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HADAMARD-TYPE INEQUALITIES ON THE COORDINATES FOR (h_1, h_2, h_3)-PREINVEX FUNCTIONS

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Abstract. In the present paper, we define the class of (h_1, h_2, h_3)-preinvex functions on co-ordinates and prove certain new Hermite-Hadamard and Fejér type inequalities for such mappings. As a consequence, we derive analogous Hadamard-type results on convex and s-convex functions in three co-ordinates. We also discuss some intriguing aspects of the associated H function.

1. Introduction

We define a function $\chi : J \to \mathbb{R}$, where $J \subseteq \mathbb{R}$ is an interval in $\mathbb{R}$, to be a convex function on J if

$$\chi(\sigma\zeta + (1-\sigma)\mu) \leq \sigma\chi(\zeta) + (1-\sigma)\chi(\mu) \tag{1}$$

is true for every $\zeta, \mu \in J$ and $\sigma \in [0,1]$. χ is concave if the reverse inequality holds. The Hermite-Hadamard inequality is one of the most significant inequalities for the class of convex functions. This twofold inequality is expressed as follows

$$\chi\left(\frac{\zeta_1+\zeta_2}{2}\right) \leq \frac{1}{\zeta_2-\zeta_1}\int_{\zeta_1}^{\zeta_2}\chi(\zeta)d\zeta \leq \frac{\chi(\zeta_1)+\chi(\zeta_2)}{2}, \tag{2}$$

where $\chi : [\zeta_1, \zeta_2] \to \mathbb{R}$ is a convex function. If χ is concave, the inequalities are in reverse order.

An s-convex function is a generalization of a convex function which was first introduced by Breckner [3]. A function $\chi : [0,\infty) \to \mathbb{R}$ is s-convex in the second sense if $\chi(\sigma\zeta + (1-\sigma)\mu) \leq \sigma^s\chi(\zeta) + (1-\sigma)^s\chi(\mu)$ holds for all $\zeta, \mu \in [0,\infty), \sigma \in [0,1]$ and for some fixed $s \in [0,1]$. Obviously, s-convexity reduces to convexity when $s = 1$.

Dragomir and Fitzpatrick [4] established the following Hermite-Hadamard type inequality, true for s-convex functions in the second sense.

$$2^{s-1}\chi\left(\frac{\zeta_1+\zeta_2}{2}\right) \leq \int_{\zeta_1}^{\zeta_2}\chi(x)dx \leq \frac{\chi(\zeta_1)+\chi(\zeta_2)}{s+1}. \tag{3}$$

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Later Varosanec [13] introduced the class of h-convex functions. Some familiar types of functions in this class are non-negative convex functions and s-convex in the second sense functions. A non-negative function $\chi : J \to \mathbb{R}, J \subseteq \mathbb{R}$ in an interval, is called h-convex if $\chi(\sigma\zeta + (1-\sigma)\mu) \leq h(\sigma)\chi(\zeta) + h(1-\sigma)\chi(\mu)$ holds for all $\zeta, \mu \in J, \sigma \in (0,1)$, where $h : J \to \mathbb{R}$ is a non-negative function, $h \not\equiv 0$ and J is an interval, $(0,1) \subseteq J$. However, the methods and text that follows examines functions h and χ without making any nonnegativity-related assumptions. Sarikaya et al. [12] demonstrated that the following variation of the Hadamard inequality holds for an h-convex function.

$$\frac{1}{2h(\frac{1}{2})}\chi\left(\frac{\zeta_1+\zeta_2}{2}\right) \leq \frac{1}{\zeta_2-\zeta_1}\int_{\zeta_1}^{\zeta_2}\chi(\zeta)dx \leq \left\{\chi(\zeta_1)+\chi(\zeta_2)\right\}\int_0^1 h(\sigma)d\sigma. \tag{4}$$

Also Bombardelli and Varošanec [2] showed that for an h-convex function the following Hermite-Hadamard-Fejer type inequality holds:

$$\begin{aligned} \frac{\int_{\zeta_1}^{\zeta_2} w(\zeta)d\zeta}{2h(\frac{1}{2})}\chi\left(\frac{\zeta_1+\zeta_2}{2}\right) &\leq \int_{\zeta_1}^{\zeta_2}\chi(\zeta)w(\zeta)dx \\ \leq (\zeta_2-\zeta_1)\Big(\chi(\zeta_1)+\chi(\zeta_2)\Big)&\int_0^1 h(\sigma)w(\sigma\zeta_1+(1-\sigma)\zeta_2)d\sigma, \end{aligned} \tag{5}$$

where $w : [\zeta_1, \zeta_2] \to \mathbb{R}$. Here $w \geq 0$ is the weight function and is symmetric with respect to $\frac{\zeta_1+\zeta_2}{2}$.

Dragomir [5] also suggested a variation for convex functions called co-ordinated convex functions, which is as follows. Consider a bidimensional rectangle $\Omega = [\zeta_1, \zeta_2] \times [\mu_1, \mu_2]$ in $\mathbb{R}^2$ with $\zeta_1 < \zeta_2$ and $\mu_1 < \mu_2$. A mapping $\chi : \Omega \to \mathbb{R}$ is said to be convex on the co-ordinates on Ω if the partial mappings $\chi_\mu : [\zeta_1, \zeta_2] \to \mathbb{R}, \chi_\mu(u) = \chi(u, \mu)$ and $\chi_\zeta : [\mu_1, \mu_2] \to \mathbb{R}, \chi_\zeta(v) = \chi(\zeta, v)$ are convex for all $\zeta \in [\zeta_1, \zeta_2]$ and $\mu \in [\mu_1, \mu_2]$.

Subsequently, Dragomir [5] established the following Hadamard-type inequality for convex functions on the co-ordinates.

$$\begin{aligned} \chi\left(\frac{\zeta_1+\zeta_2}{2}, \frac{\mu_1+\mu_2}{2}\right) &\leq \frac{1}{(\zeta_2-\zeta_1)(\mu_2-\mu_1)}\int_{\zeta_1}^{\zeta_2}\int_{\mu_1}^{\mu_2}\chi(\zeta,\mu)d\zeta d\mu \\ &\leq \frac{\chi(\zeta_1,\mu_1)+\chi(\zeta_2,\mu_1))+\chi(\zeta_1,\mu_2)+\chi(\zeta_2,\mu_2)}{4}. \end{aligned} \tag{6}$$

Alomari and Darus [1] proposed a natural extension of convex functions on the co-ordinates to the concept of s-convex functions on the co-ordinates.
The mapping $\chi : \Omega \to \mathbb{R}$ is s-convex in the second sense if the partial mappings $f_\mu : [\zeta_1, \zeta_2] \to \mathbb{R}$ and $f_\zeta : [\mu_1, \mu_2] \to \mathbb{R}$ are s-convex in the second sense.
They also demonstrated that the following inequality holds for an s-convex function:

$$\begin{aligned} 4^{s-1}\chi\left(\frac{\zeta_1+\zeta_2}{2}, \frac{\mu_1+\mu_2}{2}\right) &\leq \frac{1}{(\zeta_2-\zeta_1)(\mu_2-\mu_1)}\int_{\zeta_1}^{\zeta_2}\int_{\mu_1}^{\mu_2}\chi(\zeta,\mu)d\zeta d\mu \\ &\leq \frac{\chi(\zeta_1,\mu_1)+\chi(\zeta_2,\mu_1))+\chi(\zeta_1,\mu_2)+\chi(\zeta_2,\mu_2)}{(s+1)^2}. \end{aligned} \tag{7}$$

For other approaches and analogous results on convex and s-convex functions on the co-ordinates, see [6-10].
The primary goal of this work is to present the class of (h_1, h_2, h_3)-preinvex functions on co-ordinates and to establish new inequalities similar to those given by Dragomir in [5] and Matloka in [8]. Also, we present some interesting properties of the related

H function. We will assume that the considered integrals exist throughout the paper. Let us first recall some key concepts.

2. Preliminaries

Let $\chi : S \to \mathbb{R}$ and $\alpha : S \times S \to \mathbb{R}^n$, where S is a nonempty closed set in $\mathbb{R}^n$, be continuous functions. First, we recall the following well-known results and concepts; see [7-11] and the references therein.

Definition 2.1. [6] A set S is said to be invex at u with respect to α if $u + \sigma\alpha(v, u) \in S$ for all $u, v \in S$ and $\sigma \in [0, 1]$. S is an invex set with respect to α if S is invex at each $u \in S$.

Definition 2.2. [6] A function χ on the invex set S is preinvex with respect to α if $\chi(u + \sigma\alpha(v, u)) \leq (1 - \sigma)\chi(u) + \sigma\chi(v)$ for all $u, v \in S$ and $\sigma \in [0, 1]$.

We also need the following assumption on the function α, which is owed to Mohan and Neogy [9].

Condition A. Let $S \subseteq \mathbb{R}$ be an open invex subset with respect to α. For any $\zeta, \mu \in S$ and any $\sigma \in [0, 1]$,
$\alpha(\mu, \mu + \sigma\alpha(\zeta, \mu)) = -\sigma\alpha(\zeta, \mu)$,
$\alpha(\zeta, \mu + \sigma\alpha(\zeta, \mu)) = (1 - \sigma)\alpha(\zeta, \mu)$.

Definition 2.3. [8] Let $h : [0, 1] \to \mathbb{R}$ be a non-negative function, $h \not\equiv 0$. A non-negative function χ on the invex set S is h-preinvex with respect to α if $\chi(u + \sigma\alpha(v, u)) \leq h(1 - \sigma)\chi(u) + h(\sigma)\chi(v)$ for each $u, v \in S$ and $\sigma \in [0, 1]$.

Matloka [8] introduced the following concept of (h_1, h_2)-preinvex functions on the co-ordinates:

Definition 2.4. [8] Let h_1 and h_2 be non-negative functions on $[0, 1], h_1 \not\equiv 0, h_2 \not\equiv 0$. The nonnegative function χ on the invex set $S_1 \times S_2$ is said to be co-ordinated (h_1, h_2)-preinvex with respect to α_1 and α_2 if the partial mappings $\chi_\mu : S_1 \to \mathbb{R}, \chi_\mu(\zeta) = \chi(\zeta, \mu)$ and $\chi_\zeta : S_2 \to \mathbb{R}, \chi_\zeta(\mu) = \chi(\zeta, \mu)$ are h_1-preinvex with respect to α_1 and h_2-preinvex with respect to α_2, respectively, for all $\mu \in S_2$ and $\zeta \in S_1$.

As a result of the above definition, if f is a co-ordinated (h_1, h_2)-preinvex function, then

$$\begin{aligned}
&\chi\big(\zeta + \sigma_1\alpha_1(\zeta_2, \zeta), \mu + \sigma_2\alpha_2(\mu_2, \mu)\big) \\
&\leq h_1(1 - \sigma_1)\chi\big(\zeta, \mu + \sigma_2\alpha_2(d, \mu)\big) + h_1(\sigma_1)\chi\big(\zeta_2, \mu + \sigma_2\alpha_2(\mu_2, \mu)\big) \\
&\leq h_1(1 - \sigma_1)h_2(1 - \sigma_2)\chi(\zeta, \mu) + h_1(1 - \sigma_1)h_2(\sigma_2)\chi(\zeta, \mu_2) \\
&\quad + h_1(\sigma_1)h_2(1 - \sigma_2)\chi(\zeta_2, \mu) + h_1(\sigma_1)h_2(c_2)\chi(\zeta_2, \mu_2).
\end{aligned}$$

Now let S_1, S_2 and S_3 be nonempty subsets of $\mathbb{R}^n$ and $\alpha_1 : S_1 \times S_1 \to \mathbb{R}^n, \alpha_2 : S_2 \times S_2 \to \mathbb{R}^n$ and $\alpha_3 : S_3 \times S_3 \to \mathbb{R}^n$.

3. Main Results

Definition 3.1. Let $(u, v, w) \in S_1 \times S_2 \times S_3$. We say $S_1 \times S_2 \times S_3$ is invex at (u, v, w) with respect to α_1, α_2 and α_3 if for each $(\zeta, \mu, \eta) \in S_1 \times S_2 \times S_3$ and $\sigma_1, \sigma_2, \sigma_3 \in [0, 1]$, $(u + \sigma_1\alpha_1(\zeta, u), v + \sigma_2\alpha_2(\mu, u), w + \sigma_3\alpha_3(\eta, w)) \in S_1 \times S_2 \times S_3$.

$S_1 \times S_2 \times S_3$ is said to be an invex set with respect to α_1, α_2 and α_3 if it is invex at each $(u, v, w) \in S_1 \times S_2 \times S_3$.

DEFINITION 3.2. Let h_1, h_2 and h_3 be non-negative functions on $[0,1], h_1 \not\equiv 0, h_2 \not\equiv 0, h_3 \not\equiv 0$. The non-negative function f on the invex set $S_1 \times S_2 \times S_3$ is said to be co-ordinated (h_1, h_2, h_3)-preinvex with respect to α_1, α_2 and α_3, if the partial mappings $\chi_\zeta : S_2 \times S_3 \to \mathbb{R}, \chi_\zeta(\mu, \eta) = \chi(\zeta, \mu, \eta); \chi_\mu : S_1 \times S_3 \to \mathbb{R}, \chi_\mu(\zeta, \eta) = \chi(\zeta, \mu, \eta); \chi_\eta : S_1 \times S_2 \to \mathbb{R}, \chi_\eta(\zeta, \mu) = \chi(\zeta, \mu, \eta)$ are (h_2, h_3)-preinvex with respect to α_1, (h_1, h_3)-preinvex with respect to α_2 and (h_1, h_2)-preinvex with respect to α_3, respectively, for all $\zeta \in S_1, \mu \in S_2, \eta \in S_3$.
If $\alpha(\zeta, u) = \zeta - u, \alpha(\mu, v) = \mu - v$ and $\alpha(\eta, w) = \eta - w$ then the function χ is called (h_1, h_2, h_3)-convex on the co-ordinates.

REMARK 1. From the above definition it follows that if χ is co-ordinated (h_1, h_2, h_3)-preinvex, then

$$\begin{aligned}
&\chi\Big(\zeta_1 + \sigma_1\alpha_1(\zeta_2, \zeta_1), \mu_1 + \sigma_2\alpha(\mu_2, \mu_1), \eta_1 + \sigma_3\alpha_3(\eta_2, \eta_1)\Big) \\
&\qquad \le h_1(1-\sigma_1)\chi\Big(\zeta_1, \mu_1 + \sigma_2\alpha_2(\mu_2, \mu_1), \eta_1 + \sigma_3\alpha_3(\eta_2, \eta_1)\Big) \\
&\qquad\quad + h_1(\sigma_1)\chi\Big(\zeta_2, \mu_1 + \sigma_2\alpha_2(\mu_2, \mu_1), \eta_1 + \sigma_3\alpha_3(\eta_2, \eta_1)\Big) \\
&\le h_1(1-\sigma_1)\Big[h_2(1-\sigma_1)\chi\Big(\zeta_1, \mu_1, \eta_1 + \sigma_3\alpha_3(\eta_2, \eta_1)\Big) + h_2(\sigma_2)\chi\Big(\zeta_1, \mu_2, \eta_1) + \sigma_3\alpha_3(\eta_2, \eta_1)\Big)\Big] \\
&+ h_1(\sigma_1)\Big[h_2(1-\sigma_1)\chi\Big(\zeta_2, \mu_1, \eta_1 + \sigma_3\alpha_3(\eta_2, \eta_1)\Big) + h_2(\sigma_2)\chi\Big(\zeta_2, \mu_2, \eta_1 + \sigma_3\alpha_3(\eta_2, \eta_1)\Big)\Big] \\
&\le h_1(1-\sigma_1)h_2(1-\sigma_2)h_3(1-\sigma_3)\chi(\zeta_1, \mu_1, \eta_1) + h_1(1-\sigma_1)h_2(\sigma_2)h_3(1-\sigma_3)\chi(\zeta_1, \mu_2, \eta_1) \\
&\quad + h_1(\sigma_1)h_2(1-\sigma_2)h_3(1-\sigma_3)\chi(\zeta_2, \mu_1, \eta_1) + h_1(\sigma_1)h_2(\sigma_2)h_3(1-\sigma_3)\chi(\zeta_2, \mu_2, \eta_1) \\
&\quad + h_1(1-\sigma_1)h_2(1-\sigma_2)h_3(\sigma_3)\chi(\zeta_1, \mu_1, \eta_2) + h_1(1-\sigma_1)h_2(\sigma_2)h_3(\sigma_3)\chi(\zeta_1, \mu_2, \eta_2) \\
&\qquad + h_1(\sigma_1)h_2(1-\sigma_2)h_3(\sigma_3)\chi(\zeta_2, \mu_1, \eta_2) + h_1(\sigma_1)h_2(\sigma_2)h_3(\sigma_3)\chi(\zeta_2, \mu_2, \eta_2).
\end{aligned}$$

As a consequence of the preceding remark, we arrive at the following result.

THEOREM 1. *Let* $\chi : \Big[\zeta_1, \zeta_1 + \alpha_1(\zeta_2, \zeta_1)\Big] \times \Big[\mu_1, \mu_1 + \alpha_2(\mu_2, \mu_1)\Big] \times \Big[\eta_1, \eta_1 + \alpha_3(\eta_2, \eta_1)\Big] \to \mathbb{R}$ *with* $\zeta_1 < \zeta_1 + \alpha_1(\zeta_2, \zeta_1), \mu_1 < \mu_1 + \alpha_2(\mu_2, \mu_1)$ *and* $\eta_1 < \eta_1 + \alpha_2(\eta_2, \eta_1)$, *be* (h_1, h_2, h_3)*-preinvex on the co-ordinates with respect to* α_1, α_2 *and* α_3*;* $w : \Big[\zeta_1, \zeta_1 + \alpha_1(\zeta_2, \zeta_1)\Big] \times \Big[\mu_1, \mu_1 + \alpha_2(\mu_2, \mu_1)\Big] \times \Big[\eta_1, \eta_1 + \alpha_3(\eta_2, \eta_1)\Big] \to \mathbb{R}, w \ge 0$, *symmetric with respect to* $\Big(\zeta_1 + \frac{1}{2}\alpha_1(\zeta_2, \zeta_1), \mu_1 + \frac{1}{2}\alpha_2(\mu_2, \mu_1), \eta_1 + \frac{1}{2}\alpha_3(\eta_2, \eta_1)\Big)$. *Then if Condition A for* α_1, α_2, *and* α_3 *is fulfilled, we have*

$$\begin{aligned}
&\chi\Big(\zeta_1 + \frac{1}{2}\alpha_1(\zeta_2, \zeta_1), \mu_1 + \frac{1}{2}\alpha_2(\mu_2, \mu_1), \eta_1 + \frac{1}{2}\alpha_3(\eta_2, \eta_1)\Big) \\
&\int_{\zeta_1}^{\zeta_1+\alpha_1(\zeta_2,\zeta_1)} \int_{\mu_1}^{\mu_1+\alpha_2(\mu_2,\mu_1)} \int_{\eta_1}^{\eta_1+\alpha_3(\eta_2,\eta_1)} w(\zeta, \mu, \eta) d\zeta d\mu d\eta \\
&\qquad \le 8h_1\left(\frac{1}{2}\right) h_2\left(\frac{1}{2}\right) h_3\left(\frac{1}{2}\right) \\
&\int_{\zeta_1}^{\zeta_1+\alpha_1(\zeta_2,\zeta_1)} \int_{\mu_1}^{\mu_1+\alpha_2(\mu_2,\mu_1)} \int_{\eta_1}^{\eta_1+\alpha_3(\eta_2,\eta_1)} \chi(\zeta, \mu, \eta) \cdot w(\zeta, \mu, \eta) d\zeta d\mu d\eta.
\end{aligned} \tag{8}$$

Proof. Using the definition of an (h_1, h_2, h_3)-preinvex function on the co-ordinartes and Condition A for α_1, α_2 and α_3, we have

$$\chi\Big(\zeta_1 + \tfrac{1}{2}\alpha(\zeta_2, \zeta_1), \mu_1 + \tfrac{1}{2}\alpha(c, d), \eta_1 + \tfrac{1}{2}\alpha(\eta_2, \eta_1)\Big) \le$$
$$h_1\Big(\tfrac{1}{2}\Big)h_2\Big(\tfrac{1}{2}\Big)h_3\Big(\tfrac{1}{2}\Big)\Big\{\chi\Big(\zeta_1 + \sigma_1\alpha_1(\zeta_2, \zeta_1), \mu_1 + \sigma_2\alpha_2(\mu_2, \mu_1), \eta_1 + \sigma_3\alpha_3(\eta_2, \eta_1)\Big) +$$
$$\chi\Big(\zeta_1 + \sigma_1\alpha_1(\zeta_2, \zeta_1), \mu_1 + (1 - \sigma_2)\alpha_2(\mu_2, \mu_1), \eta_1 + \sigma_3\alpha_3(\eta_2, \eta_1)\Big) +$$
$$\chi\Big(\zeta_1 + (1 - \sigma_1)\alpha_1(\zeta_2, \zeta_1), \mu_1 + \sigma_2\alpha_2(\mu_2, \mu_1), \eta_1 + \sigma_3\alpha_3(\eta_2, \eta_1)\Big) +$$
$$\chi\Big(a + (1 - \sigma_1)\alpha_1(\zeta_2, \zeta_1), c + (1 - \sigma_2)\alpha_2(\mu_2, \mu_1), e + \sigma_3\alpha_3(\eta_2, \eta_1)\Big) +$$
$$\chi\Big(\zeta_1 + \sigma_1\alpha_1(\zeta_2, \zeta_1), \mu_1 + \sigma_2\alpha_2(\mu_2, \mu_1), \eta_1 + (1 - \sigma_3)\alpha_3(\eta_2, \eta_1)\Big) +$$
$$\chi\Big(\zeta_1 + \sigma_1\alpha_1(\zeta_2, \zeta_1), \mu_1 + (1 - \sigma_2)\alpha_2(\mu_2, \mu_1), \eta_1 + (1 - \sigma_3)\alpha_3(\eta_2, \eta_1)\Big) +$$
$$\chi\Big(\zeta_1 + (1 - \sigma_1)\alpha_1(\zeta_2, \zeta_1), \mu_1 + \sigma_2\alpha_2(\mu_2, \mu_1), \eta_1 + (1 - \sigma_3)\alpha_3(\eta_2, \eta_1)\Big) +$$
$$\chi\Big(\zeta_1 + (1 - \sigma_1)\alpha_1(\zeta_2, \zeta_1), \mu_1 + (1 - \sigma_2)\alpha_2(\mu_2, \mu_1), \eta_1 + (1 - \sigma_3)\alpha_3(\eta_2, \eta_1)\Big)\Big\}.$$

Now, multiplying the above inequality by

$$w\Big(\zeta_1 + \sigma_1\alpha_1(\zeta_2, \zeta_1), \mu_1 + \sigma_2\alpha_2(\mu_2, \mu_1), \eta_1 + \sigma_3\alpha_3(\eta_2, \eta_1)\Big) =$$
$$w\Big(\zeta_1 + \sigma_1\alpha_1(\zeta_2, \zeta_1), \mu_1 + (1 - \sigma_2)\alpha_2(\mu_2, \mu_1), \eta_1 + \sigma_3\alpha_3(\eta_2, \eta_1)\Big) =$$
$$w\Big(\zeta_1 + (1 - \sigma_1)\alpha_1(\zeta_2, \zeta_1), \mu_1 + \sigma_2\alpha_2(\mu_2, \mu_1), \eta_1 + \sigma_3\alpha_3(\eta_2, \eta_1)\Big) =$$
$$w\Big(\zeta_1 + (1 - \sigma_1)\alpha_1(\zeta_2, \zeta_1), \mu_1 + (1 - \sigma_2)\alpha_2(\mu_2, \mu_1), \eta_1 + \sigma_3\alpha_3(\eta_2, \eta_1)\Big) =$$
$$w\Big(\zeta_1 + \sigma_1\alpha_1(\zeta_2, \zeta_1), \mu_1 + \sigma_2\alpha_2(\mu_2, \mu_1), \eta_1 + (1 - \sigma_3)\alpha_3(\eta_2, \eta_1)\Big) =$$
$$w\Big(\zeta_1 + \sigma_1\alpha_1(\zeta_2, \zeta_1), \mu_1 + (1 - \sigma_2)\alpha_2(\mu_2, \mu_1), \eta_1 + (1 - \sigma_3)\alpha_3(\eta_2, \eta_1)) =$$
$$w\Big(\zeta_1 + (1 - \sigma_1)\alpha_1(\zeta_2, \zeta_1), \mu_1 + \sigma_2\alpha_2(\mu_2, \mu_1), \eta_1 + (1 - \sigma_3)\alpha_3(\eta_2, \eta_1)\Big) =$$
$$w\Big(\zeta_1 + (1 - \sigma_1)\alpha_1(\zeta_2, \zeta_1), \mu_1 + (1 - \sigma_2)\alpha_2(\mu_2, \mu_1), \eta_1 + (1 - \sigma_3)\alpha_3(\eta_2, \eta_1)\Big),$$

and integrating over $[0, 1] \times [0, 1] \times [0, 1]$, we obtain

$$\chi(\zeta_1 + \frac{1}{2}\alpha_1(\zeta_2, \zeta_1), \mu_1 + \frac{1}{2}\alpha_2(\mu_2, \mu_1), \eta_1 + \frac{1}{2}\alpha_3(\eta_2, \eta_1))$$
$$\int_0^1 \int_0^1 \int_0^1 w(\zeta_1 + \sigma_1\alpha_1(\zeta_2, \zeta_1), \mu_1 + \sigma_2\alpha_2(\mu_2, \mu_1), \eta_1 + \sigma_3\alpha_3(\eta_2, \eta_1))d\sigma_1 d\sigma_2 d\sigma_3$$
$$= \chi(\zeta_1 + \frac{1}{2}\alpha_1(\zeta_2, \zeta_1), \mu_1 + \frac{1}{2}\alpha_2(\mu_2, \mu_1), \eta_1 + \frac{1}{2}\alpha_3(\eta_2, \eta_1)) \frac{1}{\alpha_1(\zeta_2, \zeta_1) \cdot \alpha_2(\mu_2, \mu_1) \cdot \alpha_3(\eta_2, \eta_1)}$$
$$\int_{\zeta_1}^{\zeta_1 + \alpha_1(\zeta_2, \zeta_1)} \int_{\mu_1}^{\mu_1 + \alpha_2(\mu_2, \mu_1)} \int_{\eta_1}^{\eta_1 + \alpha_3(\eta_2, \eta_1)} w(\zeta, \mu, \eta) d\zeta d\mu d\eta$$
$$\le 8h_1\left(\frac{1}{2}\right) h_2\left(\frac{1}{2}\right) h_3\left(\frac{1}{2}\right) \frac{1}{\alpha_1(\zeta_2, \zeta_1) \cdot \alpha_2(\mu_2, \mu_1) \cdot \alpha_3(\eta_2, \eta_1)}$$
$$\int_{\zeta_1}^{\zeta_1 + \alpha_1(\zeta_2, \zeta_1)} \int_{\mu_1}^{\mu_1 + \alpha_2(\mu_2, \mu_1)} \int_{\eta_1}^{\eta_1 + \alpha_3(\eta_2, \eta_1)} \chi(\zeta, \mu, \eta) \cdot w(\zeta, \mu, \eta) d\zeta d\mu d\eta,$$

which is the required inequality. □

Remark 2. If $\alpha_1(\zeta_2,\zeta_1)=\zeta_2-\zeta_1, \alpha_2(\mu_2,\mu_1)=\mu_2-\mu_1, \alpha_3(\eta_2,\eta_1)=\eta_2-\eta_1, h_1(\sigma_1)=h_2(\sigma_2)=h_3(\sigma_3)=\sigma$, then we get the following inequality for functions convex on the co-ordinates.

$$\begin{aligned}\chi\left(\frac{\zeta_1+\zeta_2}{2},\frac{\mu_1+\mu_2}{2},\frac{\eta_1+\eta_2}{2}\right)&\int_{\zeta_1}^{\zeta_2}\int_{\mu_1}^{\mu_2}\int_{\eta_1}^{\eta_2}w(\zeta,\mu,\eta)d\zeta d\mu d\eta\\ &\le\int_{\zeta_1}^{\zeta_2}\int_{\mu_1}^{\mu_2}\int_{\eta_1}^{\eta_2}\chi(\zeta,\mu,\eta)\cdot w(\zeta,\mu,\eta)d\zeta d\mu d\eta.\end{aligned}\tag{9}$$

Further if $w(\zeta,\mu,\eta)\equiv 1$, then

$$\begin{aligned}&\chi\left(\frac{\zeta_1+\zeta_2}{2},\frac{\mu_1+\mu_2}{2},\frac{\eta_1+\eta_2}{2}\right)\\ &\le\frac{1}{(\zeta_2-\zeta_1)(\mu_2-\mu_1)(\eta_2-\eta_1)}\int_{\zeta_1}^{\zeta_2}\int_{\mu_1}^{\mu_2}\int_{\eta_1}^{\eta_2}\chi(\zeta,\mu,\eta)d\zeta d\mu d\eta,\end{aligned}\tag{10}$$

which is analogous to the inequality (6) on convex functions on co-ordinates in two dimensions by Dragomir [5].

Remark 3. If $\alpha_1(\zeta_2,\zeta_1)=\zeta_2-\zeta_1, \alpha_2(\mu_2,\mu_1)=\mu_2-\mu_1, \alpha_3(\eta_2,\eta_1)=\eta_2-\eta_1, h_1(\sigma_1)=h_2(\sigma_2)=h_3(\sigma_3)=\sigma^s$, then we get the following inequality for functions s-convex on the co-ordinates.

$$\begin{aligned}\chi\left(\frac{\zeta_1+\zeta_2}{2},\frac{\mu_1+\mu_2}{2},\frac{\eta_1+\eta_2}{2}\right)&\int_{\zeta_1}^{\zeta_2}\int_{\mu_1}^{\mu_2}\int_{\eta_1}^{\eta_2}w(\zeta,\mu,\eta)d\zeta d\mu d\eta\\ &\le 8^{1-s}\int_{\zeta_1}^{\zeta_2}\int_{\mu_1}^{\mu_2}\int_{\eta_1}^{\eta_2}\chi(\zeta,\mu,\eta)\cdot w(\zeta,\mu,\eta)d\zeta d\mu d\eta.\end{aligned}\tag{11}$$

Further if $w(x,y,z)\equiv 1$, then

$$\begin{aligned}&8^{s-1}\chi\left(\frac{\zeta_1+\zeta_2}{2},\frac{\mu_1+\mu_2}{2},\frac{\eta_1+\eta_2}{2}\right)\\ &\le\frac{1}{(\zeta_2-\zeta_1)(\mu_2-\mu_1)(\eta_2-\eta_1)}\int_{\zeta_1}^{\zeta_2}\int_{\mu_1}^{\mu_2}\int_{\eta_1}^{\eta_2}\chi(\zeta,\mu,\eta)d\zeta d\mu d\eta,\end{aligned}\tag{12}$$

which is analogous to the inequality (3) on s-convex functions by Dragomir and Fitzpatrick [4] and inequality (7) on s-convex functions on co-ordinates in two dimensions by Alomari and Darus [1].

Theorem 2. *Let* $\chi:\Big[\zeta_1,\zeta_1+\alpha_1(\zeta_2,\zeta_1)v\Big]\times\Big[\mu_1,\mu_1+\alpha_2(\mu_2,\mu_1)\Big]\times\Big[\eta_1,\eta_1+\alpha_3(\eta_2,\eta_1)\Big]\to\mathbb{R}$ *with* $\zeta_1<\zeta_1+\alpha_1(\zeta_2,\zeta_1), \mu_1<\mu_1+\alpha_2(\mu_2,\mu_1)$ *and* $\eta_1<\eta_1+\alpha_2(\eta_2,\eta_1)$, *be* (h_1,h_2,h_3)*-preinvex on the co-ordinates with respect to* α_1, α_2 *and* α_3*;* $w:\Big[\zeta_1,\zeta_1+\alpha_1(\zeta_2,\zeta_1)\Big]\times\Big[\mu_1,\mu_1+\alpha_2(\mu_2,\mu_1)\Big]\times\Big[\eta_1,\eta_1+\alpha_3(\eta_2,\eta_1)\Big]\to\mathbb{R}, w\ge 0$, *symmetric with* *respect to* $\Big(\zeta_1 + \frac{1}{2}\alpha_1(\zeta_2, \zeta_1), \mu_1 + \frac{1}{2}\alpha_2(\mu_2, \mu_1), \eta_1 + \frac{1}{2}\alpha_3(\eta_2, \eta_1)\Big)$. *Then,*

$$\frac{1}{\alpha_1(\zeta_2, \zeta_1)\alpha_2(\mu_2, \mu_1)\alpha_3(\eta_2, \eta_1)} \int_{\zeta_1}^{\zeta_1+\alpha_1(\zeta_2,\zeta_1)} \int_{\mu_1}^{\mu_1+\alpha_1(\mu_2,\mu_1)} \int_{\eta_1}^{\eta_1+\alpha_1(\eta_2,\eta_1)} \chi(\zeta, \mu, \eta) \cdot$$
$$w(\zeta, \mu, \eta)\zeta d\mu d\eta \leq \Big\{\chi(\zeta_1, \mu_1, \eta_1) + \chi(\zeta_1, \mu_2, \eta_1) + \chi(\zeta_2, \mu_1, \eta_1) + \chi(\zeta_2, \mu_2, \eta_1)$$
$$+\chi(\zeta_1, \mu_1, \eta_2) + \chi(\zeta_1, \mu_2, \eta_2) + \chi(\zeta_2, \mu_1, \eta_2) + \chi(\zeta_2, \mu_2, \eta_2)\Big\}$$
$$\int_0^1 \int_0^1 \int_0^1 h_1(\sigma)h_2(\sigma)h_3(\sigma) \cdot w\Big(\zeta_1 + \sigma_1\alpha_1(\zeta_2, \zeta_1), \mu_1 + \sigma_2\alpha_2(\mu_2, \mu_1),$$
$$\eta_1 + \sigma_3\alpha_3(\eta_2, \eta_1)\Big) d\sigma_1 d\sigma_2 d\sigma_3.$$

Proof. From the definition of (h_1, h_2, h_3)-preinvex function on the co-ordinates with respect to α_1, α_2 and α_3, we have the following inequalities,

(a)

$$\chi\Big(\zeta_1 + \sigma_1\alpha_1(\zeta_2, \zeta_1), \mu_1 + \sigma_2\alpha_2(\mu_2, \mu_1), \eta_1 + \sigma_3\alpha_3(\eta_2, \eta_1)\Big) \leq h_1(1 - \sigma_1)$$
$$h_2(1 - \sigma_2)h_3(1 - \sigma_3)\chi(\zeta_1, \mu_1, \eta_1) + h_1(1 - \sigma_1)h_2(\sigma_2)h_3(1 - \sigma_3)\chi(\zeta_1, \mu_2, \eta_1)$$
$$+h_1(\sigma_1)h_2(1 - \sigma_2)h_3(1 - \sigma_3)\chi(\zeta_2, \mu_1, \eta_1) + h_1(\sigma_1)h_2(\sigma_2)h_3(1 - \sigma_3)\chi(\zeta_2, \mu_2, \eta_1)$$
$$+h_1(1 - \sigma_1)h_2(1 - \sigma_2)h_3(\sigma_3)\chi(\zeta_1, \mu_1, \eta_2) + h_1(1 - \sigma_1)h_2(\sigma_2)h_3(\sigma_3)\chi(\zeta_1, \mu_2, \eta_2)$$
$$+h_1(\sigma_1)h_2(1 - \sigma_2)h_3(\sigma_3)\chi(\zeta_2, \mu_1, \eta_2) + h_1(\sigma_1)h_2(\sigma_2)h_3(\sigma_3)\chi(\zeta_2, \mu_2, \eta_2),$$

(b)

$$\chi\Big(\zeta_1 + \sigma_1\alpha_1(\zeta_2, \zeta_1), \mu_1 + (1 - \sigma_2)\alpha_2(\mu_2, \mu_1), \eta_1 + \sigma_3\alpha_3(\eta_2, \eta_1)\Big) \leq h_1(1 - \sigma_1)$$
$$h_2(\sigma_2)h_3(1 - \sigma_3)\chi(\zeta_1, \mu_1, \eta_1) + h_1(1 - \sigma_1)h_2(1 - \sigma_2)h_3(1 - \sigma_3)\chi(\zeta_1, \mu_2, \eta_1)$$
$$+h_1(\sigma_1)h_2(\sigma_2)h_3(1 - \sigma_3)\chi(\zeta_2, \mu_1, \eta_1) + h_1(\sigma_1)h_2(1 - \sigma_2)h_3(1 - \sigma_3)\chi(\zeta_2, \mu_2, \eta_1)$$
$$+h_1(1 - \sigma_1)h_2(\sigma_2)h_3(\sigma_3)\chi(\zeta_1, \mu_1, \eta_2) + h_1(1 - \sigma_1)h_2(1 - \sigma_2)h_3(\sigma_3)\chi(\zeta_1, \mu_2, \eta_2)$$
$$+h_1(\sigma_1)h_2(\sigma_2)h_3(\sigma_3)\chi(\zeta_2, \mu_1, \eta_2) + h_1(\sigma_1)h_2(1 - \sigma_2)h_3(\sigma_3)\chi(\zeta_2, \mu_2, \eta_2),$$

(c)

$$\chi\Big(\zeta_1 + (1 - \sigma_1)\alpha_1(\zeta_2, \zeta_1), \mu_1 + \sigma_2\alpha_2(\mu_2, \mu_1), \eta_1 + \sigma_3\alpha_3(\eta_2, \eta_1)\Big) \leq h_1(\sigma_1)h_2(1 - \sigma_2)$$
$$h_3(1 - \sigma_3)\chi(\zeta_1, \mu_1, \eta_1) + h_1(\sigma_1)h_2(\sigma_2)h_3(1 - \sigma_3)\chi(\zeta_1, \mu_2, \eta_1) + h_1(1 - \sigma_1)h_2(1 - \sigma_2)$$
$$h_3(1 - \sigma_3)\chi(\zeta_2, \mu_1, \eta_1) + h_1(1 - \sigma_1)h_2(\sigma_2)h_3(1 - \sigma_3)\chi(\zeta_2, \mu_2, \eta_1) + h_1(\sigma_1)h_2(1 - \sigma_2)$$
$$h_3(\sigma_3)\chi(\zeta_1, \mu_1, \eta_2) + h_1(\sigma_1)h_2(\sigma_2)h_3(\sigma_3)\chi(\zeta_1, \mu_2, \eta_2) + h_1(1 - \sigma_1)h_2(1 - \sigma_2)h_3(\sigma_3)$$
$$\chi(\zeta_2, \mu_1, \eta_2) + h_1(1 - \sigma_1)h_2(\sigma_2)h_3(\sigma_3)\chi(\zeta_2, \mu_2, \eta_2),$$

(d)

$$\chi\Big(\zeta_1 + (1-\sigma_1)\alpha_1(\zeta_2,\zeta_1), \mu_1 + (1-\sigma_2)\alpha_2(\mu_2,\mu_1), \eta_1 + \sigma_3\alpha_3(\eta_2,\eta_1)\Big) \leq h_1(\sigma_1)h_2(\sigma_2) h_3(1-\sigma_3)\chi(\zeta_1,\mu_1,\eta_1) + h_1(\sigma_1)h_2(1-\sigma_2)h_3(1-\sigma_3)\chi(\zeta_1,\mu_2,\eta_1) + h_1(1-\sigma_1)h_2(\sigma_2) h_3(1-\sigma_3)\chi(\zeta_2,\mu_1,\eta_1) + h_1(1-\sigma_1)h_2(1-\sigma_2)h_3(1-\sigma_3)\chi(\zeta_2,\mu_2,\eta_1) + h_1(\sigma_1)h_2(\sigma_2) h_3(\sigma_3)\chi(\zeta_1,\mu_1,\eta_2) + h_1(\sigma_1)h_2(1-\sigma_2)h_3(\sigma_3)\chi(\zeta_1,\mu_2,\eta_2) + h_1(1-\sigma_1)h_2(\sigma_2) h_3(\sigma_3)\chi(\zeta_2,\mu_1,\eta_2) + h_1(1-\sigma_1)h_2(1-\sigma_2)h_3(\sigma_3)\chi(\zeta_2,\mu_2,\eta_2),$$

(e)

$$\chi\Big(\zeta_1 + \sigma_1\alpha_1(\zeta_2,\zeta_1), \mu_1 + \sigma_2\alpha_2(\mu_2,\mu_1), \eta_1 + (1-\sigma_3)\alpha_3(\eta_2,\eta_1)\Big) \leq h_1(1-\sigma_1)h_2(1-\sigma_2) h_3(\sigma_3)\chi(\zeta_1,\mu_1,\eta_1) + h_1(1-\sigma_1)h_2(\sigma_2)h_3(\sigma_3)\chi(\zeta_1,\mu_2,\eta_1) + h_1(\sigma_1)h_2(1-\sigma_2)h_3(\sigma_3) \chi(\zeta_2,\mu_1,\eta_1) + h_1(\sigma_1)h_2(\sigma_2)h_3(\sigma_3)\chi(\zeta_2,\mu_2,\eta_1) + h_1(1-\sigma_1)h_2(1-\sigma_2)h_3(1-\sigma_3) \chi(\zeta_1,\mu_1,\eta_2) + h_1(1-\sigma_1)h_2(\sigma_2)h_3(1-\sigma_3)\chi(\zeta_1,\mu_2,\eta_2) + h_1(\sigma_1)h_2(1-\sigma_2)h_3(1-\sigma_3) \chi(\zeta_2,\mu_1,\eta_2) + h_1(\sigma_1)h_2(\sigma_2)h_3(1-\sigma_3)\chi(\zeta_2,\mu_2,\eta_2),$$

(f)

$$\chi\Big(\zeta_1 + \sigma_1\alpha_1(\zeta_2,\zeta_1), \mu_1 + (1-\sigma_2)\alpha_2(\mu_2,\mu_1), \eta_1 + (1-\sigma_3)\alpha_3(\eta_2,\eta_1)\Big) \leq h_1(1-\sigma_1)h_2(\sigma_2) h_3(\sigma_3)\chi(\zeta_1,\mu_1,\eta_1) + h_1(1-\sigma_1)h_2(1-\sigma_2)h_3(\sigma_3)\chi(\zeta_1,\mu_2,\eta_1) + h_1(\sigma_1)h_2(\sigma_2)h_3(\sigma_3) \chi(\zeta_2,\mu_1,\eta_1) + h_1(\sigma_1)h_2(1-\sigma_2)h_3(\sigma_3)\chi(\zeta_2,\mu_2,\eta_1) + h_1(1-\sigma_1)h_2(\sigma_2)h_3(1-\sigma_3) \chi(\zeta_1,\mu_1,\eta_2) + h_1(1-\sigma_1)h_2(1-\sigma_2)h_3(1-\sigma_3)\chi(\zeta_1,\mu_2,\eta_2) + h_1(\sigma_1)h_2(\sigma_2)h_3(1-\sigma_3) \chi(\zeta_2,\mu_1,\eta_2) + h_1(\sigma_1)h_2(1-\sigma_2)h_3(1-\sigma_3)\chi(\zeta_2,\mu_2,\eta_2),$$

(g)

$$\chi\Big(\zeta_1 + (1-\sigma_1)\alpha_1(\zeta_2,\zeta_1), \mu_1 + \sigma_2\alpha_2(\mu_2,\mu_1), \eta_1 + (1-\sigma_3)\alpha_3(\eta_2,\eta_1)\Big) \leq h_1(\sigma_1)h_2(1-\sigma_2) h_3(\sigma_3)\chi(\zeta_1,\mu_1,\eta_1) + h_1(\sigma_1)h_2(\sigma_2)h_3(\sigma_3)\chi(\zeta_1,\mu_2,\eta_1) + h_1(1-\sigma_1)h_2(1-\sigma_2)h_3(\sigma_3) \chi(\zeta_2,\mu_1,\eta_1) + h_1(1-\sigma_1)h_2(\sigma_2)h_3(\sigma_3)\chi(\zeta_2,\mu_2,\eta_1) + h_1(\sigma_1)h_2(1-\sigma_2)h_3(1-\sigma_3) \chi(\zeta_1,\mu_1,\eta_2) + h_1(\sigma_1)h_2(\sigma_2)h_3(1-\sigma_3)\chi(\zeta_1,\mu_2,\eta_2) + h_1(1-\sigma_1)h_2(1-\sigma_2)h_3(1-\sigma_3) \chi(\zeta_2,\mu_1,\eta_2) + h_1(1-\sigma_1)h_2(\sigma_2)h_3(1-\sigma_3)\chi(\zeta_2,\mu_2,\eta_2),$$

(h)

$$\chi\Big(\zeta_1 + (1-\sigma_1)\alpha_1(\zeta_2,\zeta_1), \mu_1 + (1-\sigma_2)\alpha_2(\mu_2,\mu_1), \eta_1 + (1-\sigma_3)\alpha_3(\eta_2,\eta_1)\Big) \leq h_1(\sigma_1) h_2(\sigma_2)h_3(\sigma_3)\chi(\zeta_1,\mu_1,\eta_1) + h_1(\sigma_1)h_2(1-\sigma_2)h_3(\sigma_3)\chi(\zeta_1,\mu_2,\eta_1) + h_1(1-\sigma_1)h_2(\sigma_2) h_3(\sigma_3)\chi(\zeta_2,\mu_1,\eta_1) + h_1(1-\sigma_1)h_2(1-\sigma_2)h_3(\sigma_3)\chi(\zeta_2,\mu_2,\eta_1) + h_1(\sigma_1)h_2(\sigma_2)h_3(1-\sigma_3) \chi(\zeta_1,\mu_1,\eta_2) + h_1(\sigma_1)h_2(1-\sigma_2)h_3(1-\sigma_3)\chi(\zeta_1,\mu_2,\eta_2) + h_1(1-\sigma_1)h_2(\sigma_2)h_3(1-\sigma_3) \chi(\zeta_2,\mu_1,\eta_2) + h_1(1-\sigma_1)h_2(1-\sigma_2)h_3(1-\sigma_3)\chi(\zeta_2,\mu_2,\eta_2).$$

Multiplying both sides of the preceding inequalities by
$w\Big(\zeta_1 + \sigma_1\alpha_1(\zeta_2,\zeta_1), \mu_1 + \sigma_2\alpha_2(\mu_2,\mu_1), \eta_1 + \sigma_3\alpha_3(\eta_2,\eta_1)\Big)$,
$w\Big(\zeta_1 + \sigma_1\alpha_1(\zeta_2,\zeta_1), \mu_1 + (1-\sigma_2)\alpha_2(\mu_2,\mu_1), \eta_1 + \sigma_3\alpha_3(\eta_2,\eta_1)\Big)$,

$$w\Big(\zeta_1 + (1-\sigma_1)\alpha_1(\zeta_2,\zeta_1), \mu_1 + \sigma_2\alpha_2(\mu_2,\mu_1), \eta_1 + \sigma_3\alpha_3(\eta_2,\eta_1)\Big),$$
$$w\Big(\zeta_1 + (1-\sigma_1)\alpha_1(\zeta_2,\zeta_1), \mu_1 + (1-\sigma_2)\alpha_2(\mu_2,\mu_1), \eta_1 + \sigma_3\alpha_3(\eta_2,\eta_1)\Big),$$
$$w\Big(\zeta_1 + \sigma_1\alpha_1(\zeta_2,\zeta_1), \mu_1 + \sigma_2\alpha_2(\mu_2,\mu_1), \eta_1 + (1-\sigma_3)\alpha_3(\eta_2,\eta_1)\Big),$$
$$w\Big(\zeta_1 + \sigma_1\alpha_1(\zeta_2,\zeta_1), \mu_1 + (1-\sigma_2)\alpha_2(\mu_2,\mu_1), \eta_1 + (1-\sigma_3)\alpha_3(\eta_2,\eta_1)\Big),$$
$$w\Big(\zeta_1 + (1-\sigma_1)\alpha_1(\zeta_2,\zeta_1), \mu_1 + \sigma_2\alpha_2(\mu_2,\mu_1), \eta_1 + (1-\sigma_3)\alpha_3(\eta_2,\eta_1)\Big),$$
$$w\Big(\zeta_1 + (1-\sigma_1)\alpha_1(\zeta_2,\zeta_1), \mu_1 + (1-\sigma_2)\alpha_2(\mu_2,\mu_1), \eta_1 + (1-\sigma_3)\alpha_3(\eta_2,\eta_1)\Big)$$

respectively, adding and integrating over $[0,1]\times[0,1]\times[0,1]$, we obtain the required inequality. □

THEOREM 3. *Let* $\chi, \pi : [\zeta_1, \zeta_1 + \alpha_1(\zeta_2,\zeta_1)] \times [\mu_1, \mu_1 + \alpha_1(\mu_2,\mu_1)] \times [\eta_1, \eta_1 + \alpha_1(\eta_2,\eta_1)] \to \mathbb{R}$ *with* $\zeta_1 < \zeta_1 + \alpha_1(\zeta_2,\zeta_1)$, $\mu_1 < \mu_1 + \alpha_2(\mu_2,\mu_1)$ *and* $\eta_1 < \eta_1 + \alpha_2(\eta_2,\eta_1)$. *If* χ *is* (h_1,h_2,h_3)*-preinvex on the co-ordinates and* π *is* (k_1,k_2,k_3)*-preinvex on the co-ordinates with respect* α_1, α_2 *and* α_3*, then*

$$\begin{aligned}
&\frac{1}{\alpha_1(\zeta_2,\zeta_1)\alpha_2(\mu_2,\mu_1)\alpha_3(\eta_2,\eta_1))}\\
&\qquad\int_{\zeta}^{\zeta+\alpha_1(\zeta_2,\zeta_1)}\int_{\mu}^{\mu+\alpha_2(\mu_2,\mu_1)}\int_{\eta}^{\eta+\alpha_(\eta_2,\eta_1)} \chi(\zeta,\mu,\eta)\cdot\pi(\zeta,\mu,\eta)d\zeta d\mu d\eta\\
&\le M_1\int_0^1\int_0^1\int_0^1 h_1(\sigma_1)h_2(\sigma_2)h_3(\sigma_3)k_1(\sigma_1)k_2(\sigma_2)k_3(\sigma_3)d\sigma_1 d\sigma_2 d\sigma_3\\
&+M_2\int_0^1\int_0^1\int_0^1 h_1(\sigma_1)h_2(\sigma_2)h_3(\sigma_3)k_1(\sigma_1)k_2(1-\sigma_2)k_3(\sigma_3)d\sigma_1 d\sigma_2 d\sigma_3\\
&+M_3\int_0^1\int_0^1\int_0^1 h_1(\sigma_1)h_2(\sigma_2)h_3(\sigma_3)k_1(1-\sigma_1)k_2(\sigma_2)k_3(\sigma_3)d\sigma_1 d\sigma_2 d\sigma_3\\
&+M_4\int_0^1\int_0^1\int_0^1 h_1(\sigma_1)h_2(\sigma_2)h_3(\sigma_3)k_1(1-\sigma_1)k_2(1-\sigma_2)k_3(\sigma_3)d\sigma_1 d\sigma_2 d\sigma_3\\
&+M_5\int_0^1\int_0^1\int_0^1 h_1(\sigma_1)h_2(\sigma_2)h_3(\sigma_3)k_1(\sigma_1)k_2(\sigma_2)k_3(1-\sigma_3)d\sigma_1 d\sigma_2 d\sigma_3\\
&+M_6\int_0^1\int_0^1\int_0^1 h_1(\sigma_1)h_2(\sigma_2)h_3(\sigma_3)k_1(\sigma_1)k_2(1-\sigma_2)k_3(1-\sigma_3)d\sigma_1 d\sigma_2 d\sigma_3\\
&+M_7\int_0^1\int_0^1\int_0^1 h_1(\sigma_1)h_2(\sigma_2)h_3(\sigma_3)k_1(1-\sigma_1)k_2(\sigma_2)k_3(1-1-\sigma_3)d\sigma_1 d\sigma_2 d\sigma_3\\
&+M_8\int_0^1\int_0^1\int_0^1 h_1(\sigma_1)h_2(\sigma_2)h_3(\sigma_3)k_1(1-\sigma_1)k_2(1-\sigma_2)k_3(1-\sigma_3)d\sigma_1 d\sigma_2 d\sigma_3
\end{aligned}$$

where,

$$\begin{aligned}
M_1 &= \chi(\zeta_1,\mu_1,\eta_1)\pi(\zeta_1,\mu_1,\eta_1) + \chi(\zeta_1,\mu_2,\eta_1)\pi(\zeta_1,\mu_2,\eta_1) + \chi(\zeta_2,\mu_1,\eta_1)\pi(\zeta_2,\mu_1,\eta_1)\\
&\quad+\chi(\zeta_2,\mu_2,\eta_1)\pi(\zeta_2,\mu_2,\eta_1) + \chi(\zeta_1,\mu_1,\eta_2)\pi(\zeta_1,\mu_1,\eta_2) + \chi(\zeta_1,\mu_2,\eta_2)\pi(\zeta_1,\mu_2,\eta_2)\\
&\quad+\chi(\zeta_2,\mu_1,\eta_2)\pi(\zeta_2,\mu_1,\eta_2) + \chi(\zeta_2,\mu_2,\eta_2)\pi(\zeta_2,\mu_2,\eta_2),
\end{aligned}$$

$$M_2 = \chi(\zeta_1,\mu_1,\eta_1)\pi(\zeta_1,\mu_2,\eta_1) + \chi(\zeta_1,\mu_2,\eta_1)\pi(\zeta_1,\mu_1,\eta_1) + \chi(\zeta_2,\mu_1,\eta_1)\pi(\zeta_2,\mu_2,\eta_1) + \chi(\zeta_2,\mu_2,\eta_1)\pi(\zeta_2,\mu_1,\eta_1) + \chi(\zeta_1,\mu_1,\eta_2)\pi(\zeta_1,\mu_2,\eta_2) + \chi(\zeta_1,\mu_2,\eta_2)\pi(\zeta_1,\mu_1,\eta_2) + \chi(\zeta_2,\mu_1,\eta_2)\pi(\zeta_2,\mu_2,\eta_2) + \chi(\zeta_2,\mu_2,\eta_2)\pi(\zeta_2,\mu_1,\eta_2),$$

$$M_3 = \chi(\zeta_1,\mu_1,\eta_1)\pi(\zeta_2,\mu_1,\eta_1) + \chi(\zeta_1,\mu_2,\eta_1)\pi(\zeta_2,\mu_2,\eta_1) + \chi(\zeta_2,\mu_1,\eta_1)\pi(\zeta_1,\mu_1,\eta_1) + \chi(\zeta_2,\mu_2,\eta_1)\pi(\zeta_1,\mu_2,\eta_1) + \chi(\zeta_1,\mu_1,\eta_2)\pi(\zeta_2,\mu_1,\eta_2) + \chi(\zeta_1,\mu_2,\eta_2)\pi(\zeta_2,\mu_2,\eta_2) + \chi(\zeta_2,\mu_1,\eta_2)\pi(\zeta_1,\mu_1,\eta_2) + \chi(\zeta_2,\mu_2,\eta_2)\pi(\zeta_1,\mu_2,\eta_2),$$

$$M_4 = \chi(\zeta_1,\mu_1,\eta_1)\pi(\zeta_2,\mu_2,\eta_1) + \chi(\zeta_1,\mu_2,\eta_1)\pi(\zeta_2,\mu_1,\eta_1) + \chi(\zeta_2,\mu_1,\eta_1)\pi(\zeta_1,\mu_2,\eta_1) + \chi(\zeta_2,\mu_2,\eta_1)\pi(\zeta_1,\mu_1,\eta_1) + \chi(\zeta_1,\mu_1,\eta_2)\pi(\zeta_2,\mu_2,\eta_2) + \chi(\zeta_1,\mu_2,\eta_2)\pi(\zeta_2,\mu_1,\eta_2) + \chi(\zeta_2,\mu_1,\eta_2)\pi(\zeta_1,\mu_2,\eta_2) + \chi(\zeta_2,\mu_2,\eta_2)\pi(\zeta_1,\mu_1,\eta_2),$$

$$M_5 = \chi(\zeta_1,\mu_1,\eta_1)\pi(\zeta_1,\mu_1,\eta_2) + \chi(\zeta_1,\mu_2,\eta_1)\pi(\zeta_1,\mu_2,\eta_2) + \chi(\zeta_2,\mu_1,\eta_1)\pi(\zeta_2,\mu_1,\eta_2) + \chi(\zeta_2,\mu_2,\eta_1)\pi(\zeta_2,\mu_2,\eta_2) + \chi(\zeta_1,\mu_1,\eta_2)\pi(\zeta_1,\mu_1,\eta_1) + \chi(\zeta_1,\mu_2,\eta_2)\pi(\zeta_1,\mu_2,\eta_1) + \chi(\zeta_2,\mu_1,\eta_2)\pi(\zeta_2,\mu_1,\eta_1) + \chi(\zeta_2,\mu_2,\eta_2)\pi(\zeta_2,\mu_2,\eta_1),$$

$$M_6 = \chi(\zeta_1,\mu_1,\eta_1)\pi(\zeta_1,\mu_2,\eta_2) + \chi(\zeta_1,\mu_2,\eta_1)\pi(\zeta_1,\mu_1,\eta_2) + \chi(\zeta_2,\mu_1,\eta_1)\pi(\zeta_2,\mu_2,\eta_2) + \chi(\zeta_2,\mu_2,\eta_1)\pi(\zeta_2,\mu_1,\eta_2) + \chi(\zeta_1,\mu_1,\eta_2)\pi(\zeta_1,\mu_2,\eta_1) + \chi(\zeta_1,\mu_2,\eta_2)\pi(\zeta_1,\mu_1,\eta_1) + \chi(\zeta_2,\mu_1,\eta_2)\pi(\zeta_2,\mu_2,\eta_1) + \chi(\zeta_2,\mu_2,\eta_2)\pi(\zeta_2,\mu_1,\eta_1),$$

$$M_7 = \chi(\zeta_1,\mu_1,\eta_1)\pi(\zeta_2,\mu_1,\eta_2) + \chi(\zeta_1,\mu_2,\eta_1)\pi(\zeta_2,\mu_2,\eta_2) + \chi(\zeta_2,\mu_1,\eta_1)\pi(\zeta_1,\mu_1,\eta_2) + \chi(\zeta_2,\mu_2,\eta_1)\pi(\zeta_1,\mu_2,\eta_2) + \chi(\zeta_1,\mu_1,\eta_2)\pi(\zeta_2,\mu_1,\eta_1) + \chi(\zeta_1,\mu_2,\eta_2)\pi(\zeta_2,\mu_2,\eta_1) + \chi(\zeta_2,\mu_1,\eta_2)\pi(\zeta_1,\mu_1,\eta_1) + \chi(\zeta_2,\mu_2,\eta_2)\pi(\zeta_1,\mu_2,\eta_1)$$

and

$$M_8 = \chi(\zeta_1,\mu_1,\eta_1)\pi(\zeta_2,\mu_2,\eta_2) + \chi(\zeta_1,\mu_2,\eta_1)\pi(\zeta_2,\mu_1,\eta_2) + \chi(\zeta_2,\mu_1,\eta_1)\pi(\zeta_1,\mu_2,\eta_2) + \chi(\zeta_2,\mu_2,\eta_1)\pi(\zeta_1,\mu_1,\eta_2) + \chi(\zeta_1,\mu_1,\eta_2)\pi(\zeta_2,\mu_2,\eta_1) + \chi(\zeta_1,\mu_2,\eta_2)\pi(\zeta_2,\mu_1,\eta_1) + \chi(\zeta_2,\mu_1,\eta_2)\pi(\zeta_1,\mu_2,\eta_1) + \chi(\zeta_2,\mu_2,\eta_2)\pi(\zeta_1,\mu_1,\eta_1).$$

Proof. Since χ is (h_1,h_2,h_3)-preinvex on the co-ordinates and π is (k_1,k_2,k_3)-preinvex on the co-ordinates with respect to α_1,α_2 and α_3, it follows that

$$\chi(\zeta_1+\sigma_1\alpha_1(\zeta_2,\zeta_1),\mu_1+\sigma_2\alpha(\mu_2,\mu_1),\eta_1+\sigma_3\alpha_3(\eta_2,\eta_1)) \le h_1(1-\sigma_1)h_2(\sigma_2)h_3(1-\sigma_3)\chi(\zeta_1,\mu_2,\eta_1) + h_1(\sigma_1)h_2(1-\sigma_2)h_3(1-\sigma_3)\chi(\zeta_2,\mu_1,\eta_1) + h_1(\sigma_1)h_2(\sigma_2)h_3(1-\sigma_3)\chi(\zeta_2,\mu_2,\eta_1) + h_1(1-\sigma_1)h_2(1-\sigma_2)h_3(\sigma_3)\chi(\zeta_1,\mu_1,\eta_2) + h_1(1-\sigma_1)h_2(\sigma_2)h_3(\sigma_3)\chi(\zeta_1,\mu_2,\eta_2) + h_1(\sigma_1)h_2(1-\sigma_2)h_3(\sigma_3)\chi(\zeta_2,\mu_1,\eta_2) + h_1(\sigma_1)h_2(\sigma_2)h_3(\sigma_3)\chi(\zeta_2,\mu_2,\eta_2)$$

and

$$\pi(\zeta_1+\sigma_1\alpha_1(\zeta_2,\zeta_1),\mu_1+\sigma_2\alpha(\mu_2,\mu_1),\eta_1+\sigma_3\alpha_3(\eta_2,\eta_1)) \le k_1(1-\sigma_1)k_2(\sigma_2)k_3(1-\sigma_3)\pi(\zeta_1,\mu_2,\eta_1) + k_1(\sigma_1)k_2(1-\sigma_2)k_3(1-\sigma_3)\pi(\zeta_2,\mu_1,\eta_1) + k_1(\sigma_1)k_2(\sigma_2)k_3(1-\sigma_3)\pi(\zeta_2,\mu_2,\eta_1) + k_1(1-\sigma_1)k_2(1-\sigma_2)k_3(\sigma_3)\pi(\zeta_1,\mu_1,\eta_2) + k_1(1-\sigma_1)k_2(\sigma_2)k_3(\sigma_3)\pi(\zeta_1,\mu_2,\eta_2) + k_1(\sigma_1)k_2(1-\sigma_2)k_3(\sigma_3)\pi(\zeta_2,\mu_1,\eta_2) + k_1(\sigma_1)k_2(\sigma_2)k_3(\sigma_3)\pi(\zeta_2,\mu_2,\eta_2).$$

Multiplying the preceding inequalities and integrating over $[0,1]\times[0,1]\times[0,1]$ and using the equality

$$\int_0^1\int_0^1\int_0^1 \chi\Big(\zeta_1+\sigma_1\alpha_1(\zeta_2,\zeta_1), \mu_1+\sigma_2\alpha_2(\mu_2,\mu_1), \eta_1+\sigma_3\alpha_3(\eta_2,\eta_1)\Big)\cdot$$
$$\pi\Big(\zeta_1+\sigma_1\alpha_1(\zeta_2,\zeta_1), \mu_1+\sigma_2\alpha_2(\mu_2,\mu_1), \eta_1+\sigma_3\alpha_3(\eta_2,\eta_1)\Big)d\sigma_1 d\sigma_2 d\sigma_3 =$$
$$\frac{1}{\alpha_1(\zeta_2,\zeta_1)\alpha_2(\mu_2,\mu_1)\alpha_3(\eta_2,\eta_1)}\int_{\zeta_1}^{\zeta_1+\alpha(\zeta_2,\zeta_1)}\int_{\mu_1}^{\mu_1+\alpha_2(\mu_2,\mu_1)}\int_{\eta_1}^{\eta_1+\alpha_3(\eta_2,\eta_1)}$$
$$\chi(\zeta,\mu,\zeta)\cdot\pi(\zeta,\mu,\zeta)d\zeta d\mu d\eta,$$

we obtain our required inequality. □

4. H function and its properties

In this section we discuss a closely related function to convex and preinvex functions on the co-ordinates, namely the H function and derive some key connecting results. Now for a function $\chi : [\zeta_1,\zeta_2]\times[\mu_1,\mu_2]\times[\eta_1,\eta_2]\to\mathbb{R}$, let us define a mapping $H : [0,1]\times[0,1]\times[0,1]\to\mathbb{R}$ in the following way:

$$H(t,r,m) = \frac{1}{(\zeta_2-\zeta_1)(\mu_2-\mu_1)(\eta_2-\eta_1)}\int_{\zeta_1}^{\zeta_2}\int_{\mu_1}^{\mu_2}\int_{\eta_1}^{\eta_2}\chi\Big(\zeta+(1-t)\frac{\zeta_1+\zeta_2}{2},$$
$$r\mu+(1-r)\frac{\mu_1+\mu_2}{2}, m\eta+(1-m)\frac{\eta_1+\eta_2}{2}\Big)d\zeta d\mu d\eta$$

Note that,

$$H(0,0,0) = \chi\left(\frac{\zeta_1+\zeta_2}{2}, \frac{\mu_1+\mu_2}{2}, \frac{\eta_1+\eta_2}{2}\right)$$

and

$$H(1,1,1) = \frac{1}{(\zeta_2-\zeta_1)(\mu_2-\mu_1)(\eta_2-\eta_1)}\int_{\zeta_1}^{\zeta_2}\int_{\mu_1}^{\mu_2}\int_{\eta_1}^{\eta_2}\chi(\zeta,\mu,\eta)d\zeta d\mu d\eta.$$

In [1], [5] and [8], some features of an analogous mapping are provided for a convex on the co-ordinates function, s-convex on the co-ordinates function and (h_1,h_2)-convex on the co-ordinates functions respectively. Here we explore which of these qualities can be generalised for (h_1,h_2,h_3)-convex on the co-ordinates functions.

Theorem 4. *Suppose that $\chi : [\zeta_1,\zeta_2]\times[\mu_1,\mu_2]\times[\eta_1,\eta_2]\to\mathbb{R}$ is (h_1,h_2,h_3)-convex on the co-ordinates. Then,*
(a). The mapping H is (h_1,h_2,h_3)-convex on the co-ordinates on $[0,1]\times[0,1]\times[0,1]$.
(b). $H(0,0,0)\le 8h_1\big(\frac{1}{2}\big)h_2\big(\frac{1}{2}\big)h_3\big(\frac{1}{2}\big)H(t,r,m)$ for any $(t,r,m)\in[0,1]\times[0,1]\times[0,1]$.

Proof. (a).The (h_1,h_2,h_3)-convexity on the co-ordinates of the mapping H is a result of the (h_1,h_2,h_3)-convexity on the co-ordinates of the f. In other words, for

$r, m \in [0,1]$ and for all $\alpha, \beta \geq 0$ with $\alpha + \beta = 1$ and $t_1, t_2 \in [0,1]$, we have

$$\begin{aligned}
H(\alpha t_1 + \beta t_2, r, m) &= \frac{1}{(\zeta_2 - \zeta_1)(\mu_2 - \mu_1)(\eta_2 - \eta_1)} \int_{\zeta_1}^{\zeta_2} \int_{\mu_1}^{\mu_2} \int_{\eta_1}^{\eta_2} \\
&\quad \chi\Big((\alpha t_1 + \beta t_2)\zeta + (1 - (\alpha t_1 + \beta t_2))\frac{\zeta_1 + \zeta_2}{2}, r\mu + (1-r)\frac{\mu_1 + \mu_2}{2}, \\
&\qquad m\eta + (1-m)\frac{\eta_1 + \eta_2}{2}\Big) d\zeta d\mu d\eta \\
&= \frac{1}{\zeta_2 - \zeta_1)(\mu_2 - \mu_1)(\eta_2 - \eta_1)} \int_{\zeta_1}^{\zeta_2} \int_{\mu_1}^{\mu_2} \int_{\eta_1}^{\eta_2} \chi\Big(\alpha\{t_1\zeta + (1-t_1)\frac{\zeta_1 + \zeta_2}{2}\} + \\
&\quad \beta\{t_2\zeta + (1-t_2)\frac{\zeta_1 + \zeta_2}{2}\}, r\mu + (1-r)\frac{\mu_1 + \mu_2}{2}, m\eta + (1-m)\frac{\eta_1 + \eta_2}{2}\Big) d\zeta d\mu d\eta \\
&\leq \frac{h_1(\alpha)}{(\zeta_2 - \zeta_1)(\mu_2 - \mu_1)(\eta_2 - \eta_1)} \int_{\zeta_1}^{\zeta_2} \int_{\mu_1}^{\mu_2} \int_{\eta_1}^{\eta_2} \chi\Big(t_1\zeta + (1-t_1)\frac{\zeta_1 + \zeta_2}{2}, \\
&\quad r\mu + (1-r)\frac{\mu_1 + \mu_2}{2}, m\eta + (1-m)\frac{\eta_1 + \eta_2}{2}\Big) d\zeta d\mu d\eta \\
&\quad + \frac{h_1(\beta)}{(\zeta_2 - \zeta_1)(\mu_2 - \mu_1)(\eta_2 - \eta_1)} \int_{\zeta_1}^{\zeta_2} \int_{\mu_1}^{\mu_2} \int_{\eta_1}^{\eta_2} \chi\Big(t_2\zeta + (1-t_2)\frac{\zeta_1 + \zeta_2}{2}, \\
&\quad r\mu + (1-r)\frac{\mu_1 + \mu_2}{2}, m\eta + (1-m)\frac{\eta_1 + \eta_2}{2}\Big) d\zeta d\mu d\eta \\
&= h_1(\alpha) H(t_1, r, m) + h_2(\beta) H(t_2, r, m).
\end{aligned}$$

Similarly, for $t, m \in [0,1]$ and for all $\alpha, \beta \geq 0$ with $\alpha + \beta = 1$ and $r_1, r_2 \in [0,1]$,
$H(t, \alpha r_1 + \beta r_2, m) = h_2(\alpha) H(t, r_1, m) + h_2(\beta) H(t, r_2, m)$
and for $t, r \in [0,1]$ and for all $\alpha, \beta \geq 0$ with $\alpha + \beta = 1$ and $m_1, m_2 \in [0,1]$,
$H(t, r, \alpha m_1 + \beta m_2) = h_3(\alpha) H(t, r, m_1) + h_3(\beta) H(t, r, m_2)$.

(b). In the definition of $H(t, r, m)$, we make the following change of variables:

$$x = t\zeta + (1-t)\frac{\zeta_1 + \zeta_2}{2}, y = r\mu + (1-r)\frac{\mu_1 + \mu_2}{2} \quad \& z = m\eta + (1-m)\frac{\eta_1 + \eta_2}{2}.$$

Also setting,

$$\begin{aligned}
x_1 = t\zeta_1 + (1-t)\frac{\zeta_1 + \zeta_2}{2}, x_2 = t\zeta_2 + (1-t)\frac{\zeta_1 + \zeta_2}{2}, \\
y_1 = r\mu_1 + (1-r)\frac{\mu_1 + \mu_2}{2}, y_2 = r\mu_2 + (1-r)\frac{\mu_1 + \mu_2}{2}, \\
z_1 = m\eta_1 + (1-m)\frac{\eta_1 + \eta_2}{2}, z_2 = m\eta_2 + (1-m)\frac{\eta_1 + \eta_2}{2},
\end{aligned}$$

we have

$$\begin{aligned}
H(t, r, m) &= \frac{1}{(u_2 - u_1)(v_2 - v_1)(w_2 - w_1)} \int_{u_1}^{u_2} \int_{v_1}^{v_2} \int_{w_1}^{w_2} \chi(u, v, w) du dv dw \\
&\geq \frac{1}{8h_1(\frac{1}{2})h_2(\frac{1}{2})h_3(\frac{1}{2})} \chi\left(\frac{\zeta_1 + \zeta_2}{2}, \frac{\mu_1 + \mu_2}{2}, \frac{\eta_1 + \eta_2}{2}\right).
\end{aligned}$$

□

Remark 4. If χ is convex on the co-ordinates, then $h(t) = t$ and we have the inequality $H(t, r, m) \geq H(0, 0, 0)$, which leads us back to the inequality (10) of Remark 2.

Remark 5. If χ is s-convex on the co-ordinates in the second sense, then $h(t) = t^s$ and we have the inequality $H(t, r, m) \geq 8^{s-1} H(0, 0, 0)$, which leads us back to the inequality (12) of Remark 3.

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