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SUFFICIENT CONDITIONS FOR STARLIKENESS OF RECIPROCAL ORDER

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Abstract. A normalized analytic function f defined on the unit disk $\mathbb{D}$ is starlike of reciprocal order α, $0 \leq \alpha < 1$, if $\operatorname{Re}(f(z)/(zf'(z))) > \alpha$ for all $z \in \mathbb{D}$. Such functions are starlike and therefore univalent in $\mathbb{D}$. Using the well-known Miller-Mocanu differential subordination theory, sufficient conditions involving differential inclusions are obtained for a normalized analytic function to be starlike of reciprocal order α. Furthermore, a few conditions are derived for a function f to belong to a subclass of reciprocal starlike functions, satisfying $|f(z)/(zf'(z)) - 1| < 1 - \alpha$.

1. Introduction

Let $\mathbb{D}_r$ be the open disk in the complex plane with center at the origin and radius r and let $\mathbb{D} := \mathbb{D}_1$. In view of Riemann mapping theorem, we shall restrict our focus to the class $\mathcal{H}$ of all analytic functions defined on $\mathbb{D}$. For $a \in \mathbb{C}$ and $n \in \mathbb{N}$, let $\mathcal{H}[a, n] = \left\{f \in \mathcal{H} : f(z) = a + \sum_{k=n}^{\infty} a_k z^k\right\}$. Let $\mathcal{A}$ be the class of functions in $\mathcal{H}[0, 1]$ satisfying $f'(0) = 1$ and its subclass consisting of univalent functions is denoted by $\mathcal{S}$. Many subclasses of $\mathcal{S}$ are characterized by some geometric property such as convexity or starlikeness of the image of $\mathbb{D}$ under the mapping $Q_{ST}(z) := zf'(z)/f(z)$ or $Q_{CV}(z) := 1+zf''(z)/f'(z)$. A set D in the complex plane is said to be starlike with respect to its interior point w_0 if the line segments joining w_0 to any other points of the set are contained in D. A function $f \in \mathcal{S}$ is said to be starlike if the image domain $f(\mathbb{D})$ is starlike with respect to the origin and the class of all such functions is denoted by $\mathcal{ST}$. The functions in this class are characterized by the property $\operatorname{Re}(Q_{ST}(z)) > 0$. A class which is closely associated with the class $\mathcal{ST}$ is the class $\mathcal{P}$ of all analytic functions $p(z) = 1+\sum_{k=1}^{\infty} c_k z^k$ in $\mathbb{D}$ with positive real part. For $0 \leq \alpha < 1$, let $\mathcal{ST}(\alpha)$ be the subclass of $\mathcal{ST}$ consisting of functions characterized by $\operatorname{Re}(Q_{ST}(z)) > \alpha$. A function $f \in \mathcal{S}$ satisfying $\operatorname{Re}(1/Q_{ST}(z)) > \alpha$ $(0 \leq \alpha < 1)$ for every $z \in \mathbb{D}$ are said to be starlike of reciprocal order α and the class of all such functions is denoted by $\mathcal{RST}(\alpha)$. This class has been studied by many authors in [1–5, 7, 11, 15–17, 21, 24].

Let $\mathcal{B}$ denote the class of all analytic functions $w : \mathbb{D} \to \mathbb{D}$ which fixes the origin. For $f, g \in \mathcal{H}$, the function f is said to be subordinate to the function g if there exists

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a function $w \in \mathcal{B}$ such that $f(z) = g(w(z))$ for all $z \in \mathbb{D}$, and it is written as $f \prec g$. The general theory of second order differential subordinations is developed by Miller and Mocanu [12] in which for given domains Ω, Δ in $\mathbb{C}$ and an analytic function $p : \mathbb{D} \to \mathbb{C}$, they determined the class of functions $\psi : \mathbb{C}^3 \times \mathbb{D} \to \mathbb{C}$ for which the following implication holds.

$$\left\{\psi\left(p(z), zp'(z), z^2p''(z); z\right) : z \in \mathbb{D}\right\} \subset \Omega \implies \{p(z) : z \in \mathbb{D}\} \subset \Delta. \tag{1}$$

If $\Delta \neq \mathbb{C}$ is a simply connected domain in $\mathbb{C}$, then Riemann mapping theorem ensures the existence of a function q which maps $\mathbb{D}$ univalently onto Δ, satisfying $q(0) = p(0)$. Then the implication (1) can be rewritten as

$$\left\{\psi\left(p(z), zp'(z), z^2p''(z); z\right) : z \in \mathbb{D}\right\} \subset \Omega \implies p \prec q.$$

The analytic functions q that are univalent on $\overline{\mathbb{D}} \backslash E(q)$ where

$$E(q) = \{\zeta \in \partial\mathbb{D} : q(z) \to \infty \text{ as } z \to \zeta\}$$

are called the functions with nice boundary and this class of all functions with nice boundary is denoted by $\mathcal{Q}$.

Definition 1.1. [14] Let Ω be a subset of $\mathbb{C}$, q be a function with nice boundary and n be a positive integer. The class of admissible functions $\Psi_n(\Omega, q)$ consists of all functions $\psi : \mathbb{C}^3 \times \mathbb{D} \to \mathbb{C}$ satisfying the admissibility condition $\psi(r, s, t; z) \notin \Omega$ when $r = q(\zeta)$, $s = m\zeta q'(\zeta)$, $\operatorname{Re}(t/s) + 1 \geq m \operatorname{Re}(\zeta q''(\zeta)/q'(\zeta) + 1)$ for $\zeta \in \partial\mathbb{D} \backslash E(q)$ and $m \geq n$. Further, let $\Psi(\Omega, q) := \Psi_1(\Omega, q)$.

The following theorem of Miller and Mocanu serves the base for the first and second order differential subordination theory.

Theorem 1.2 (Miller-Mocanu Theorem). *Let $q \in \mathcal{Q}$ with $q(0) = a$, $\Omega \subset \mathbb{C}$ and let $\psi \in \Psi_n(\Omega, q)$. If the function $p \in \mathcal{H}[a, n]$ satisfies the differential inclusion*

$$\left\{\psi(p(z), zp'(z), z^2p''(z); z) : z \in \mathbb{D}\right\} \subset \Omega,$$

then $p \prec q$.

In [13], the authors investigated the first order differential subordinations by considering $\psi : \mathbb{C}^2 \times \mathbb{D} \to \mathbb{C}$. For first order differential subordinations, the admissibility condition in Definition 1.1 becomes $\psi(r, s; z) \notin \Omega$ where $r = q(\zeta)$, $s = m\zeta q'(\zeta)$ for $\zeta \in \partial\mathbb{D} \backslash E(q)$ and $m \geq n$. Moreover, Miller-Mocanu theorem can be restated as follows. Let $q \in \mathcal{Q}$ with $q(0) = a$ and let $\psi \in \Psi_n(\Omega, q)$. If $p \in \mathcal{H}[a, n]$ satisfies $\psi(p(z), zp'(z); z) \in \Omega$ for every $z \in \mathbb{D}$, then $p \prec q$. Using the theory of differential subordinations, sufficient conditions for the starlikeness of functions are studied by various authors [8–10, 18, 20, 22, 25].

Certain differential inequalities which serve as sufficient conditions for starlikeness of reciprocal order are derived in [4], [15] and [7]. In this paper, we aim to obtain certain differential inclusions involving Q_{ST} and Q_{CV} for functions to belong to the class of starlike functions of reciprocal order α $(0 \leq \alpha < 1)$ and its subclass of functions satisfying $f(z)/(zf'(z)) \prec 1+(1-\alpha)z$. First, we define a class $\Psi(\Omega)$ of admissible functions and prove that $f \in \mathcal{RST}(\alpha)$ whenever $\psi \in \Psi(\Omega)$ and $\psi\left(Q_{ST}(z), Q_{CV}(z); z\right) \in \Omega$ for every $z \in \mathbb{D}$. Then using this class of admissible functions, for various choices of ψ, we obtain several sufficient conditions in terms of Q_{ST} and Q_{CV} for functions to be in $\mathcal{RST}(\alpha)$ and its subclass of functions satisfying $|f(z)/(zf'(z)) - 1| < 1 - \alpha$.

Recently, in [6], [19] and [23], similar studies on analytic and meromorphic functions are done by the authors.

2. Sufficient conditions for starlikeness of reciprocal order α

In this section, using the first order differential subordination theory, certain sufficient conditions for functions to be starlike of reciprocal order α are obtained. For this, consider the case when $q(z) = (1+z)/(1-z)$ for which it can be seen that $E(q) = \{1\}$ and for $\zeta \in \partial\mathbb{D}\backslash\{1\}$, $r = q(\zeta) = i\rho$ where $\rho \in \mathbb{R}$ and $s = m\zeta q'(\zeta) = -m(1+\rho^2)/2$. Thus the admissibility condition simplifies to

$$\psi(i\rho, \sigma; z) \notin \Omega \text{ when } \rho \in \mathbb{R},\ \sigma \leq -n(1+\rho^2)/2,$$

for every $z \in \mathbb{D}$. Let the class of all such functions satisfying the above admissibility condition with respect to $q(z) = (1+z)/(1-z)$ be denoted by $\mathcal{P}_n(\Omega)$ and let $\mathcal{P}(\Omega) :=$ $\mathcal{P}_1(\Omega)$. In this case, Miller-Mocanu theorem becomes

Lemma 2.1. *[14, Theorem 2.3i] Let $\psi \in \mathcal{P}(\Omega)$. If the function $p \in \mathcal{H}[1, n]$ satisfies*

$$\{\psi(p(z), zp'(z); z) : z \in \mathbb{D}\} \subset \Omega,$$

then $p \in \mathcal{P}$.

Using Lemma 2.1, the following result is obtained which gives a condition in terms of Q_{ST} and Q_{CV} for a function f to be in $\mathcal{RST}(\alpha)$.

Lemma 2.2. *Let $\alpha \in [0, 1)$ and $f \in \mathcal{A}$ with $f'(z) \neq 0$. For $\Omega \subset \mathbb{C}$, let $\Psi(\Omega)$ be the class of all functions $\psi : \mathbb{C}^2 \times \mathbb{D} \to \mathbb{C}$ satisfying*

$$\psi\left(\frac{1}{\alpha + i\tau}, \zeta + i\eta; z\right) \notin \Omega,$$

for $\tau \in \mathbb{R}$, $\zeta + i\eta \in \mathbb{C}$ with $(\alpha + i\tau)(\zeta + i\eta) \in \mathbb{R}$ and $(\alpha + i\tau)(\zeta + i\eta) \geq (3-\alpha)/2 + \tau^2/(2(1-\alpha))$. If $\psi \in \Psi(\Omega)$ and $\psi(Q_{ST}(z), Q_{CV}(z); z) \in \Omega$ for every $z \in \mathbb{D}$, then $f \in \mathcal{RST}(\alpha)$.

Proof. If the function $p : \mathbb{D} \to \mathbb{C}$ is defined by

$$p(z) = \frac{1}{1-\alpha}\left(\frac{f(z)}{zf'(z)} - \alpha\right), \tag{2}$$

then p is analytic and $p(0) = 1$. It follows that $f \in \mathcal{RST}(\alpha)$ is equivalent to $p \in \mathcal{P}$. Rewriting (2) it can be seen that

$$Q_{ST}(z) = \frac{1}{(1-\alpha)p(z) + \alpha}. \tag{3}$$

A simple computation using (3), we get

$$Q_{CV}(z) = \frac{1 - (1-\alpha)zp'(z)}{(1-\alpha)p(z) + \alpha}. \tag{4}$$

Consider the transformation defined by

$$u = \frac{1}{(1-\alpha)r + \alpha} \text{ and } v = \frac{1 - (1-\alpha)s}{(1-\alpha)r + \alpha}.$$

Let the function $\Theta : \mathbb{C}^2 \times \mathbb{D} \to \mathbb{C}$ be related to ψ as

$$\Theta(r,s;z) = \psi(u,v;z) = \psi\left(\frac{1}{(1-\alpha)r+\alpha}, \frac{1-(1-\alpha)s}{(1-\alpha)r+\alpha}; z\right). \tag{5}$$

Therefore, by using equations (3), (4) and (5), it follows that

$$\Theta(p(z), zp'(z); z) = \psi(Q_{ST}(z), Q_{CV}(z); z),$$

and so by the hypothesis, it can be seen that $\Theta(p(z), zp'(z); z) \in \Omega$. Now, to prove the result, it suffices to show that $\Theta \in \mathcal{P}(\Omega)$. If $r = i\rho$ and $s = \sigma$ with $\rho \in \mathbb{R}$ and $\sigma \leq -(1+\rho^2)/2$, then it follows that $u = 1/((1-\alpha)i\rho+\alpha)$ and $v = (1-(1-\alpha)\sigma)/((1-\alpha)i\rho+\alpha)$. If we let $\tau = (1-\alpha)\rho$ and $\zeta + i\eta = (1-(1-\alpha)\sigma)/((1-\alpha)i\rho+\alpha)$, then $u = 1/(\alpha+i\tau)$ and $v = \zeta + i\eta$. Further, it can be observed that

$$\frac{v}{u} = (\alpha+i\tau)(\zeta+i\eta) = 1-(1-\alpha)\sigma.$$

Thus, it follows that $v/u \in \mathbb{R}$ and

$$(\alpha+i\tau)(\zeta+i\eta) \geq 1 + \frac{(1-\alpha)(1+\rho^2)}{2} = \frac{3-\alpha}{2} + \frac{(1-\alpha)\rho^2}{2} = \frac{3-\alpha}{2} + \frac{\tau^2}{2(1-\alpha)}.$$

Therefore, by the hypothesis, it follows that

$$\psi\left(\frac{1}{\alpha+i\tau}, \zeta+i\eta; z\right) = \Theta(i\rho, \sigma; z) \notin \Omega$$

which shows that $\Theta \in \mathcal{P}(\Omega)$ and hence the result follows by Lemma 2.1. □

For various choices of ψ, corresponding domain Ω is obtained such that $\psi \in \Psi(\Omega)$ and with the aid of this lemma, many differential inequalities and inclusions are obtained that provide sufficient conditions for functions to be starlike of reciprocal order α.

THEOREM 2.3. *Let $0 \leq \alpha < 1$. If the function $f \in \mathcal{A}$ satisfies*

$$\frac{Q_{CV}(z)}{Q_{ST}(z)} \in \mathbb{C}\backslash\left[\frac{3-\alpha}{2}, \infty\right)$$

for every $z \in \mathbb{D}$, then $f \in \mathcal{RST}(\alpha)$.

Proof. Let the function $\psi : \mathbb{C}^2 \times \mathbb{D} \to \mathbb{C}$ be defined by $\psi(r,s;z) := s/r$. Then for $\tau \in \mathbb{R}$ and $\zeta + i\eta \in \mathbb{C}$ with $(\alpha+i\tau)(\zeta+i\eta) \geq (3-\alpha)/2 + \tau^2/(2(1-\alpha))$, it can be seen that

$$\psi\left(\frac{1}{\alpha+i\tau}, \zeta+i\eta; z\right) = (\alpha+i\tau)(\zeta+i\eta) \geq \frac{3-\alpha}{2} + \frac{\tau^2}{2(1-\alpha)} \geq \frac{3-\alpha}{2}.$$

Therefore, it follows that $\psi \in \Psi(\Omega_1)$, where Ω_1 is the region defined by $\Omega_1 := \mathbb{C}\backslash[(3-\alpha)/2, \infty)$. Also, by the hypothesis, we have $\psi(Q_{ST}(z), Q_{CV}(z); z) \in \Omega_1$. Therefore, by Lemma 2.2, it follows that $f \in \mathcal{RST}(\alpha)$. □

THEOREM 2.4. *Let $\gamma \in (-3/2, \infty)$ and $\alpha \in (\alpha_0, 1)$ where $\alpha_0 = \max\{2(\gamma+1)/(2\gamma+3), 0\}$. Let*

$$\delta_\gamma(\alpha) = \begin{cases} \dfrac{\alpha}{2(1-\alpha)}, & \alpha_0 < \alpha \leq \dfrac{2\gamma+3}{2(\gamma+2)}, \\[2ex] \dfrac{2\gamma+3-\alpha}{2\alpha}, & \dfrac{2\gamma+3}{2(\gamma+2)} < \alpha < 1. \end{cases} \tag{6}$$

If the function $f \in \mathcal{A}$ *satisfies*

$$\operatorname{Re}(\gamma Q_{ST}(z) + Q_{CV}(z)) < \delta_\gamma(\alpha)$$

for all $z \in \mathbb{D}$*, then* $f \in \mathcal{RST}(\alpha)$.

Proof. If a function $\psi : \mathbb{C}^2 \times \mathbb{D} \to \mathbb{C}$ is defined by

$$\psi(r, s; z) = r\left(\gamma + \frac{s}{r}\right),$$

then, for $\tau \in \mathbb{R}$, $\zeta + i\eta \in \mathbb{C}$ with $(\alpha + i\tau)(\zeta + i\eta) \geq (3-\alpha)/2 + \tau^2/(2(1-\alpha))$, we have

$$\psi\left(\frac{1}{\alpha + i\tau}, \zeta + i\eta; z\right) = \frac{1}{\alpha + i\tau}\left(\gamma + (\alpha + i\tau)(\zeta + i\eta)\right),$$

and its real part is given by

$$\operatorname{Re}\psi\left(\frac{1}{\alpha + i\tau}, \zeta + i\eta; z\right) = \frac{\alpha}{\alpha^2 + \tau^2}\left(\gamma + (\alpha + i\tau)(\zeta + i\eta)\right). \tag{7}$$

Then it follows that

$$\operatorname{Re}\psi\left(\frac{1}{\alpha + i\tau}, \zeta + i\eta; z\right) \geq \frac{\alpha}{\alpha^2 + \tau^2}\left(\gamma + \frac{3-\alpha}{2} + \frac{\tau^2}{2(1-\alpha)}\right).$$

Define $\varphi : [0, \infty) \to \mathbb{R}$ by

$$\varphi(t) = \frac{\alpha}{\alpha^2 + t}\left(\gamma + \frac{3-\alpha}{2} + \frac{t}{2(1-\alpha)}\right). \tag{8}$$

Then the first derivative of this function is given by

$$\varphi'(t) = \frac{\alpha}{(\alpha^2 + t)^2}\left(\frac{2\alpha(\gamma+2) - 2\gamma - 3}{2(1-\alpha)}\right).$$

Thus φ is increasing when $\alpha > (2\gamma+3)/(2(\gamma+2))$ and hence attains minimum at $t = 0$, which is given by $\min\varphi(t) = (2\gamma + 3 - \alpha)/(2\alpha)$. When $\alpha \leq (2\gamma+3)/(2(\gamma+2))$, φ is decreasing and hence $\min\varphi(t) = \lim_{t\to\infty}\varphi(t) = \alpha/(2(1-\alpha))$. Therefore, from (6), it follows that $\operatorname{Re}\psi\left(1/(\alpha + i\tau), \zeta + i\eta; z\right) \geq \delta_\gamma(\alpha)$, which shows that $\psi \in \Psi(\Omega_2)$, where $\Omega_2 := \{z : \operatorname{Re} z < \delta_\gamma(\alpha)\}$. From the hypothesis, we have $\psi(Q_{ST}(z), Q_{CV}(z); z) \in \Omega_2$. Thus by Lemma 2.2, $f \in \mathcal{RST}(\alpha)$ and hence the result follows. The existence of functions satisfying the hypothesis is ensured if the inequality $\operatorname{Re}(\gamma Q_{ST}(z) + Q_{CV}(z)) < \delta_\gamma(\alpha)$ is satisfied at the origin. Then we must have $\gamma + 1 < \delta_\gamma(\alpha)$, that is $\gamma + 1 < \alpha/(2(1-\alpha))$ and $\gamma + 1 < (2\gamma + 3 - \alpha)/(2\alpha)$. The inequality $\gamma + 1 < \alpha/(2(1-\alpha))$ can be rewritten as $2(\gamma+1) < \alpha(2\gamma+3)$ which gets satisfied since it is assumed that $\alpha > \alpha_0 \geq 2(\gamma+1)/(2\gamma+3)$. Also, the inequality $\gamma + 1 < (2\gamma + 3 - \alpha)/(2\alpha)$ can be rewritten as $(1-\alpha)(2\gamma+3) > 0$ which holds since $\gamma > -3/2$. □

The sufficient condition for a function to be starlike of reciprocal order α in terms of Q_{CV} is obtained by taking $\gamma = 0$ in Theorem 2.4.

Corollary 2.5. *Let* $2/3 \leq \alpha < 1$. *If the function* $f \in \mathcal{A}$ *satisfies the inequality*

$$\operatorname{Re} Q_{CV}(z) < \begin{cases} \dfrac{\alpha}{2(1-\alpha)}, & \dfrac{2}{3} < \alpha \leq \dfrac{3}{4} \\ \dfrac{3-\alpha}{2\alpha}, & \dfrac{3}{4} < \alpha < 1 \end{cases}$$

for all $z \in \mathbb{D}$*, then* $f \in \mathcal{RST}(\alpha)$.

In [3], Banga and Sivaprasad Kumar have proved that if the function $f \in \mathcal{A}$ satisfies

$$\operatorname{Re} Q_{CV}(z) < \begin{cases} \dfrac{3-\alpha}{2\alpha}, & 0 < \alpha \leq \dfrac{3}{4} \\ \dfrac{\alpha}{2(1-\alpha)}, & \dfrac{3}{4} < \alpha < 1 \end{cases} \tag{9}$$

for all $z \in \mathbb{D}$, then $f \in \mathcal{RST}(\alpha)$. In the proof of this result, they considered the function $S(z) := (z^2 - 2(2-\alpha)z + 1)/((1-z)(1+(1-2\alpha)z)$. Instead of finding a half plane contained in the image domain $S(\mathbb{D})$, they have found a half plane containing $S(\mathbb{D})$. Corollary 2.5 serves as the correct version of their result.

THEOREM 2.6. *Let $\alpha \in [0,1)$ and $\gamma \in \mathbb{R}\backslash[0,2]$. If the function $f \in \mathcal{A}$ satisfies either the differential inequality*

$$\operatorname{Re} \frac{1+\gamma Q_{CV}(z)}{Q_{ST}(z)} < \alpha + \frac{(3-\alpha)\gamma}{2} \qquad \text{for } \gamma > 2 \tag{10}$$

or

$$\operatorname{Re} \frac{1+\gamma Q_{CV}(z)}{Q_{ST}(z)} > \alpha + \frac{(3-\alpha)\gamma}{2} \qquad \text{for } \gamma < 0 \tag{11}$$

for every $z \in \mathbb{D}$, then $f \in \mathcal{RST}(\alpha)$.

Proof. Let the function $\psi : \mathbb{C}^2 \times \mathbb{D} \to \mathbb{C}$ be defined by

$$\psi(r, s; z) := \frac{1+\gamma s}{r}.$$

For $\tau \in \mathbb{R}$ and $\zeta + i\eta \in \mathbb{C}$ with $(\alpha + i\tau)(\zeta + i\eta) \geq (3-\alpha)/2 + \tau^2/(2(1-\alpha))$, it can be observed that

$$\operatorname{Re} \psi \left(\frac{1}{\alpha + i\tau}, \zeta + i\eta; z \right) = \operatorname{Re} (\alpha + i\tau + \gamma(\alpha + i\tau)(\zeta + i\eta)) = \alpha + \gamma(\alpha + i\tau)(\zeta + i\eta).$$

Then for $\gamma \geq 0$,

$$\operatorname{Re} \psi \left(\frac{1}{\alpha + i\tau}, \zeta + i\eta; z \right) \geq \alpha + \gamma \left(\frac{3-\alpha}{2} + \frac{\tau^2}{2(1-\alpha)} \right) \geq \alpha + \frac{(3-\alpha)\gamma}{2}. \tag{12}$$

Correspondingly, for $\gamma < 0$,

$$\operatorname{Re} \psi \left(\frac{1}{\alpha + i\tau}, \zeta + i\eta; z \right) \leq \alpha + \gamma \left(\frac{3-\alpha}{2} + \frac{\tau^2}{2(1-\alpha)} \right) \leq \alpha + \frac{(3-\alpha)\gamma}{2}. \tag{13}$$

If Ω_3 and Ω_4 are the halfplanes defined by $\Omega_3 := \{z \in \mathbb{C} : \operatorname{Re} z < \alpha + (3-\alpha)\gamma/2\}$ and $\Omega_4 := \{z \in \mathbb{C} : \operatorname{Re} z > \alpha + (3-\alpha)\gamma/2\}$, then from the inequalities (12) and (13), it is clear that for $\gamma \geq 0$, $\psi \in \Psi(\Omega_3)$ and for $\gamma < 0$, $\psi \in \Psi(\Omega_4)$. Further, from the hypothesis, we have $\psi(Q_{ST}(z), Q_{CV}(z); z) \in \Omega_3$ for $\gamma \geq 0$ and $\psi(Q_{ST}(z), Q_{CV}(z); z) \in \Omega_4$ for $\gamma < 0$. Thus, the result follows by an application of Lemma 2.2. At the origin, the inequality (10) reduces to $(1-\alpha)(\gamma - 2) > 0$ which holds when $\gamma > 2$. Similarly, at the origin, the inequality (11) reduces to $(1-\alpha)(\gamma-2) < 0$ which is true for every $\gamma < 0$. This ensures the existence of functions satisfying the hypothesis for $\gamma \in \mathbb{R}\backslash[0,2]$. ☐

THEOREM 2.7. *Let $\alpha \in [0,1)$ and $f \in \mathcal{A}$. For $\gamma \geq -\alpha$, if the function f satisfies the differential inequality*

$$\operatorname{Re} \frac{1+\gamma Q_{ST}(z)}{Q_{CV}(z)} > \frac{2(\alpha+\gamma)}{3-\alpha} \tag{14}$$

for every $z \in \mathbb{D}$, then $f \in \mathcal{RST}(\alpha)$.

Proof. Define the function $\psi : \mathbb{C}^2 \times \mathbb{D} \to \mathbb{C}$ by

$$\psi(r,s;z) := \frac{r}{s}\left(\frac{1}{r}+\gamma\right). \tag{15}$$

From the hypothesis, it follows by the definition of ψ that

$$\operatorname{Re} \psi(Q_{ST}(z), Q_{CV}(z); z) > \frac{2(\alpha+\gamma)}{3-\alpha}.$$

For $\tau \in \mathbb{R}$ and $\zeta + i\eta \in \mathbb{C}$ with $(\alpha+i\tau)(\zeta+i\eta) \geq (3-\alpha)/2 + \tau^2/(2(1-\alpha))$, it follows from (15) that

$$\operatorname{Re} \psi\left(\frac{1}{\alpha+i\tau}, \zeta+i\eta; z\right) = \frac{\alpha+\gamma}{(\alpha+i\tau)(\zeta+i\eta)} \leq \frac{2(1-\alpha)(\alpha+\gamma)}{(3-\alpha)(1-\alpha)+\tau^2} \leq \frac{2(\alpha+\gamma)}{3-\alpha}.$$

since $\gamma \geq -\alpha$. This shows that $\psi \in \Psi(\Omega_5)$ where $\Omega_5 = \{z \in \mathbb{C} : \operatorname{Re} z > 2(\alpha+\gamma)/(3-\alpha)\}$. Thus, the result follows by an application of Lemma 2.2. Furthermore, about the origin, inequality (14) reduces to $(3+\gamma)(1-\alpha) > 0$ which holds since $\gamma \geq -\alpha > -3$. This ensures the existence of functions satisfying the hypothesis. □

THEOREM 2.8. *Let $\alpha \in [0,1)$. For $\gamma > (\alpha-1)/(3-\alpha)$, if the function $f \in \mathcal{A}$ satisfies either the differential inequality*

$$\operatorname{Re} \frac{Q^2{}_{ST}(z) + \gamma Q_{CV}(z)}{Q_{ST}(z) Q_{CV}(z)} > \frac{2}{3-\alpha} + \alpha\gamma$$

for all $z \in \mathbb{D}$, then $f \in \mathcal{RST}(\alpha)$.

Proof. Define a function $\psi : \mathbb{C}^2 \times \mathbb{D} \to \mathbb{C}$ by

$$\psi(r,s;z) := \frac{r}{s} + \frac{\gamma}{r}.$$

For $\tau \in \mathbb{R}$, $\zeta + i\eta \in \mathbb{C}$ with $(\alpha+i\tau)(\zeta+i\eta) \geq (3-\alpha)/2 + \tau^2/(2(1-\alpha))$, it can be seen that

$$\begin{aligned}\operatorname{Re} \psi\left(\frac{1}{\alpha+i\tau}, \zeta+i\eta; z\right) &= \operatorname{Re}\left(\frac{1}{(\alpha+i\tau)(\zeta+i\eta)} + \gamma(\alpha+i\tau)\right) \\ &= \alpha\gamma + \frac{1}{(\alpha+i\tau)(\zeta+i\eta)} \leq \frac{2(1-\alpha)}{(3-\alpha)(1-\alpha)+\tau^2} + \alpha\gamma.\end{aligned}$$

If the function $\varphi : [0,\infty) \to \mathbb{R}$ is defined by

$$\varphi(t) = \frac{2(1-\alpha)}{(3-\alpha)(1-\alpha)+t}, \tag{16}$$

then its derivative is given by

$$\varphi'(t) = -\frac{2(1-\alpha)}{((3-\alpha)(1-\alpha)+t)^2}$$

which is negative. Therefore, φ is decreasing and its maximum is given by $\varphi(0) = 2/(3-\alpha)$. Thus, it follows that $\psi(1/(\alpha+i\tau), \zeta+i\eta; z) \in \mathbb{C}\backslash\Omega_6$ where $\Omega_6 = \{z \in \mathbb{C} : \operatorname{Re} z > \alpha\gamma + 2/(3-\alpha)\}$. From the hypothesis, we have $\psi(Q_{ST}(z), Q_{CV}(z); z) \in \Omega_6$ for every $z \in \mathbb{D}$. Hence, by an application of Lemma 2.2, the result follows. The existence of functions satisfying hypothesis is ensured since the inequality in the hypothesis holds at the origin for $\gamma > (\alpha-1)/(3-\alpha)$. □

THEOREM 2.9. *Let $\alpha \in [0,1)$. For $\gamma \in (-2/(2-\alpha), 0]$, if the function f satisfies the differential inequality*

$$\operatorname{Re} \frac{Q_{ST}(z) + \gamma Q_{CV}(z)}{Q^2{}_{ST}(z)} > \alpha + \frac{\alpha\gamma(3-\alpha)}{2} \tag{17}$$

for every $z \in \mathbb{D}$, then $f \in \mathcal{RST}(\alpha)$.

Proof. For $\gamma \in (-2/(2-\alpha), 0]$, let the function $\psi : \mathbb{C}^2 \times \mathbb{D} \to \mathbb{C}$ be defined by

$$\psi(r, s; z) := \frac{1}{r} + \gamma \frac{s}{r^2}.$$

For $\tau \in \mathbb{R}$, $\zeta + i\eta \in \mathbb{C}$ satisfying the condition $(\alpha+i\tau)(\zeta+i\eta) \geq (3-\alpha)/2 + \tau^2/(2(1-\alpha))$,

$$\operatorname{Re} \psi\left(\frac{1}{\alpha+i\tau}, \zeta+i\eta; z\right) = \operatorname{Re}((\alpha+i\tau) + \gamma(\alpha+i\tau)^2(\zeta+i\eta)) = \alpha + \alpha\gamma(\alpha+i\tau)(\zeta+i\eta).$$

Since $\gamma \leq 0$, it follows that

$$\operatorname{Re} \psi\left(\frac{1}{\alpha+i\tau}, \zeta+i\eta; z\right) \leq \alpha + \alpha\gamma\left(\frac{3-\alpha}{2} + \frac{\tau^2}{2(1-\alpha)}\right) \leq \alpha + \frac{\alpha\gamma(3-\alpha)}{2}.$$

Let the region Ω_γ be defined by $\Omega_\gamma := \{z \in \mathbb{C} : \operatorname{Re} z > \alpha + \alpha\gamma(3-\alpha)/2\}$. From the hypothesis, it follows that $\psi(Q_{ST}(z), Q_{CV}(z); z) \in \Omega_\gamma$ for every $z \in \mathbb{D}$. Therefore, the result follows by an application of Lemma 2.2. Since $\gamma > -2/(2-\alpha)$, it follows that $1-\alpha > -(1-\alpha)(2-\alpha)\gamma/2$ which gives $1+\gamma > \alpha + \alpha\gamma(3-\alpha)/2$. Thus, the inequality (17) hold at the origin and hence the existence of functions satisfying the hypothesis is ensured. □

THEOREM 2.10. *Let $f \in \mathcal{A}$ and for $\alpha \in [0,1)$ and $\gamma \in \mathbb{R}$, let*

$$\delta_\gamma(\alpha) = \begin{cases} \alpha + \dfrac{(3-\alpha)\gamma}{2\alpha}, & \alpha \geq \dfrac{3}{4}, \\ \alpha + \dfrac{\alpha\gamma}{2(1-\alpha)}, & \alpha < \dfrac{3}{4}. \end{cases}$$

If the function $f \in \mathcal{A}$ satisfies either

$$\operatorname{Re}(1/Q_{ST}(z) + \gamma Q_{CV}(z)) < \delta_\gamma(\alpha) \text{ for } \gamma > \max\left\{\frac{2\alpha}{3}, \frac{2(1-\alpha)^2}{3\alpha-2}\right\} \tag{18}$$

or

$$\operatorname{Re}(1/Q_{ST}(z) + \gamma Q_{CV}(z)) > \delta_\gamma(\alpha) \text{ for } \gamma < \min\left\{0, \frac{2(1-\alpha)^2}{3\alpha-2}\right\} \tag{19}$$

for all $z \in \mathbb{D}$, then $f \in \mathcal{RST}(\alpha)$.

Proof. Let the function $\psi : \mathbb{C}^2 \times \mathbb{D} \to \mathbb{C}$ be defined by

$$\psi(r, s; z) := \frac{1}{r} + \gamma s.$$

For $\tau \in \mathbb{R}$ and $\zeta + i\eta \in \mathbb{C}$ satisfying $(\alpha + i\tau)(\zeta + i\eta) \geq (3-\alpha)/2 + \tau^2/(2(1-\alpha))$, we have

(20)

$$\operatorname{Re} \psi \left(\frac{1}{\alpha + i\tau}, \zeta + i\eta; z \right) = \operatorname{Re} \left(\alpha + i\tau + \frac{\gamma}{\alpha + i\tau} (\alpha + i\tau)(\zeta + i\eta) \right) = \alpha + \frac{\alpha\gamma(\alpha + i\tau)(\zeta + i\eta)}{\alpha^2 + \tau^2}.$$

For $\gamma \geq 0$, it is evident that

$$\operatorname{Re} \psi \left(\frac{1}{\alpha + i\tau}, \zeta + i\eta; z \right) \geq \alpha + \frac{\alpha\gamma}{\alpha^2 + \tau^2} \left(\frac{3-\alpha}{2} + \frac{\tau^2}{2(1-\alpha)} \right). \tag{21}$$

Let $\varphi : [0, \infty) \to \mathbb{C}$ be defined by

$$\varphi(t) := \frac{\alpha\gamma}{\alpha^2 + t} \left(\frac{3-\alpha}{2} + \frac{t}{2(1-\alpha)} \right). \tag{22}$$

Its derivative φ' is given by

$$\varphi'(t) = \frac{\alpha\gamma(4\alpha - 3)}{2(1-\alpha)(\alpha^2 + t)^2},$$

which is non-negative for $\alpha \geq 3/4$. Then, it is clear that φ is increasing and hence attains its minimum at $t = 0$ given by $(3-\alpha)\gamma/(2\alpha)$. Further, for $\alpha < 3/4$, it can be observed that $\varphi'(t) < 0$. Thus φ is decreasing and attains minimum as t tends to ∞, which is given by $\alpha\gamma/(2(1-\alpha))$. Therefore, it follows from the definition of $\delta_\gamma(\alpha)$ and (21) that

$$\operatorname{Re} \psi \left(\frac{1}{\alpha + i\tau}, \zeta + i\eta; z \right) \geq \delta_\gamma(\alpha). \tag{23}$$

Similarly, for $\gamma < 0$, we have $\varphi'(t) \leq 0$ for $\alpha \geq 3/4$ and $\varphi'(t) > 0$ for $\alpha < 3/4$. Thus, the function φ defined in (22) is decreasing for $\alpha \geq 3/4$ and hence $\max \varphi(t) = \varphi(0) = (3-\alpha)\gamma/(2\alpha)$. For $\alpha < 3/4$, the function φ is increasing and hence attains its maximum as t tends to infinity, which is given by $\alpha\gamma/(2(1-\alpha))$. So, for $\gamma < 0$, from equation (20), it follows that

$$\operatorname{Re} \psi \left(\frac{1}{\alpha + i\tau}, \zeta + i\eta; z \right) \leq \alpha + \frac{\alpha\gamma}{\alpha^2 + \tau^2} \left(\frac{3-\alpha}{2} + \frac{\tau^2}{2(1-\alpha)} \right) \leq \delta_\gamma(\alpha). \tag{24}$$

From the inequalities (18), (19), (23) and (24), it can be seen that $\psi(Q_{ST}(z), Q_{CV}(z); z) \in \Omega_\gamma$ for every $z \in \mathbb{D}$ and $\psi \in \Psi(\Omega_\gamma)$ where $\Omega_\gamma := \{z \in \mathbb{C} : \operatorname{Re} z < \delta_\gamma(\alpha)\}$ for $\gamma > \max\{2\alpha/3, (2(1-\alpha)^2)/(3\alpha-2)\}$ and $\Omega_\gamma := \{z \in \mathbb{C} : \operatorname{Re} z > \delta_\gamma(\alpha)\}$ for $\gamma < \min\{0, (2(1-\alpha)^2)/(3\alpha-2)\}$. Therefore, it follows that $f \in \mathcal{RST}(\alpha)$ by an application of Lemma 2.2. At the origin, the inequality (18) becomes $1 + \gamma < \delta_\gamma(\alpha)$, that is,

$$1 + \gamma < \alpha + \frac{(3-\alpha)\gamma}{2\alpha} \text{ for } \alpha \geq \frac{3}{4} \tag{25}$$

and

$$1 + \gamma < \alpha + \frac{\alpha\gamma}{2(1-\alpha)} \text{ for } \alpha < \frac{3}{4}. \tag{26}$$

The inequality (25) holds if $\gamma > 2\alpha/3$ and the inequality (26) holds if $\gamma > 2(1-\alpha)^2/(3\alpha-2)$, which is true by the assumed conditions on γ. This ensures the existence of functions satisfying the inequality (18) in the hypothesis. Similarly, it can be shown that the (19) holds at origin when $\gamma < 0$ and $\gamma < 2(1-\alpha)^2/(3\alpha-2)$.

□

THEOREM 2.11. *Let $\alpha \in [0,1)$ and $f \in \mathcal{A}$. For $\gamma \in [-1/\alpha, 1/(2-\alpha))$, if the function f satisfies*

$$\operatorname{Re} \frac{Q_{CV}(z)}{Q_{ST}(z)} \left(1 + \frac{\gamma}{Q_{ST}(z)}\right) < \frac{(1+\alpha\gamma)(3-\alpha)}{2} \tag{27}$$

for every $z \in \mathbb{D}$, then $f \in \mathcal{RST}(\alpha)$.

Proof. Let the function $\psi : \mathbb{C}^2 \times \mathbb{D} \to \mathbb{C}$ be defined by

$$\psi(r, s; z) := \frac{s}{r}\left(1 + \frac{\gamma}{r}\right).$$

For $\tau \in \mathbb{R}$ and $\zeta + i\eta \in \mathbb{C}$ satisfying the condition $(\alpha + i\tau)(\zeta + i\eta) \geq (3-\alpha)/2 + \tau^2/(2(1-\alpha))$, it can be seen that

$$\operatorname{Re}\psi\left(\frac{1}{\alpha+i\tau}, \zeta + i\eta; z\right) = \operatorname{Re}\left((\alpha+i\tau)(\zeta+i\eta)(1+\gamma(\alpha+i\tau))\right) = (1+\alpha\gamma)(\alpha+i\tau)(\zeta+i\eta).$$

Since $\gamma \geq -1/\alpha$, it follows that,

$$\operatorname{Re}\psi\left(\frac{1}{\alpha+i\tau}, \zeta + i\eta; z\right) \geq (1+\alpha\gamma)\left(\frac{3-\alpha}{2} + \frac{\tau^2}{2(1-\alpha)}\right) \geq \frac{(1+\alpha\gamma)(3-\alpha)}{2}.$$

This shows that $\psi \in \Psi(\Omega_7)$ where $\Omega_7 = \{z \in \mathbb{C} : \operatorname{Re} z < (1+\alpha\gamma)(3-\alpha)/2\}$. From the hypothesis, it follows by the definition of ψ that $\psi(Q_{ST}(z), Q_{CV}(z); z) \in \Omega_7$ for all $z \in \mathbb{D}$. Therefore, the result follows from Lemma 2.2. Furthermore, at the origin, inequality (27) reduces to $(2-\alpha)(1-\alpha)\gamma < 1-\alpha$ which holds since $\gamma < 1/(2-\alpha)$. This proves the existence of functions satisfying the hypothesis. □

THEOREM 2.12. *Let $\alpha \in [0,1)$. If the function $f \in \mathcal{A}$ satisfies either*

$$\operatorname{Re} \frac{1}{Q_{ST}(z)}\left(\frac{1}{Q_{ST}(z)} + \gamma Q_{CV}(z)\right) < \alpha^2 + \frac{(3-\alpha)\gamma}{2}, \text{ for } \gamma > 2(1+\alpha) \tag{28}$$

or

$$\operatorname{Re} \frac{1}{Q_{ST}(z)}\left(\frac{1}{Q_{ST}(z)} + \gamma Q_{CV}(z)\right) > \alpha^2 + \frac{(3-\alpha)\gamma}{2}, \text{ for } \gamma < 0$$

then $f \in \mathcal{RST}(\alpha)$.

Proof. For $0 \leq \alpha < 1$, $\gamma \in \mathbb{R}\backslash[0, 2(1+\alpha)]$, define the function $\varphi : [0,\infty) \to \mathbb{R}$ by

$$\varphi(t) := \alpha^2 - t + \gamma\left(\frac{3-\alpha}{2} + \frac{t}{2(1-\alpha)}\right).$$

Its derivative is given by $\varphi'(t) = -1 + \gamma/2(1-\alpha)$ which is non-negative when $\gamma \geq 2(1-\alpha)$ and negative otherwise. Thus, when $\gamma \geq 2(1-\alpha)$, the function φ is increasing and

$$\min_{t\in[0,\infty)} \varphi(t) = \varphi(0) = \alpha^2 + \frac{(3-\alpha)\gamma}{2}.$$

When $\gamma < 2(1-\alpha)$, the function φ is decreasing and $\max\varphi(t) = \alpha^2 + (3-\alpha)\gamma/2$. Let the function $\psi : \mathbb{C}^2 \times \mathbb{D} \to \mathbb{C}$ be defined by

$$\psi(r, s; z) := \frac{1}{r^2} + \frac{\gamma s}{r}.$$

For $\tau \in \mathbb{R}$ and $\zeta + i\eta \in \mathbb{C}$ satisfying $(\alpha + i\tau)(\zeta + i\eta) \geq (3-\alpha)/2 + \tau^2/(2(1-\alpha))$,
(29)

$$\operatorname{Re}\psi\left(\frac{1}{\alpha+i\tau}, \zeta+i\eta; z\right) = \operatorname{Re}((\alpha+i\tau)^2+\gamma(\alpha+i\tau)(\zeta+i\eta)) = \alpha^2-\tau^2+\gamma(\alpha+i\tau)(\zeta+i\eta).$$

Case (i) $\gamma > 2(1+\alpha)$. It can be clearly seen that $\gamma > 2(1+\alpha) \geq 2(1-\alpha)$. Thus, in this case, it follows from equation (29) that

$$\operatorname{Re}\psi\left(\frac{1}{\alpha+i\tau}, \zeta+i\eta; z\right) \geq \alpha^2-\tau^2+\gamma\left(\frac{3-\alpha}{2}+\frac{\tau^2}{2(1-\alpha)}\right) = \varphi(\tau^2) \geq \alpha^2+\frac{(3-\alpha)\gamma}{2}.$$

Therefore, $\psi \in \Psi(\Omega_8)$ where $\Omega_8 := \{z \in \mathbb{C} : \operatorname{Re} z < \alpha^2 + (3-\alpha)\gamma/2\}$. From the hypothesis, it follows that $\psi(Q_{ST}(z), Q_{CV}(z); z) \in \Omega_8$ for every $z \in \mathbb{D}$. Thus, $f \in \mathcal{RST}(\alpha)$ by an application of Lemma 2.2. Further, at the origin, inequality (28) reduces to

$$1+\gamma < \alpha^2 + \frac{(3-\alpha)\gamma}{2}$$

which can be rewritten as $1-\alpha^2 < (1-\alpha)\gamma/2$ which holds as it is assumed that $\gamma > 2(1+\alpha)$. This proves that there exist functions satisfying the hypothesis.

Case (ii) $\gamma < 0$. In this case, since $\gamma < 0 < 2(1-\alpha)$, from equation (29), it follows that

$$\operatorname{Re}\psi\left(\frac{1}{\alpha+i\tau}, \zeta+i\eta; z\right) \leq \alpha^2-\tau^2+\gamma\left(\frac{3-\alpha}{2}+\frac{\tau^2}{2(1-\alpha)}\right) = \varphi(\tau^2) \leq \alpha^2+\frac{(3-\alpha)\gamma}{2},$$

Therefore, $\psi \in \Psi(\Omega_9)$ where $\Omega_9 := \{z \in \mathbb{C} : \operatorname{Re} z > \alpha^2 + (3-\alpha)\gamma/2\}$ and from the hypothesis, $\psi(Q_{ST}(z), Q_{CV}(z); z) \in \Omega_9$ for every $z \in \mathbb{D}$. So, the result follows by Lemma 2.2. Also, the existence of functions satisfying the hypothesis is ensured by the assumed condition. □

THEOREM 2.13. *Let $\alpha \in [0,1)$ and $\gamma \in \mathbb{R}$. If the function $f \in \mathcal{A}$ satisfies either*

$$\frac{Q_{CV}(z)}{Q_{ST}(z)}\left(1+\gamma\frac{Q_{CV}(z)}{Q_{ST}(z)}\right) \in \mathbb{C}\setminus\left[\frac{(3-\alpha)(2+(3-\alpha)\gamma)}{4}, \infty\right) \quad \text{for } \gamma \geq 0 \tag{30}$$

or

$$\frac{Q_{CV}(z)}{Q_{ST}(z)}\left(1+\gamma\frac{Q_{CV}(z)}{Q_{ST}(z)}\right) \in \mathbb{C}\setminus\left(-\infty, \frac{(3-\alpha)(2+(3-\alpha)\gamma)}{4}\right] \quad \text{for } \gamma < 0, \tag{31}$$

in $\mathbb{D}$, then $f \in \mathcal{RST}(\alpha)$.

Proof. Let the function $\psi : \mathbb{C}^2 \times \mathbb{D} \to \mathbb{C}$ be defined by

$$\psi(r,s;z) := \frac{s}{r} + \gamma\left(\frac{s}{r}\right)^2.$$

Then for $\tau \in \mathbb{R}$ and $\zeta + i\eta \in \mathbb{C}$, with $(\alpha+i\tau)(\zeta+i\eta) \geq (3-\alpha)/2 + \tau^2/(2(1-\alpha))$,

$$\psi\left(\frac{1}{\alpha+i\tau}, \zeta+i\eta; z\right) = (\alpha+i\tau)(\zeta+i\eta) + \gamma(\alpha+i\tau)^2(\zeta+i\eta)^2.$$

For $\gamma \geq 0$, it can be observed that

$$\psi\left(\frac{1}{\alpha+i\tau}, \zeta+i\eta; z\right) \geq \left(\frac{3-\alpha}{2}+\frac{\tau^2}{2(1-\alpha)}\right) + \gamma\left(\frac{3-\alpha}{2}+\frac{\tau^2}{2(1-\alpha)}\right)^2. \tag{32}$$

It can be easily seen that the function $\varphi : [0,\infty) \to \mathbb{R}$ defined by

$$\varphi(t) := \frac{3-\alpha}{2} + \frac{t}{2(1-\alpha)}$$

is increasing and non-negative. Thus, for $\gamma \geq 0$, the function $\varphi(t) + \gamma(\varphi(t))^2$ is increasing and hence attains its minimum at $t = 0$ which is given by $(3-\alpha)(2+(3-\alpha)\gamma)/4$. Therefore, it follows from the inequality (32) that

$$\psi\left(\frac{1}{\alpha+i\tau}, \zeta+i\eta; z\right) \geq \varphi(\tau^2) + \gamma(\varphi(\tau^2))^2 \geq \frac{(3-\alpha)(2+(3-\alpha)\gamma)}{4},$$

which implies that $\psi \in \Psi\left(\mathbb{C}\backslash[(3-\alpha)(2+(3-\alpha)\gamma)/4, \infty)\right)$. Further, from the hypothesis, it can be seen that $\psi(Q_{ST}(z), Q_{CV}(z); z) \in \mathbb{C}\backslash[(3-\alpha)(2+(3-\alpha)\gamma)/4, \infty)$. Thus, it follows that $f \in \mathcal{RST}(\alpha)$ by Lemma 2.2. Also, since $\gamma \geq 0 > 2/(\alpha-5)$, it follows that $2(\alpha-1) < ((3-\alpha)^2 - 4)\gamma$ which gives

$$1 + \gamma < \frac{3-\alpha}{2} + \frac{(3-\alpha)^2\gamma}{4}.$$

This shows that the containment (30) holds at $z = 0$ which ensures the existence of functions satisfying the hypothesis. Similarly, for $\gamma < 0$, it follows that

$$\psi\left(\frac{1}{\alpha+i\tau}, \zeta+i\eta; z\right) \leq \varphi(\tau^2) + \gamma(\varphi(\tau^2))^2 \leq \frac{(3-\alpha)(2+(3-\alpha)\gamma)}{4},$$

which shows that $\psi \in \Psi\left(\mathbb{C}\backslash(-\infty, (3-\alpha)(2+(3-\alpha)\gamma)/4]\right)$. Also, from the hypothesis, it can be seen that $\psi(Q_{ST}(z), Q_{CV}(z); z) \in \mathbb{C}\backslash(-\infty, (3-\alpha)(2+(3-\alpha)\gamma)/4]$, from which the result follows by Lemma 2.2. Further, it can be shown that the containment (31) holds at origin which proves the existence of functions satisfying the hypothesis. □

THEOREM 2.14. *Let $\alpha \in [0,1)$ and $\gamma \in \mathbb{R}$. If the function $f \in \mathcal{A}$ satisfies either*

$$\frac{Q_{ST}(z)}{Q_{CV}(z)}\left(1+\gamma\frac{Q_{ST}(z)}{Q_{CV}(z)}\right) \in \mathbb{C}\backslash\left(-\infty, \frac{2}{3-\alpha} + \frac{4\gamma}{(3-\alpha)^2}\right] \quad \text{for } \gamma \geq 0 \tag{33}$$

or

$$\frac{Q_{ST}(z)}{Q_{CV}(z)}\left(1+\gamma\frac{Q_{ST}(z)}{Q_{CV}(z)}\right) \in \mathbb{C}\backslash\left[\frac{2}{3-\alpha} + \frac{4\gamma}{(3-\alpha)^2}, \infty\right) \quad \text{for } \gamma < 0 \tag{34}$$

for every $z \in \mathbb{D}$, then $f \in \mathcal{RST}(\alpha)$.

Proof. Define the function $\psi : \mathbb{C}^2 \times \mathbb{D} \to \mathbb{C}$ by

$$\psi(r,s;z) := \frac{r}{s} + \gamma\left(\frac{r}{s}\right)^2.$$

For $\tau \in \mathbb{R}$ and $\zeta + i\eta \in \mathbb{C}$ with $(\alpha+i\tau)(\zeta+i\eta) \geq (3-\alpha)/2 + \tau^2/(2(1-\alpha))$, we have

$$\psi\left(\frac{1}{\alpha+i\tau}, \zeta+i\eta; z\right) = \frac{1}{(\alpha+i\tau)(\zeta+i\eta)} + \frac{\gamma}{(\alpha+i\tau)^2(\zeta+i\eta)^2}.$$

For $\gamma \geq 0$, it can be observed that

$$\psi\left(\frac{1}{\alpha+i\tau}, \zeta+i\eta; z\right) \leq \frac{2(1-\alpha)}{(3-\alpha)(1-\alpha)+\tau^2} + \frac{4\gamma(1-\alpha)^2}{((3-\alpha)(1-\alpha)+\tau^2)^2}.$$

If the function $\varphi : [0,\infty) \to \mathbb{R}$ is defined by

$$\varphi(t) := \frac{2(1-\alpha)}{(3-\alpha)(1-\alpha)+t},$$

it can be seen that φ is non-negative and its derivative is given by

$$\varphi'(t) = \frac{-2(1-\alpha)}{((3-\alpha)(1-\alpha)+t)^2} < 0.$$

Thus, the function φ is decreasing and attains its maximum at $t = 0$ which is given by $2/(3-\alpha)$. Therefore, it follows that

$$\psi\left(\frac{1}{\alpha+i\tau}, \zeta+i\eta; z\right) \leq \varphi(\tau^2) + \gamma\left(\varphi(\tau^2)\right)^2 \leq \frac{2}{3-\alpha} + \frac{4\gamma}{(3-\alpha)^2},$$

which implies that $\psi \in \Psi\left(\mathbb{C}\setminus\left(-\infty, 2/(3-\alpha)+4\gamma/(3-\alpha)^2\right]\right)$. Further, from the hypothesis, we have $\psi(Q_{ST}(z), Q_{CV}(z); z) \in \mathbb{C}\setminus\left(-\infty, 2/(3-\alpha)+4\gamma/(3-\alpha)^2\right]$. Thus, it follows that $f \in \mathcal{RST}(\alpha)$ by an application of Lemma 2.2. Similarly, for $\gamma < 0$, it follows that

$$\begin{aligned}\psi\left(\frac{1}{\alpha+i\tau}, \zeta+i\eta; z\right) &\geq \frac{2(1-\alpha)}{(3-\alpha)(1-\alpha)+\tau^2} + \frac{4\gamma(1-\alpha)^2}{\left((3-\alpha)(1-\alpha)+\tau^2\right)^2} \\ &= \varphi(\tau^2) + \gamma\left(\varphi(\tau^2)\right)^2 \geq \frac{2}{3-\alpha} + \frac{4\gamma}{(3-\alpha)^2}.\end{aligned}$$

This shows that $\psi \in \Psi\left(\mathbb{C}\setminus\left[2/(3-\alpha)+4\gamma/(3-\alpha)^2, \infty\right)\right)$. It can be seen from the hypothesis that $\psi(Q_{ST}(z), Q_{CV}(z); z) \in \mathbb{C}\setminus\left[2/(3-\alpha)+4\gamma/(3-\alpha)^2, \infty\right)$. Thus, the result follows by Lemma 2.2. Further, it can be shown that the containments (33) and (34) hold at origin which proves the existence of functions satisfying the hypothesis. □

3. Sufficient conditions for functions to be in a subclass of $\mathcal{RST}(\alpha)$

If the function q is defined by $w(z) := z$, then $q \in \mathcal{B}$ and $E(q) = \emptyset$. For $\zeta \in \partial\mathbb{D}$, it can be observed that $r = q(\zeta) = e^{i\theta}$ and $s = m\zeta q'(\zeta) = me^{i\theta}$ where $\theta \in \mathbb{R}$. The class of admissible functions with respect to $q(z) = z$ is the set of all functions $\psi : \mathbb{C}^2 \times \mathbb{D} \to \mathbb{C}$ satisfying $\psi(e^{i\theta}, Ke^{i\theta}; z) \notin \Omega$ for every $z \in \mathbb{D}$, where $\theta \in \mathbb{R}$ and $K \geq n$. Let this class be denoted by $\mathcal{B}_n(\Omega)$ with $\mathcal{B}(\Omega) := \mathcal{B}_1(\Omega)$. For this choice of q, Theorem 1.2 provides the sufficient condition for a function to be bounded by unity.

LEMMA 3.1. *Let $\psi \in \mathcal{B}(\Omega)$. If the function $w \in \mathcal{H}[0,n]$ satisfies the differential inclusion*

$$\{\psi(w(z), zw'(z); z) : z \in \mathbb{D}\} \subset \Omega,$$

then $w \in \mathcal{B}$.

Using this lemma, the following results are obtained that give sufficient conditions for normalized analytic functions to satisfy the condition $|f(z)/(zf'(z)) - 1| < 1-\alpha$ in unit disk $\mathbb{D}$.

THEOREM 3.2. *Let $f \in \mathcal{A}$ and $0 \leq \alpha < 1$. If the function f satisfies the differential inequality $|Q_{CV}(z)/Q_{ST}(z) - 1| < 1-\alpha$ for all $z \in \mathbb{D}$, then $|f(z)/(zf'(z)) - 1| < 1-\alpha$.*

Proof. For the function $f \in \mathcal{A}$, let $w : \mathbb{D} \to \mathbb{C}$ be defined by

$$w(z) = \frac{1}{1-\alpha}\left(\frac{f(z)}{zf'(z)} - 1\right). \tag{35}$$

Then it can be noted that $|f(z)/(zf'(z)) - 1| < 1 - \alpha$ is equivalent to $w \in \mathcal{B}$. A straightforward computation involving logarithmic derivative gives

$$1 + \frac{zf''(z)}{f'(z)} = \frac{1-(1-\alpha)zw'(z)}{(1-\alpha)w(z)+1}. \tag{36}$$

Therefore, from equations (35) and (36), it follows that

$$\frac{Q_{CV}(z)}{Q_{ST}(z)} - 1 = -(1-\alpha)zw'(z). \tag{37}$$

Let the function $\psi : \mathbb{C}^2 \times \mathbb{D} \to \mathbb{C}$ be defined by

$$\psi(r, s; z) := -(1-\alpha)s.$$

For $\theta \in \mathbb{R}$ and $K \geq 1$, we have $\psi(e^{i\theta}, Ke^{i\theta}; z) = -(1-\alpha)Ke^{i\theta}$. Thus, it follows that

$$\left|\psi(e^{i\theta}, Ke^{i\theta}; z)\right| = |1-\alpha|K \geq 1-\alpha,$$

and hence $\psi \in \mathcal{B}(\mathbb{D}_{1-\alpha})$. Further, from (37) and hypothesis of the theorem, it can be observed that $\psi(w(z), zw'(z); z) \in \mathbb{D}_{1-\alpha}$ for all $z \in \mathbb{D}$. Thus, by Lemma 3.1, it can be concluded that $w \in \mathcal{B}$ and accordingly the result follows. □

Theorem 3.3. *Let $0 \leq \alpha < 1$. If the function $f \in \mathcal{A}$ satisfies $|Q_{CV}(z) - 1| < 2(1-\alpha)/(2-\alpha)$ for all $z \in \mathbb{D}$, then $|f(z)/(zf'(z)) - 1| < 1 - \alpha$.*

Proof. For $f \in \mathcal{A}$, let $w : \mathbb{D} \to \mathbb{C}$ be defined as in equation (35). It can be seen that

$$\frac{f(z)}{zf'(z)} = (1-\alpha)w(z) + 1.$$

By simple computations, $Q_{CV} - 1$ can be written in terms of w as

$$Q_{CV}(z) - 1 = \frac{-(1-\alpha)(zw'(z) + w(z))}{1 + (1-\alpha)w(z)}. \tag{38}$$

If the function $\psi : \mathbb{C}^2 \times \mathbb{D} \to \mathbb{C}$ is defined by

$$\psi(r, s; z) := \frac{-(1-\alpha)(r+s)}{1+(1-\alpha)r},$$

then from equation (38) and the hypothesis, it follows that $\psi(w(z), zw'(z); z) \in \mathbb{D}_{2(1-\alpha)/(2-\alpha)}$. For $\theta \in \mathbb{R}$ and $K \geq 1$,

$$\left|\psi(e^{i\theta}, Ke^{i\theta}; z)\right| = \left|\frac{-(1-\alpha)(1+K)e^{i\theta}}{1+(1-\alpha)e^{i\theta}}\right| = \frac{(1-\alpha)(1+K)}{|1+(1-\alpha)e^{i\theta}|} \geq \frac{2(1-\alpha)}{2-\alpha},$$

which implies that $\psi \in \mathcal{B}\left(\mathbb{D}_{2(1-\alpha)/(2-\alpha)}\right)$. Thus, by Lemma 3.1, we have $w \in \mathcal{B}$ and as a consequence, it follows that $|f(z)/(zf'(z)) - 1| < 1 - \alpha$. □

Theorem 3.4. *Let $0 \leq \alpha < 1$ and $f \in \mathcal{A}$. If the function f satisfies the inequality $|Q_{CV}(z) - Q_{ST}(z)| < (1-\alpha)/(2-\alpha)$ for all $z \in \mathbb{D}$, then $|f(z)/(zf'(z)) - 1| < 1 - \alpha$.*

Proof. Let $f \in \mathcal{A}$ and $w : \mathbb{D} \to \mathbb{C}$ be defined by equation (35), so that $|f(z)/zf'(z) - 1| < 1 - \alpha$ is equivalent to $w \in \mathcal{B}$. The functions Q_{ST} and Q_{CV} are given by

$$Q_{ST}(z) = \frac{1}{(1-\alpha)w(z)+1} \text{ and } Q_{CV}(z) = \frac{1-(1-\alpha)zw'(z)}{(1-\alpha)w(z)+1}.$$

Thus, it can be seen that

$$Q_{CV}(z) - Q_{ST}(z) = \frac{-(1-\alpha)zw'(z)}{(1-\alpha)w(z)+1}. \tag{39}$$

If a function $\psi : \mathbb{C}^2 \times \mathbb{D} \to \mathbb{C}$ is defined by

$$\psi(r, s; z) = \frac{-(1-\alpha)s}{(1-\alpha)r+1},$$

then for $\theta \in \mathbb{R}$ and $K \geq 1$,

$$\left|\psi(e^{i\theta}, Ke^{i\theta}; z)\right| = \left|\frac{-(1-\alpha)Ke^{i\theta}}{(1-\alpha)e^{i\theta}+1}\right| = \frac{(1-\alpha)K}{|(1-\alpha)e^{i\theta}+1|} \geq \frac{1-\alpha}{2-\alpha}.$$

This implies that $\psi \in \mathcal{B}\left(\mathbb{D}_{(1-\alpha)/(2-\alpha)}\right)$. Also, from equation (39) and hypothesis, it can be noted that $|\psi(w(z), w'(z); z)| < (1-\alpha)/(2-\alpha)$. Thus, the result follows by an application of Lemma 3.1. □

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