a homomorphism $\forall i \in I$. To show that $\alpha : A = \prod_{i\in I} A_i \to \prod_{i\in I} B_i$ is a homomorphism. Let $a,\ b \in A$. That is, $a = (a(1),\ a(2),\ \ldots)$ and $b = (b(1),\ b(2),\ \ldots)$. Now consider;

(i)**:**

$$\begin{aligned}\alpha(a+b)(i) &= \alpha_i((a+b)(i)).\\ &= \alpha_i(a(i)+b(i)).\\ &= \alpha_i(a(i)) + \alpha_i(b(i)).[\textit{Since } \alpha_i \textit{ is a homomorphism}]\\ &= \alpha(a)(i) + \alpha(b)(i).\\ &= (\alpha(a)+\alpha(b))(i).\end{aligned}$$

Therefore, $\alpha(a+b) = \alpha(a) + \alpha(b)$.

(ii)**:**

$$\begin{aligned}\alpha(a*b)(i) &= \alpha_i((a*b)(i)).\\ &= \alpha_i(a(i)*b(i)).\\ &= \alpha_i(a(i)) * \alpha_i(b(i)).[\textit{Since } \alpha_i \textit{ is a homomorphism}]\\ &= \alpha(a)(i) * \alpha(b)(i).\\ &= (\alpha(a)*\alpha(b))(i).\end{aligned}$$

Therefore, $\alpha(a*b) = \alpha(a) * \alpha(b)$.

(iii)**:** Suppose $a \le b$. Therefore, $a(i) \le b(i)$. Since α_i are homomorphisms;

$$\begin{aligned}&\Rightarrow \alpha_i(a(i)) \le \alpha_i(b(i)).\\ &\Rightarrow \alpha(a)(i) \le \alpha(b)(i).\\ &\Rightarrow \alpha(a) \le \alpha(b).\end{aligned}$$

Hence, α is a homomorphism. □

Definition 3.7. Let $A_1,\ A_2$ be autometrized algebras. Define

$$\pi_1 : A_1 \times A_2 \to A_1 \textit{ by } \pi_1(a(1),\ a(2)) = a(1) \textit{ and}$$

$$\pi_2 : A_1 \times A_2 \to A_2 \textit{ by } \pi_2(a(1),\ a(2)) = a(2).$$

These two maps are called projection maps. It is clear that the projection maps $\pi_1,\ \pi_2$ are epimorphisms.

Definition 3.8. Let $\{A_i\}_{i\in I}$ be a family of autometrized algebras. Define

$$\pi_j : \prod_{i\in I} A_i \to A_j \textit{ by } \pi_j(a) = a(j).$$

This map is called a projection map. It is clear that the projection maps π_j is an epimorphism.

Theorem 3.9. *Let $A_1, \ldots, A_k$ are autometrized algebras and let $A = A_1 \times \ldots \times A_k$. Let $\mathscr{I}(A_i)$ is the set of all ideals of A_i for $i = 1, \ldots, k$. If $I_i \in \mathscr{I}(A_i)$, then $I = I_1 \times \ldots \times I_k$ is an ideal of A.*

Conversely, if $I = I_1 \times \ldots \times I_k$ is an ideal of A, then for $i = 1, \ldots, k$, $I_i = \pi_i(I)$ is an ideal of A_i.

Proof. Suppose that $I_i \in \mathscr{I}(A_i)$. To show that $I = I_1 \times \ldots \times I_k$ is an ideal of A.

(i)**:** Let $a = (a_1, \ldots, a_k)$, $b = (b_1, \ldots, b_k) \in I$. Then $a + b = (a_1, \ldots, a_k) + (b_1, \ldots, b_k) = (a_1 + b_1, \ldots, a_k + b_k)$. Since $a_i + b_i \in I_i \; for \; i = 1, \ldots, k$; implies that $(a_1 + b_1, \ldots, a_k + b_k) \in I$. Therefore, $a + b \in I$.

(ii)**:** Let $a = (a_1, \ldots, a_k) \in I$ and $b = (b_1, \ldots, b_k) \in A$. Suppose $b * 0 \leq a * 0$. Therefore, $(b_1, \ldots, b_k) * (0_1, \ldots, 0_k) \leq (a_1, \ldots, a_k) * (0_1, \ldots, 0_k)$. By the definition of product; $(b_1 * 0_1, \ldots, b_k * 0_k) \leq (a_1 * 0_1, \ldots, a_k * 0_k)$. This implies that $b_i * 0_i \leq a_i * 0_i \;\; for \; i = 1, \ldots, k$. Since each I_i are ideals and $a_i \in I_i \; for \; i = 1, \ldots, k$; implies that $b_i \in I_i \; for \; i = 1, \ldots, k$. Therefore, $b = (b_1, \ldots, b_k) \in I \; for \; i = 1, \ldots, k$. Hence I is ideal.

Conversely, suppose that $I = I_1 \times \ldots \times I_k$ is an ideal of A.

(i)**:** Let $a_i, b_i \in I_i$. Since π_i is on to; there exists $a = (a_1, \ldots, a_k)$, $b = (b_1, \ldots, b_k) \in I$ such that: $\pi_i((a_1, \ldots, a_k)) = a_i$ and $\pi_i((b_1, \ldots, b_k)) = b_i \; for \; i = 1, \ldots, k$. Therefore, $\pi_i(a+b) = \pi_i((a_1, \ldots, a_k)) + \pi_i((b_1, \ldots, b_k)) = a_i + b_i \in I_i \; for \; i = 1, \ldots, k$.

(ii)**:** Let $a_i \in I_i$ and $b_i \in A_i \; for \; i = 1, \ldots, k$. Suppose $b_i * 0_i \leq a_i * 0_i \; for \; i = 1, \ldots, k$. Since π_i is on to; there exists $a = (a_1, \ldots, a_k) \in I$ such that: $\pi_i((a_1, \ldots, a_k)) = a_i$. Therefore, by the definition of product $(b_1, \ldots, b_k) * (0_1, \ldots, 0_k) \leq (a_1, \ldots, a_k) * (0_1, \ldots, 0_k)$. Therefore; $(b_1, \ldots, b_k) \in I$. Clearly, $b_i \in I_i \; for \; i = 1, \ldots, k$. Hence $I_i = \pi_i(I) \; for \; i = 1, \ldots, k$ is an ideal of A_i.

□

Remark 3.10. Let $A_1, \ldots, A_k$ are monoid autometrized algebras and let $A = A_1 \times \ldots \times A_k$. Let $\mathscr{I}(A_i)$ is the set of all ideals of A_i for $i = 1, \ldots, k$. If $I_i \in \mathscr{I}(A_i)$, then $I = I_1 \times \ldots \times I_k$ is a strong ideal of A. Indeed, the ideals of monoid autometrized algebras are strong.

Corollary 3.11. *Let $A_1, \ldots, A_k$ are autometrized algebras and let $A = A_1 \times \ldots \times A_k$. Let $\mathscr{I}(A)$ be the set of all ideals of A. Then $\mathscr{I}(A) = \mathscr{I}(A_1) \times \ldots \times \mathscr{I}(A_k) \; for \; i = 1, \ldots, k$.*

Proof. It follows from the above theorem (3.9). □

Theorem 3.12. *Let A be an autometrized algebra. Let $\{A_i\}_{i \in I}$ be a family of autometrized algebras. Suppose $\alpha_i : A \to A_i$ is a homomorphism for each $i \in I$. Then $\alpha : A \to \prod_{i \in I} A_i$ is an embedding if and only if $\cap_{i \in I} \ker \alpha_i = \{0\}$.*

Proof. Suppose α is both homomorphism and one-to-one.

To show that $\cap_{i \in I} \ker \alpha_i = \{0\}$. It is clear that $0 \in \cap_{i \in I} \ker \alpha_i$. Let $x \in \cap_{i \in I} \ker \alpha_i$. This implies that $x \in \ker \alpha_i \; \forall i \in I$. So, $\alpha_i(x) = 0(i) \; \forall i \in I$. Then by definition; $\alpha(x)(i) = 0(i) \; \forall i \in I$. Therefore, $\alpha(x) = 0$. Since α is one-to-one; $x = 0$. Thus, $\cap_{i \in I} \ker \alpha_i = \{0\}$.

Conversely, suppose that $\cap_{i \in I} \ker \alpha_i = \{0\}$. To show that α is an embedding. Since each α_i is a homomorphism by theorem (3.4); $\alpha : A \to \prod_{i \in I} A_i$ is also a homomorphism. Now, we shall show that α is one-to-one. Let $a_1,\ a_2 \in A$. Suppose $\alpha(a_1) = \alpha(a_2)$. Therefore,

$$\begin{aligned} \alpha(a_1)(i) &= \alpha(a_2)(i). \\ \alpha_i(a_1) &= \alpha_i(a_2). \\ \alpha_i(a_1) * \alpha_i(a_2) &= 0. \\ \alpha_i(a_1 * a_2) &= 0. \end{aligned}$$

Therefore, $a_1 * a_2 \in \ker \alpha_i\ \forall\ i \in I$. This implies $a_1 * a_2 \in \cap_{i \in I} \ker \alpha_i$. So, $a_1 * a_2 = 0$. Clearly, $a_1 = a_2$. Therefore, α is one-to-one. Hence α is an embedding. □

Let $A_1,\ A_2$ be monoid autometrized algebras. It is obvious that $A_1 \times A_2$ is also a monoid autometrized algebra.

Theorem 3.13. *Let $A_1,\ A_2$ be monoid autometrized algebras. Then* $\ker \pi_1,\ \ker \pi_2$ *are distant ideals. That is;* $\ker \pi_1 * \ker \pi_2 = A_1 \times A_2$ *and* $\ker \pi_1 \cap \ker \pi_2 = \{0\}$.

Proof. Clearly, $\ker \pi_1,\ \ker \pi_2$ are strong ideals. To show that $\ker \pi_1,\ \ker \pi_2$ are distant ideals.

- (i)**:** To show that $\ker \pi_1 * \ker \pi_2 = A_1 \times A_2$. Clearly, $\ker \pi_1 * \ker \pi_2 \subseteq A_1 \times A_2$. Conversely, let $(x,\ y) \in A_1 \times A_2$. Therefore, $x \in A_1,\ y \in A_2$. We know that $(0,\ y) \in \ker \pi_1$ and $(x,\ 0) \in \ker \pi_2$; hence $(0,\ y) * (x,\ 0) \in \ker \pi_1 * \ker \pi_2$. Since A is monoid; $(0 * x,\ y * 0) = (x,\ y)$. Therefore, $(x,\ y) \in \ker \pi_1 * \ker \pi_2$. Whence $A_1 \times A_2 \subseteq \ker \pi_1 * \ker \pi_2$. Thus, $\ker \pi_1 * \ker \pi_2 = A_1 \times A_2$.
- (ii)**:** To show that $\ker \pi_1 \cap \ker \pi_2 = \{0\}$. Clearly, $\{0\} \in \ker \pi_1,\ \ker \pi_2$. Therefore, $\{0\} \in \ker \pi_1 \cap \ker \pi_2$. Conversely, let $a = (a_1,\ a_2) \in \ker \pi_1 \cap \ker \pi_2$. So, $a \in \ker \pi_1$ and $a \in \ker \pi_2$. This implies $\pi_1(a) = \pi_1(a_1,\ a_2) = a_1 = 0$ and $\pi_2(a) = \pi_2(a_1,\ a_2) = a_2 = 0$. Therefore, $a = (a_1,\ a_2) = (0,\ 0)$. Hence $\ker \pi_1 \cap \ker \pi_2 = \{0\}$.

□

Theorem 3.14. *Let A be a monoid autometrized algebra. Let $I,\ J$ be distant ideals of A. Then $A \cong A/I \times A/J$.*

Proof. Clearly, I, J and $I \cap J$ are strong ideals. Define a map $f : A \to A/I \times A/J$ by $f(a) = (a * I,\ a * J)$. To show that f is well-defined. Let $a,\ b \in A$. Suppose $a = b$.

$$\begin{aligned} &\Rightarrow a * I = b * I \textit{ and } a * J = b * J. \\ &\Rightarrow (a * I,\ a * J) = (b * I,\ b * J). \\ &\Rightarrow f(a) = f(b). \end{aligned}$$

Hence, f is well-defined.

To show that f is a homomorphism. Let $a,\ b \in A$.

(i)**:**

$$\begin{aligned} f(a + b) &= ((a + b) * I,\ (a + b) * J). \\ &= ((a * I) + (b * I),\ (a * J) + (b * J)). \\ &= (a * I,\ a * J) + (b * I,\ b * J). \\ &= f(a) + f(b). \end{aligned}$$

(*ii*):

$$\begin{aligned} f(a*b) &= ((a*b)*I,\ (a*b)*J). \\ &= ((a*I)*(b*I),\ (a*J)*(b*J)). \\ &= (a*I,\ a*J)*(b*I,\ b*J). \\ &= f(a)*f(b). \end{aligned}$$

(*iii*): Suppose $a \leq b$. Therefore, $a*I \leq b*I$ and $a*J \leq b*J$. By the definition of a direct product, $(a*I,\ a*J) \leq (b*I,\ b*J)$. This implies $f(a) \leq f(b)$. Hence, f is a homomorphism.

To show that f is onto map.

Let $(x*I,\ y*J) \in A/I \times A/J$. Therefore, $x,\ y \in A = I*J$. Then there exists $a_1,\ a_2 \in I$ and $b_1,\ b_2 \in J$ such that $x = a_1*b_1,\ y = a_2*b_2$. Then

$$\begin{aligned} (b_1*a_2)*I = (b_1*I)*(a_2*I) &= (b_1*I)*I. \\ &= (b_1*I)*(0*I). \\ &= (b_1*0)*I = b_1*I. \end{aligned} \tag{1}$$

Also,

$$\begin{aligned} (a_1*b_1)*I = (a_1*I)*(b_1*I) &= I*(b_1*I). \\ &= (0*I)*(b_1*I). \\ &= (0*b_1)*I = b_1*I. \end{aligned} \tag{2}$$

From equations (1) and (2); $(b_1*a_2)*I = b_1*I = (a_1*b_1)*I = x*I$. Similarly, $(b_1*a_2)*J = a_2*J = (b_2*a_2)*J = y*J$.

So, $f(b_1*a_2) = ((b_1*a_2)*I,\ (b_1*a_2)*J) = (x*I,\ y*J)$. Hence, f is onto.

Now, we shall show that $\ker f = I \cap J$.

$$\begin{aligned} \ker f &= \{a \in A | f(a) = (I,\ J)\}. \\ &= \{a \in A | (a*I,\ a*J) = (I,\ J)\}. \\ &= \{a \in A | a*I = I \textit{ and } a*J = J\}. \\ &= \{a \in A | a \in I \textit{ and } a \in J\}. \\ &= I \cap J. \end{aligned}$$

Thus, $A/I \cap J \cong A/I \times A/J$ by the first isomorphism theorem. Since $I \cap J = \{0\}$, $A/\{0\} \cong A/I \times A/J$. Hence $A \cong A/I \times A/J$. □

Definition 3.15. Let A be an autometrized algebra. A is said to be directly indecomposable if A is not isomorphic to a direct product of two non-trivial autometrized algebras.

Theorem 3.16. *Let A be a monoid autometrized algebra. Then A is directly indecomposable if and only if the only distant ideals on A are $\{0\},\ A$.*

Proof. Suppose A is directly indecomposable. To show that the only distant ideals on A are $\{0\},\ A$.

Suppose $I,\ J$ are distant ideals on A. By theorem (3.14); $A \cong A/I \times A/J$. Since A is directly indecomposable, either A/I or A/J is trivial. Therefore, $|A/I| = 1$ or $|A/J| = 1$. This implies that either $I = A$ or $J = A$. If $I = A$, then $J = \{0\}$. Since

$I,\ J$ are distant ideals on A. If $J = A$, then $I = \{0\}$. Hence the only distant ideals on A are $\{0\},\ A$.

Conversely, suppose the only distant ideals on A are $\{0\},\ A$. To show that A is directly indecomposable. Suppose $A \cong A_1 \times A_2$. Consider $\pi_1 : A \to A_1$ and $\pi_2 : A \to A_2$ are homomorphisms. Therefore, $\ker \pi_1,\ \ker \pi_2$ are distant ideals on A; either $\ker \pi_1$ or $\ker \pi_2 = \{0\}$. If $\ker \pi_1 = \{0\}$, then π_1 is one to one. Therefore, π_1 is an isomorphism. Which implies that $A \cong A_1$. Hence $|A_2| = 1$. If $\ker \pi_2 = \{0\}$, then π_2 is one to one. Therefore, π_2 is an isomorphism. Which implies that $A \cong A_2$. Hence $|A_1| = 1$.

□

4. Subdirect Products and Simple

This section presents the concept of a subdirect product and simple autometrized algebra and its behavior. We also prove that every subdirectly irreducible monoid autometrized algebra is directly indecomposable. In particular, we show that every monoid autometrized algebra is isomorphic to the subdirect product of subdirectly irreducible autometrized algebras(homomorphic images of the given algebra).

Definition 4.1. Let A be an autometrized algebra. Let $\{A_i\}_{i\in I}$ be a family of autometrized algebras. A map $\alpha : A \to \prod_{i\in I} A_i$ is said to be a subdirect embedding if α is an embedding and $\pi_i \circ \alpha : A \to A_i$ is an epimorphism.

Definition 4.2. An autometrized algebra A is said to be a subdirect product of a family of autometrized algebras $\{A_i\}_{i\in I}$, if $\alpha : A \to \prod_{i\in I} A_i$ is subdirect embedding.

Definition 4.3. An autometrized algebra A is said to be subdirectly irreducible, if A is a subdirect product of $\{A_i\}_{i\in I}$ implies $A \cong A_i$ for some $i \in I$.

Theorem 4.4. *Every subdirectly irreducible monoid autometrized algebra is directly indecomposable.*

Proof. Let A be subdirectly irreducible monoid autometrized algebra. To show that A is directly indecomposable. To show that the only distant ideals on A are $\{0\},\ A$.

Let $I,\ J$ are distant ideals on A. Clearly, $I,\ J$ are strong ideals. By theorem (3.14); $A \cong A/I \times A/J$ where $\alpha : A \to A/I \times A/J$ by $\alpha(a) = (a * I,\ a * J)$ is an isomorphism. Therefore, $\alpha : A \to A/I \times A/J$ is an embedding, and $\alpha(A)$ is subalgebra of $A/I \times A/J$. Consider $\pi_I \circ \alpha : A \to A/I$ and $\pi_J \circ \alpha : A \to A/J$. Since $\pi_I \circ \alpha(a) = \pi_I(\alpha(a)) = \pi_I(a * I,\ a * J) = a * I$. Therefore, $\pi_I \circ \alpha$ is an onto map. Similarly, $\pi_J \circ \alpha$ is an onto map. Whence, $\alpha : A \to A/I \times A/J$ is subdirectly embedding. Since A is subdirectly irreducible; either $\pi_I \circ \alpha$ or $\pi_J \circ \alpha$ is an isomorphism. That is, either $A \cong A/I$ or $A \cong A/J$. Therefore, either $I = A$ or $J = A$. If $I = A$, then $J = \{0\}$. Since $I,\ J$ are distant ideals on A. If $J = A$, then $I = \{0\}$. Hence the only distant ideals on A are $\{0\},\ A$. By theorem (3.16); A is directly indecomposable.

□

Theorem 4.5. *Let A be an autometrized algebra. Let $\{I_i\}_{i\in I} \subseteq \mathscr{I}(A)$ where each I_i is strong ideal. Suppose $\cap_{i\in I} I_i = \{0\}$. Then the map $\alpha : A \to \prod_{i\in I} A/I_i$ by $\alpha(a)(i) = \alpha_i(a) = a * I_i$ is subdirectly embedding. Where $\alpha_i : A \to A/I_i$.*

Proof. By theorem (2.15); $\alpha_i : A \to A/I_i$ by $\alpha_i(a) = a*I_i$ is epimorphism $\forall i \in I$ and $\ker \alpha_i = I_i \ \forall i \in I$. By theorem (3.4); $\alpha : A \to \prod_{i\in I} A/I_i$ by $\alpha(a)(i) = \alpha_i(a) = a * I_i$ is also a homomorphism and $\cap_{i\in I} \ker \alpha_i = \cap_{i\in I} I_i = \{0\}$. By theorem (3.12); α is one-to-one. Therefore, α is an embedding. Clearly, $\alpha(A)$ is subalgebra of $\prod_{i\in I} A/I_i$.

Consider $\pi_i \circ \alpha : A \to A/I_i$. Then, $\pi_i \circ \alpha(a) = \pi_i(\alpha(a)) = \alpha(a)(i) = a * I_i$. Therefore, $\pi_i \circ \alpha = \alpha_i$. This implies $\pi_i \circ \alpha$ is onto $\forall i \in I$. Hence α is a subdirectly embedding.

□

THEOREM 4.6. *Let A be a non-trivial monoid autometrized algebra, that is; $|A| > 1$. Let $\mathscr{I}(A)$ be the set of all ideals of A. Then A is subdirectly irreducible if and only if the intersection of all nonzero ideals is a nonzero ideal (or $\mathscr{I}(A) \setminus \{0\}$ has a minimal element).*

Proof. Suppose A is subdirectly irreducible. Let $J = \mathscr{I}(A) \setminus \{0\}$. To show that the intersection of all nonzero ideals is a nonzero ideal. Let $I_i \in J$. To show that $\cap_{I_i\in J} I_i \in J$. Assume that $\cap_{I_i\in J} I_i \notin J$. We know that $\cap_{I_i\in J} I_i \in \mathscr{I}(A)$. Therefore, $\cap_{I_i\in J} I_i = \{0\}$. By theorem (4.5); $\alpha : A \to \prod_{I_i\in J} A/I_i$ is subdirectly embedding. Since A is subdirectly irreducible; there exists $I_i \in J$ such that $\pi_i \circ \alpha : A \to A/I_i$ is an isomorphism. Therefore, $A \cong A/I_i$. Clearly, $I_i = \{0\}$. This is a contradiction. Thus, $\cap_{I_i\in J} I_i \in J$.

Conversely, suppose that the intersection of all nonzero ideals is a nonzero ideal. Let $I_i \in J$. Therefore, $\cap_{I_i\in J} I_i \neq \{0\}$ and $\cap_{I_i\in J} I_i \in \mathscr{I}(A)$. Since $|A| > 1$; there exists $a \in A$ and $a \neq 0$ such that $a \in \cap_{I_i\in J} I_i$. To show that A is subdirectly irreducible. Suppose $\alpha : A \to \prod_{i\in I} A_i$ is subdirectly embedding. Therefore, α is one-to-one. Since $a \neq 0$; $\alpha(a) \neq \alpha(0)$. Then there exists $i \in I$ such that $\alpha(a)(i) \neq \alpha(0)(i)$. Consider $\pi_i \circ \alpha : A \to A_i$. Then, $\pi_i \circ \alpha(a) \neq \pi_i \circ \alpha(0)$. Therefore, $(\pi_i \circ \alpha(a)) * (\pi_i \circ \alpha(0)) = \pi_i \circ \alpha(a * 0) \neq 0$.

$$a * 0 \notin \ker \pi_i \circ \alpha. \tag{3}$$

If $\ker \pi_i \circ \alpha \in J = \mathscr{I}(A) \setminus \{0\}$, then $\cap_{I_i\in J} I_i \subseteq \ker \pi_i \circ \alpha$. Therefore, $a \in \ker \pi_i \circ \alpha$. Clearly, $a*0 \in \ker \pi_i \circ \alpha$. This is a contradiction by (3). As a result, $\ker \pi_i \circ \alpha \notin J$. So, $\ker \pi_i \circ \alpha = \{0\}$, and $\pi_i \circ \alpha$ is one to one. Therefore, $\pi_i \circ \alpha : A \to A_i$ is an isomorphism and imply $A \cong A_i$ for some $i \in I$. Hence A is subdirectly irreducible.

□

THEOREM 4.7. *Let A be a monoid autometrized algebra. Then A is isomorphic to the subdirect product of subdirectly irreducible autometrized algebras(homomorphic images of given algebra).*

Proof. If $|A| = 1$, that is; A is trivial, then the theorem is true. Suppose $|A| > 1$. Then, there exists $a \in A$ such that $a \neq 0$.

Let $H = \{I \in \mathscr{I}(A) | a \notin I\}$. So, $\{0\} \in H$ and $H \neq \emptyset$. Therefore, $(H, \subseteq)$ is non-empty poset.

Let $\{I_i\}_{i\in I} \subseteq H$ be a chain in H. Let $\Psi = \cup_{i\in I} I_i = \bigvee_{i\in I} I_i$ be a chain in H. Then, $\Psi \in \mathscr{I}(A)$ and $I_i \subseteq \Psi \ \forall i \in I$. Since $a \notin I_i \ \forall i \in I$; implies that $a \notin \bigvee_{i\in I} I_i = \Psi \in H$. Therefore, Ψ is an upper bound of $\{I_i\}_{i\in I}$ in H. Hence every chain in H has an upper bound in H. By Zorn's lemma, H has a maximal element. Say I_a. That is; $I_a \in H$ is maximal in H. Therefore, $I_a \in \mathscr{I}(A)$ is maximal with respect to not containing a. That is; $a \notin I_a$.

To show that $I(a) \vee I_a$ is minimal element of $[I_a,\ A] \setminus \{I_a\}$. Clearly, $I_a \subseteq I(a) \vee I_a \subseteq \mathscr{I}(A)$.

If $I_a = I(a) \vee I_a$, then $I(a) \subseteq I_a$. Therefore, $a \in I_a$. This is contradiction. Since $a \notin I_a$. Therefore, $I_a \neq I(a) \vee I_a$. Whence, $I(a) \vee I_a \in [I_a,\ A] \setminus \{I_a\}$.

Let $J \in [I_a,\ A] \setminus \{I_a\}$. Therefore, $I_a \subseteq J \subseteq A$. Clearly, $I(a) \vee I_a \subseteq I(a) \vee J$.

If $a \notin J$, then $J \in H$. We know that I_a is maximal in H and $I_a \subseteq J$. This is a contradiction. Therefore, $a \in J$ and implies $I(a) \subseteq J$. Clearly, $I(a) \vee I_a \subseteq J$. Therefore, $I(a) \vee I_a$ is minimal element of $[I_a,\ A] \setminus \{I_a\}$. By correspondence theorem we have; $[I_a,\ A] \cong \mathscr{I}(A/I_a)$. Therefore, $I(a) \vee I_a$ is minimal element of $\mathscr{I}(A/I_a) \setminus \{I_a\}$. That is; $\mathscr{I}(A/I_a) \setminus \{I_a\}$ has minimal element. By theorem (4.6); A/I_a is subdirectly irreducible. We know that A/I_a is a homomorphic image of A. Therefore, $\{A/I_a\}_{a \in A;\ a \neq 0}$ is a collection of subdirectly irreducible autometrized algebras.

Now, to show that $\cap\{I_a | a \in A \text{ and } a \neq 0\} = \{0\}$.

Clearly, $\{0\} \subseteq \cap I_a$. Let $a \in \cap I_a$. If $a \neq 0$, then $a \in I_a$. This is a contradiction. Therefore, $a = 0$.

By theorem (4.5); $\alpha : A \to \prod_{a \in A;\ a \neq 0} A/I_a$ is subdirectly embedding. Hence $A \cong \alpha(A)$ and $\alpha(A)$ is a subdirect product of $\{A/I_a\}_{a \in A;\ a \neq 0}$.

□

DEFINITION 4.8. Let A be an autometrized algebra. A is said to be simple if $\mathscr{I}(A)$ contains exactly two elements $\{0\},\ A$. That is $\mathscr{I}(A) = \{\{0\},\ A\}$.

DEFINITION 4.9. Let A be an autometrized algebra. Then, A is nilradical if and only if $Rad(A) = \{0\}$.

REMARK 4.10. Let A be an autometrized algebra. If A is simple, then A is nilradical.

EXAMPLE 4.11. Let $A = \{0,\ a,\ b,\ c\}$ with $0 \leq a,\ b \leq c$ and elements $a,\ b$ are incomparable. Define $*$ and $+$ by the following tables.

$*$	0	a	b	c
0	0	a	b	c
a	a	0	c	b
b	b	c	0	a
c	c	b	a	0

$+$	0	a	b	c
0	0	a	b	c
a	a	a	c	c
b	b	c	b	c
c	c	c	c	c

Clearly, A is an autometrized algebra. Here $I_1 = \{0,\ a\}$ and $I_2 = \{0,\ b\}$ are maximal ideals. And $I_1 \cap I_2 = \{0\}$. Thus, A is nilradical but not simple.

DEFINITION 4.12. Let A be an autometrized algebra. Let $M \in \mathscr{I}(A)$. Then M is said to be a maximal ideal if $[M,\ A]$ contains exactly two distinct elements. That is $[M,\ A] = \{M,\ A\}$.

THEOREM 4.13. *Let A be an autometrized algebra. Let M be a strong ideal of A. Then M is maximal if and only if A/M is simple.*

Proof. By correspondence theorem; we have $[M,\ A] = \mathscr{I}_M(A)$ and $\mathscr{I}(A/M)$ are in one-to-one correspondence. Therefore, $|[M,\ A]| = |\mathscr{I}(A/M)|$. Since $|[M,\ A]| = 2$; implies that $|\mathscr{I}(A/M)| = 2$. Thus, the only ideals of A/M are $\{M\}$ and A/M itself. Hence A/M is simple.

Conversely, suppose A/M is simple. So, $|\mathscr{I}(A/M)| = 2$. Therefore, $|[M,\ A]| = 2$. Hence M is maximal. □

5. Chain and Representability

This section discusses different properties of chain autometrized algebra and introduces the representability in autometrized algebra. We also prove that if a weak chain monoid normal autometrized l-algebra is nilradical, then it is representable.

DEFINITION 5.1. Let $\{A_i\}_{i\in I}$ be a family of autometrized l-algebras. Let $A = \prod_{i\in I} A_i = \{a = (a(1),\ a(2),\ \ldots)|a(i) \in A_i\}$. Define for any $a = (a_i)_{i\in I},\ b = (b_i)_{i\in I}$:

$$a + b = (a_i + b_i)_{i\in I}.$$
$$a \wedge b = (a_i \wedge b_i)_{i\in I}.$$
$$a \vee b = (a_i \vee b_i)_{i\in I}.$$
$$a * b = (a_i * b_i)_{i\in I}.$$
$$a \leq b \Leftrightarrow a_i \leq b_i \forall i \in I.$$

Then $A = \prod_{i\in I} A_i$ is an autometrized l-algebra under these operations. This is called the direct product of $\{A_i\}_{i\in I}$.

DEFINITION 5.2. Let A be an autometrized l-algebra. If A satisfies either $[a * (a \vee b)] \wedge [b * (a \vee b)] = 0$ or $[a * (a \wedge b)] \wedge [b * (a \wedge b)] = 0\ \forall a,\ b \in A$, then A is said to be a weak chain.

THEOREM 5.3. *Let A be a chain autometrized l-algebra. Then*

(i): *A is a weak chain.*

(ii): $[a * (a \vee b)] + [b * (a \vee b)] = [a * (a \vee b)] \vee [b * (a \vee b)] = a * b\ \forall a,\ b \in A$.

(iii): *If A is semiregular, then* $[a * (a \vee b)] * [b * (a \vee b)] = [a * (a \vee b)] + [b * (a \vee b)] = [a * (a \vee b)] \vee [b * (a \vee b)] = a * b\ \forall a,\ b \in A$.

Proof. Suppose A is a chain. Let $a,\ b \in A$. Then, either $a \leq b$ or $b \leq a$. Suppose $a \leq b$.

(i): Since $a \vee b = b$; implies that $[a*(a\vee b)]\wedge[b*(a\vee b)] = (a*b)\wedge(b*b) = (a*b)\wedge 0$. Since $a * b \geq 0$; and hence $[a * (a \vee b)] \wedge [b * (a \vee b)] = 0$. Similar for the case $b \leq a$.

(ii): Since $a\vee b = b$; implies that $[a*(a\vee b)]+[b*(a\vee b)] = (a*b)+(b*b) = (a*b)+0 = a * b$. Similarly, $[a * (a \vee b)] \vee [b * (a \vee b)] = (a * b) \vee (b * b) = (a * b) \vee 0 = a * b$; since $a * b \geq 0$. Similar for the case $b \leq a$.

(iii): Suppose A is semiregular. Since $a\vee b = b$; implies that $[a*(a\vee b)]*[b*(a\vee b)] = (a * b) * (b * b) = (a * b) * 0 = a * b$; since A is semiregular. Similar for the case $b \leq a$.

□

THEOREM 5.4. *Let A be a chain autometrized l-algebra. Then*

(i): *A is a weak chain.*
(ii): $[a*(a\wedge b)]+[b*(a\wedge b)]=[a*(a\wedge b)]\wedge[b*(a\wedge b)]=a*b\ \forall a,\ b\in A.$
(iii): *If A is semiregular, then* $[a*(a\wedge b)]*[b*(a\wedge b)]=[a*(a\wedge b)]+[b*(a\wedge b)]=[a*(a\wedge b)]\wedge[b*(a\wedge b)]=a*b\ \forall a,\ b\in A.$

Proof. Similar to theorem (5.3).

□

Theorem 5.5. *Let A be a weak chain autometrized l-algebra. Then, A is a chain if and only if $a\wedge b=0\Rightarrow$ either $a=0$ or $b=0$.*

Proof. Suppose A is a chain. Let $a,\ b\in A$. Then, either $a\leq b$ or $b\leq a$. Therefore, either $a\wedge b=a$ or $b\wedge a=b$. Suppose $a\wedge b=0$. Hence, either $a=0$ or $b=0$.

Conversely, suppose $a\wedge b=0\Rightarrow$ either $a=0$ or $b=0$. To show that A is a chain. Let $a,\ b\in A$. Since A is a weak chain, we have either $a*(a\vee b)=0$ or $b*(a\vee b)=0$. As a result, either $a=a\vee b$ or $b=a\vee b$. Therefore, either $b\leq a$ or $a\leq b$. Hence, A is a chain. □

Theorem 5.6. *Let A be a weak chain normal autometrized l-algebra. Let M be a strong ideal of A. Then the quotient algebra A/M is an autometrized l-algebra chain if and only if M is prime.*

Proof. Suppose A/M is a chain. Let $a,\ b\in A$. Suppose $a\wedge b=0$. To show that either $a\in M$ or $b\in M$. Since M is an ideal, $a\wedge b=0\in M$. Since M is a strong ideal, implies that $(a\wedge b)*M=M$, and it follows that $(a*M)\wedge(b*M)=M$. Since A/M is a chain, either $a*M\leq b*M$ or $b*M\leq a*M$. That implies that $(a*M)\wedge(b*M)=a*M$ or $(a*M)\wedge(b*M)=b*M$. Therefore, $a*M=M$ or $b*M=M$. As a result, either $a\in M$ or $b\in M$. Hence M is prime.

Conversely, suppose M is prime. To show that A/M is a chain. Let $a*M,\ b*M\in A/M$ where $a,\ b\in A$. Since A is a weak chain, we have $[a*(a\vee b)]\wedge[b*(a\vee b)]=0$. Since M is prime; either $a*(a\vee b)\in M$ or $b*(a\vee b)\in M$. Therefore,

$$\begin{aligned}
&[a*(a\vee b)]*M=M \text{ or } [b*(a\vee b)]*M=M.\\
&\Rightarrow (a*M)*[(a\vee b)*M]=M \text{ or } (b*M)*[(a\vee b)*M]=M.\\
&\Rightarrow a*M=(a\vee b)*M \text{ or } b*M=(a\vee b)*M.\\
&\Rightarrow a*M=(a*M)\vee(b*M) \text{ or } b*M=(a*M)\vee(b*M).
\end{aligned}$$

As a result $b*M\leq a*M$ or $a*M\leq b*M$. Hence A/M is a chain. □

Theorem 5.7. *Let A be a weak chain normal autometrized l-algebra. If P is a prime strong ideal, then $\{I\in\mathscr{I}(A)|P\subseteq I\}$ is chain under inclusion.*

Proof. Suppose that J and K are incomparable ideals containing P. That is, $P\subseteq J$ and $P\subseteq K$ such that $J\nsubseteq K$ and $K\nsubseteq J$. Therefore there exists $a\in J$ such that $a\notin K$ and there exists $b\in K$ such that $b\notin J$. Clearly, $a*0\in J$ and $b*0\in K$.

Now consider $(a*0)*P$ and $(b*0)*P$. Since A/P is chain; either $(a*0)*P\leq(b*0)*P$ or $(b*0)*P\leq(a*0)*P$. Which implies that $a*0\leq b*0$ or $b*0\leq a*0$. Since $a\in J$ and $b\in K$, implies that $b\in J$ and $a\in K$. This is a contradiction. Hence $\{I\in\mathscr{I}(A)|P\subseteq I\}$ is chain under inclusion.

□

Theorem 5.8. *Let A be a weak chain monoid normal autometrized l-algebra. Let $\alpha : A \to B$ be a homomorphism. Then* $\ker(\alpha)$ *is a prime ideal if and only if* $Im(\alpha)$ *is a chain autometrized l-algebra.*

Proof. We know that by the fundamental theorem of homomorphism, $A/\ker(\alpha) \cong Im(\alpha)$. Clearly, $\ker(\alpha)$ is a strong ideal. If $\ker(\alpha)$ is prime, then by theorem (5.6) $A/\ker(\alpha)$ is a chain autometrized l-algebra. Hence $Im(\alpha)$ is a chain autometrized l-algebra.

Conversely, suppose $Im(\alpha)$ is a chain autometrized l-algebra. Therefore, $A/\ker(\alpha)$ is a chain autometrized l-algebra. Thus, again by theorem (5.6) $\ker(\alpha)$ is prime ideal. □

Definition 5.9. Let A be an autometrized algebra. We say that A is representable if it can be represented as a subdirect product of chain autometrized algebras.

Theorem 5.10. *Let A be a weak chain monoid normal autometrized l-algebra. Then, there is a family $\{P_i\}_{i\in I}$ of prime ideals of A with $\cap_{i\in I}P_i = \{0\}$ if and only if A is a subdirect product of chain autometrized l-algebras.*

Proof. Clearly, all the ideals of A are strong. Suppose there is a family $\{P_i\}_{i\in I}$ of prime ideals of A with $\cap_{i\in I}P_i = \{0\}$. To show that A is a subdirect product of chain autometrized l-algebras. Let $A_i = A/P_i \; for \; i \in I$. By theorem (5.6); A_i are chain autometrized l-algebras.

Now define, $\alpha : A \to \prod_{i\in I} A_i$ by $\alpha(a) = (a * P_1,\ a * P_2,\ \ldots)\ \forall a \in A$. Since $\cap_{i\in I}P_i = \{0\}$; implies that $\ker(\alpha) = \cap_{i\in I}P_i = \{0\}$. Thus α is injective.

Consider $\pi_i \circ \alpha : A \to A_i$ where π_i is the *i-th* projection map. Since $\pi_i \circ \alpha(a) = \pi_i(\alpha(a)) = \pi_i(a * P_1,\ a * P_2,\ \ldots) = a * P_i$; therefore $\pi_i \circ \alpha$ is an onto map. Thus, A is a subdirect product of the chain autometrized l-algebras $\{A_i\}_{i\in I}$.

Conversely, suppose A is a subdirect product of chain autometrized l-algebras $\{A_i\}_{i\in I}$. To show that there is a family $\{P_i\}_{i\in I}$ of prime ideals of A with $\cap_{i\in I}P_i = \{0\}$. Let $\alpha : A \to \prod_{i\in I} A_i$ be a monomorphism where A_i are chain autometrized l-algebras. Clearly, $\pi_i \circ \alpha : A \to A_i$ is onto. Let $\ker(\pi_i \circ \alpha) = P_i \; for \; i \in I$. Therefore, $A/P_i \cong A_i$. This implies that A/P_i is a chain. By theorem (4.13), P_i is a prime strong ideal.

Clearly, $\{0\} \in \cap_{i\in I}P_i$. Let $x \in \cap_{i\in I}P_i$. Therefore, $\pi_i(\alpha(x)) = 0_i\ \forall\ i \in I$. This implies that $\alpha(x) = 0$. Since α is injective; implies that $x = 0$. Hence $\cap_{i\in I}P_i = \{0\}$. Thus, there is a family $\{P_i\}_{i\in I}$ of prime ideals of A with $\cap_{i\in I}P_i = \{0\}$.

□

Theorem 5.11. *Let A be a weak chain monoid normal autometrized l-algebra. Then, the following are equivalent:*

(i)**:** *A is representable,*

(ii)**:** *A is a subdirect product of chain autometrized l-algebras,*

(iii)**:** *there exists a family $\{P_i\}_{i\in I}$ of prime ideals of A with $\cap_{i\in I}P_i = \{0\}$,*

(iv)**:** *Every subdirectly irreducible order-reversing homomorphic image of A is chain.*

Proof. $(i) \Rightarrow (ii)$**:** It follows from the definition (5.9).

$(ii) \Rightarrow (iii)$**:** It follows from theorem (5.10).

$(iii) \Rightarrow (iv)$**:** Suppose that there exists a family $\{P_i\}_{i\in I}$ of prime ideals of A with $\cap_{i\in I}P_i = \{0\}$. To show that every subdirectly irreducible order-reversing homomorphic image of A is a chain.

Let B be a subdirectly irreducible and order-reversing homomorphic image of A. Clearly, B is a monoid autometrized l-algebra.

Let $\alpha : A \to B$ be order-reversing epimorphism. Therefore, $B = \alpha(A)$. By theorem (2.11); $\{\alpha(P_i)\}_{i\in I}$ are prime ideals of B such that $\alpha(P_i) = \{\alpha(x) \in B : x \in P_i\}$. Clearly, $\{\alpha(P_i)\}_{i\in I}$ are prime strong ideals of B.

To show that $\cap_{i\in I}\alpha(P_i) = \{0\}$. Let $x \in \cap_{i\in I}\alpha(P_i)$. Then $x \in \alpha(P_i)\ \forall\ i \in I$. Since α is on to; there exists $a \in P_i\ \forall\ i \in I$ such that $\alpha(a) = x$. This implies that $a \in \cap_{i\in I}P_i = \{0\}$. Therefore, $a = 0$. It is clear that $x = \alpha(a) = \alpha(0) = 0$; and hence $\cap_{i\in I}\alpha(P_i) = \{0\}$. Thus, B satisfies theorem (5.10). That is B is a subdirect product of chain autometrized l-algebras. Say $\{B_i\}_{i\in I}$. Therefore $\gamma : B \to \prod_{i\in I} B_i$ is embedding.

Since B is subdirectly irreducible; there exists $i \in I$ such that $\pi_i \circ \gamma : A \to A_i$ is an isomorphism. Therefore, $B = B_i$ for some $i \in I$. Since B_i is chain; B is chain.

$(iv) \Rightarrow (i)$**:** Suppose that every subdirectly irreducible order-reversing homomorphic image of A is chain. To show that A is representable.

We know that A can be represented as a subdirect product of subdirectly irreducible autometrized l-algebras. Therefore, there exists an embedding; $\alpha : A \to \prod_{i\in I} A_i$ such that $\pi_i \circ \alpha : A \to A_i$ is epimorphism. This implies that each A_i is a subdirectly irreducible order-reversing homomorphic image of A. Hence each A_i is chain $\forall\ i \in I$. Therefore, A is represented as a subdirect product of chain autometrized l-algebras. Thus A is representable.

□

Theorem 5.12. *Let A be a monoid autometrized algebra. Then, the followings are equivalent:*

(i)**:** *A is nilradical,*
(ii)**:** *there is a family $\{M_i\}_{i\in I}$ of maximal ideals of A with $\cap_{i\in I}M_i = \{0\}$,*
(iii)**:** *A is a subdirect product of simple autometrized algebra.*

Proof. $(i) \Rightarrow (ii)$**:** By the definition *nilradical* $= Rad(A) = \bigcap\{M : M$ is a maximal ideal of $A\} = \{0\}$.

$(ii) \Rightarrow (iii)$**:** Suppose there is a family $\{M_i\}_{i\in I}$ of maximal ideals of A with $\cap_{i\in I}M_i = \{0\}$.

To show that A is a subdirect product of simple autometrized algebras. Let $A_i = A/M_i\ for\ i \in I$. By theorem (4.13); A_i are simple autometrized algebras. Now define, $\alpha : A \to \prod_{i\in I} A_i$ by $\alpha(a) = (a * M_1,\ a * M_2,\ \ldots)\ \forall a \in A$. Since $\cap_{i\in I}M_i = \{0\}$; implies that $\ker(\alpha) = \cap_{i\in I}M_i = \{0\}$. Thus α is injective by theorem (3.12).

Consider $\pi_i \circ \alpha : A \to A_i$ where π_i is the *i-th* projection map. Since $\pi_i \circ \alpha(a) = \pi_i(\alpha(a)) = \pi_i(a * M_1,\ a * M_2,\ \ldots) = a * M_i$; therefore $\pi_i \circ \alpha$ is an onto map. Thus, A is a subdirect product of the simple autometrized algebra $\{A_i\}_{i\in I}$.

$(iii) \Rightarrow (i)$**:** Suppose A is a subdirect product of simple autometrized algebras $\{A_i\}_{i\in I}$.

To show that A is nilradical. Let $\alpha : A \to \prod_{i\in I} A_i$ be a monomorphism where A_i are simple autometrized algebras. Clearly, $\pi_i \circ \alpha : A \to A_i$ is onto. Let $\ker(\pi_i \circ \alpha) = M_i\ for\ i \in I$. Therefore, $A/M_i \cong A_i$. This implies that A/M_i is simple. By theorem (4.13), M_i is maximal.

Clearly, $\{0\} \in Rad(A)$. Let $x \in \cap_{i\in I} M_i$. So, $x \in M_i$. Therefore, $\pi_i(\alpha(x)) = 0_i\ \forall\ i \in I$. This implies that $\alpha(x) = 0$. Since α is injective; implies that $x = 0$. Therefore $\cap_{i\in I} M_i = \{0\}$. It is easily to show that $Rad(A) = \bigcap\{M : M$ is a maximal ideal of $A\} \subseteq \cap_{i\in I} M_i$. Hence $Rad(A) = \{0\}$. Thus, A is nilradical.

□

THEOREM 5.13. *Let A be a weak chain monoid normal autometrized l-algebra. Then, the followings are equivalent:*

(i): *A is nilradical,*
(ii): *there is a family $\{M_i\}_{i\in I}$ of maximal ideals of A with $\cap_{i\in I} M_i = \{0\}$,*
(iii): *A is a subdirect product of simple chain autometrized l-algebras.*

Proof. It is a direct consequence of theorems (5.10) and (5.12). □

COROLLARY 5.14. *Let A be a weak chain monoid normal autometrized l-algebra. If A is nilradical, then it is representable.*

Proof. It follows from theorem (5.13). □

THEOREM 5.15. *Let A be a weak chain monoid normal autometrized l-algebra. If A is representable, then*

(i): $n(a \wedge b) = na \wedge nb\ \forall\ a,\ b \in A$.
(ii): $n(a \vee b) = na \vee nb\ \forall\ a,\ b \in A$

Proof. Since A is representable, A is represented as a subdirect product of chain autometrized l-algebras. Therefore, there exists an embedding; $\alpha : A \to \prod_{i\in I} A_i$ such that A_i is chain. Clearly, $A \cong \alpha(A)$.

(i): Let $a_i,\ b_i \in A_i$. If $a_i \le b_i$, then $a_i \wedge b_i = a_i$. Since A_i is translation invariant; $a_i + a_i \le b_i + a_i \le b_i + b_i$. So, $2a_i \le 2b_i$. Assume that $na_i \le nb_i$. Then $(n+1)a_i = na_i + a_i \le nb_i + a_i \le nb_i + b_i = (n+1)b_i$. By induction; $na_i \le nb_i$. Therefore, $n(a_i \wedge b_i) = na_i = na_i \wedge nb_i$.

Now consider, $a = (a_i)_{i\in I},\ b = (b_i)_{i\in I} \in \prod_{i\in I} A_i$. Then,

$$\begin{aligned} n(a \wedge b) &= n((a_i)_{i\in I} \wedge (b_i)_{i\in I}). \\ &= n((a_i \wedge b_i)_{i\in I}). \\ &= (n(a_i \wedge b_i))_{i\in I}. \\ &= (na_i \wedge nb_i)_{i\in I}. \\ &= (na_i)_{i\in I} \wedge (nb_i)_{i\in I}. \\ &= n(a_i)_{i\in I} \wedge n(b_i)_{i\in I}. \\ &= na \wedge nb. \end{aligned}$$

Therefore, it holds in $\prod_{i\in I} A_i$. Therefore, it holds in $\alpha(A)$. Hence $n(a \wedge b) = nb = na \wedge nb$ holds in A.

(ii): Let $a_i,\ b_i \in A_i$. If $a_i \le b_i$, then $a_i \vee b_i = b_i$. By similar argument as (i); $na_i \le nb_i$. Therefore, $n(a_i \vee b_i) = nb_i = na_i \vee nb_i$ holds in A_i. Also, it holds in $\prod_{i\in I} A_i$. Therefore, it holds in $\alpha(A)$. Hence $n(a \vee b) = nb = na \vee nb$ holds in A.

□

EXAMPLE 5.16. Let $A = \{0,\ a,\ b,\ c\}$ with $0 \le a,\ b \le c$ and elements $a,\ b$ are incomparable. Define $*$ and $+$ by the following tables.

*	0	a	b	c
0	0	a	b	c
a	a	0	c	b
b	b	c	0	a
c	c	b	a	0

+	0	a	b	c
0	0	a	b	c
a	a	a	c	c
b	b	c	b	c
c	c	c	c	c

Clearly, A is an autometrized l-algebra. Here A is representable. Clearly, $a \wedge b = 0$ implies that $n(a \wedge b) = 0$.

And we easily see that $na \wedge nb = a \wedge b = 0$. Therefore, $n(a \wedge b) = na \wedge nb$. Similarly, $a \vee b = c$ and implies that $n(a \vee b) = c$. And we easily see that $na \vee nb = a \vee b = c$. Therefore, $n(a \vee b) = na \vee nb$.

6. Conclusion

In this paper, we introduced the concept of direct products and discussed some basic facts about distant ideals. We also introduced the definition of directly indecomposable in an autometrized algebra. Furthermore, we presented the concept of a subdirect product and simple autometrized algebra and its behavior. We also introduced the definition of subdirectly irreducible in an autometrized algebra. We also proved that every subdirectly irreducible monoid autometrized algebra is directly indecomposable. Lastly, we discussed different properties of chain autometrized algebra and introduced the representability in autometrized algebra. We also showed that every nilradical monoid autometrized algebra is a subdirect product of simple autometrized algebras. In the future, we may explore the concepts of Archimedean autometrized algebra and varieties in autometrized algebra.

References

[1] K. N. Swamy, *A general theory of autometrized algebras*, Math. Ann. **157** (1) (1964), 65–74.
[2] L. M. Blumenthal, *Boolean geometry. 1*, Bull. New. Ser. Am. Math. Soc. **58** (4) (1952), 501–501.
[3] D. Ellis, *Autometrized boolean algebras i: Fundamental distance-theoretic properties of b*, Can. J. Math. **3** (1951), 87–93.
[4] E. Nordhaus and L. Lapidus, *Brouwerian geometry*, Can. J. Math. **6** (1954), 217–229.
[5] K. R. Roy, *Newmannian geometry i*, Bull. Calcutta Math. Soc. **52** (1960), 187–194.
[6] K. Narasimha Swamy, *Autometrized lattice ordered groups i*, Math. Ann. **154** (5) (1964), 406–412.
[7] K. Swamy and N. P. Rao, *Ideals in autometrized algebras*, J. Aust. Math. Soc. **24** (3) (1977), 362–374.
[8] J. Rachůnek, *Prime ideals in autometrized algebras*, Czechoslov. Math. J. **37** (1) (1987), 65–69.
[9] J. Rachůnek, *Polars in autometrized algebras*, Czechoslov. Math. J. **39** (4) (1989), 681–685.
[10] J. Rachůnek, *Regular ideals in autometrized algebras*, Math. Slovaca. **40** (2) (1990), 117–122.
[11] J. Rachůnek, *Spectra of autometrized lattice algebras*, Math. Bohem. **123** (1) (1998), 87–94.
[12] M. E. Hansen, *Minimal prime ideals in autometrized algebras*, Czechoslov. Math. J. **44** (1) (1994), 81–90.
[13] T. Kovář, *Normal autometrized lattice ordered algebras*, Math. Slovaca. **50** (4) (2000), 369–376.

[14] I. Chajda and J. Rachunek, *Annihilators in normal autometrized algebras*, Czechoslov. Math. J. **51** (1) (2001), 111–120.

[15] B. Subba Rao and P. Yedlapalli, *Metric spaces with distances in representable autometrized algebras*, Southeast Asian Bull. Math. **42** (3) (2018), 453–462.

[16] B. Rao, A. Kanakam, and P. Yedlapalli, *A note on representable autometrized algebras*, Thai J. Math. **17** (1) (2019), 277–281.

[17] B. Rao, A. Kanakam, and P. Yedlapalli, *Representable autometrized semialgebra*, Thai J. Math. **19** (4) (2021), 1267–1272.

[18] B. Rao, A. Kanakam, and P. Yedlapalli, *Representable autometrized algebra and mv algebra*, Thai J. Math. **20** (2) (2022), 937–943.

[19] G. Y. Tilahun, R. K. Parimi, and M. H. Melesse, *Structure of an autometrized algebra*, Research in Mathematics. **10** (1) (2023), 2192856.

[20] G. Y. Tilahun, R. K. Parimi, and M. H. Melesse, *Equivalent conditions of normal autometrized l-algebra*, Research in Mathematics. **10** (1) (2023), 2215037.

Gebrie Yeshiwas Tilahun
Department of Mathematics, Arba Minch University, Arba Minch, Ethiopia.
E-mail: `gebrieyeshiwas0@gmail.com`

Radhakrishna Kishore Parimi
Department of Mathematics, Arba Minch University, Arba Minch, Ethiopia.
E-mail: `parimirkk@gmail.com`

Mulugeta Habte Melesse
Department of Mathematics, Arba Minch University, Arba Minch, Ethiopia.
E-mail: `mulugeta.habte@amu.edu.et`