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ON PARTIAL SOLUTIONS TO CONJECTURES FOR RADIUS PROBLEMS INVOLVING LEMNISCATE OF BERNOULLI

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Abstract. Given a function f analytic in open disk centred at origin of radius unity and satisfying the condition $|f(z)/g(z)-1|<1$ for a analytic function g with certain prescribed conditions in the unit disk, radii constants R are determined for the values of $Rzf'(Rz)/f(Rz)$ to lie inside the domain enclosed by the curve $|w^2-1|=1$ (lemniscate of Bernoulli). This, in turn, provides a partial solution to the conjectures and problems for determination of sharp bounds R for such functions f.

1. Introduction

For $\alpha \in \mathbb{C}$ and $s>0$, let $\mathbb{D}(\alpha,s) := \{z \in \mathbb{C} : |z-\alpha| < s\}$ denotes the open disk centred at α and radius s. Let $\mathcal{A}$ be the class of analytic functions f with $f(0)=0=f'(0)-1$, which are defined in $\mathbb{D} := \mathbb{D}(0,1)$ and let $\mathcal{S} \subset \mathcal{A}$ consists of univalent functions. For $i=1,2,3,4,5$, consider the following class

$$\mathcal{G}_i = \left\{ f \in \mathcal{A} : \left| \frac{f(z)}{g(z)} - 1 \right| < 1 \text{ for some } g \in \mathcal{A} \text{ with } \operatorname{Re}\left(\frac{g(z)}{\psi_i(z)} \right) > 0 \, (z \in \mathbb{D}) \right\},$$

where the functions $\psi_i \in \mathcal{A}$ are given by z, $z+z^2/2$, $z/(1-z^2)$, $z/(1-z)^2$ and $z/(1+z)$ respectively. For the class $\mathcal{G}_1$, Ratti [23, Theorem 4, p. 245] determined its radius of univalence and starlikeness. Associated with various choices of ϕ in the class $\mathcal{S}^*(\phi) := \{f \in \mathcal{A} : zf'(z)/f(z) \prec \phi(z)\}$ studied in [19], further radii constants for the class $\mathcal{G}_1$ were investigated by Ali *et al.* [5, Theorem 2.3, p. 30], Gupta *et al.* [12, Theorem 6.1, p. 1170], Mendiratta *et al.* [21, Theorem 3.7, p. 383], Cho *et al.* [9, Theorem 4.1, p. 228], Arora and Kumar [6, Theorem 4.10, p. 1007], Kumar and Ravichandran [16, Theorem 3.5, p. 208], Wani and Swaminathan [35, Theorem 4.1, p. 178], Gandhi and Ravichandran [11, Theorem 3.3], Sharma *et al.* [28, Theorem 5.3, p. 936] and Gandhi [10, Theorem 3.5, p. 182]. Here the function ϕ in $\mathcal{S}^*(\phi)$ corresponds to a univalent function with $\phi(0)=1$, $\phi'(0)>0$, $\operatorname{Re}\phi>0$ in $\mathbb{D}$ and the domain $\phi(\mathbb{D})$ is starlike with respect to 1 and symmetric about the line $\operatorname{Im} z = 0$. Kanaga and Ravichandran [14], Lee *et al.* [18], Ahmad El-Faqeer *et al.* [1] and Sebastian and

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Ravichandran [26] investigated the similar radius problems for the classes $\mathcal{G}_2$, $\mathcal{G}_3$, $\mathcal{G}_4$ and $\mathcal{G}_5$ respectively.

In the present manuscript, we are concerned with the radius problems for the classes $\mathcal{G}_i$ $(i = 1, 2, 3, 4, 5)$ associated with the subclass

$$\mathcal{S}_L^* := \mathcal{S}^*(\sqrt{1+z}) = \left\{ f \in \mathcal{A} : \left| \left(\frac{zf'(z)}{f(z)} \right)^2 - 1 \right| < 1 \right\}$$

of starlike functions introduced by Sokół and Stankiewicz [33] and further investigated by several authors [2–4, 13, 15, 17, 25, 29–32, 34]. By the $\mathcal{S}_L^*$-radius for the subclass $\mathcal{G} \subseteq \mathcal{A}$, denoted by $\mathcal{R}_{\mathcal{S}_L^*}(\mathcal{G})$, we mean

$$\sup\{R : r^{-1} f(rz) \in \mathcal{S}_L^* \text{ for all } r \in (0, R] \text{ and } f \in \mathcal{G}\}.$$

Although the $\mathcal{S}_L^*$-radius has been computed for the classes $\mathcal{G}_1$, $\mathcal{G}_3$, $\mathcal{G}_4$ and $\mathcal{G}_5$ in [5, Theorem 2.2(a), p. 28], [18, Theorem 2(2), p. 4479], [1, Theorem 2(2), p. 523] and [26, Theorem 2.2(2), p. 91] respectively, the bounds did not turn out to be sharp. Their technique involved finding the disk $\mathbb{D}(\alpha, s)$ such that $zf'(z)/f(z) \in \mathbb{D}(\alpha, s)$ for $f \in \mathcal{G}_i$ $(i = 1, 3, 4, 5)$ and then applying the result of Ali *et al.* [4, Lemma 2.2, p. 6559] which embeds that disk into the domain enclosed by the curve $|w^2 - 1| = 1$ (known as lemniscate of Bernoulli). But this technique failed to yield the desired sharp radii bounds. Also, Kanaga and Ravichandran [14, Theorem 3.1, p. 422] did not compute the $\mathcal{S}_L^*$-radius for the class $\mathcal{G}_2$. In Section 2, we have employed a new technique to solve this problem of computing the upper bounds of $\mathcal{S}_L^*$-radii for the classes $\mathcal{G}_i$ $(i = 1, 2, 3, 4, 5)$, some of them turns out to be the conjectured radii constants for these classes. Moreover, explicit functions in these classes are provided to show that these upper bounds are actually attained.

In the last section, the upper bounds for $\mathcal{S}_L^*$-radii have been computed for the following subclasses of $\mathcal{A}$:

$$\mathcal{H} = \left\{ f \in \mathcal{A} : \left| \frac{f(z)}{g(z)} - 1 \right| < 1 \text{ for some } g \in \mathcal{K} \right\}$$

and

$$\mathcal{J}_\alpha = \left\{ f \in \mathcal{A} : \left| \frac{f(z)}{g(z)} - 1 \right| < 1 \text{ for some } g \in \mathcal{A} \text{ with } \operatorname{Re}\left(\frac{g(z)}{\zeta(z)} \right) > 0, \zeta \in \mathcal{S}^*(\alpha) \right\}.$$

Here, $\mathcal{K}$ and $\mathcal{S}^*(\alpha)$ are subclasses of $\mathcal{S}$ consisting of convex functions and starlike functions of order α $(0 \le \alpha < 1)$ respectively. The class $\mathcal{J}_\alpha$ was introduced by Causey and Merkes [7]. It is important to point out that $\mathcal{S}_L^*$-radius for the class $\mathcal{H}$ evaluated by Ali *et al.* [5, Theorem 2.5, p. 32] was not sharp. Moreover, the radius $\mathcal{R}_{\mathcal{S}_L^*}(\mathcal{J}_\alpha)$ was not calculated in [20] (although the other $\mathcal{S}^*(\phi)$-radii are computed there).

We shall employ the following lemmas concerning the class $\mathcal{P}(\gamma) =: \{p : \mathbb{D} \to \mathbb{C} : p(0) = 1 \text{ and } \operatorname{Re} p(z) > \gamma \text{ for all } z \in \mathbb{D}\}$, where $\gamma \in [0, 1)$.

Lemma 1.1. *[8, Lemma 4, p. 182] If $q \in \mathcal{P}(1/2)$, then*

$$\operatorname{Re}\left(\frac{zq'(z)}{q(z)} \right) \ge -\frac{|z|}{1+|z|} \quad \text{for} \quad |z| \le \frac{1}{3}.$$

Lemma 1.2. *[27, Lemma 2, p. 239] If $q \in \mathcal{P}(\gamma)$, $0 \le \gamma < 1$, then*

$$\operatorname{Re}\left(\frac{zq'(z)}{q(z)} \right) \le \left| \frac{zq'(z)}{q(z)} \right| \le \frac{2r(1-\gamma)}{(1-r)(1+(1-2\gamma)r)}, \quad |z| = r.$$

LEMMA 1.3. *[24, Lemma 2.1, p. 267] If $q \in \mathcal{P}(\gamma)$, $0 \le \gamma < 1$, then*

$$\left| q(z) - \frac{1+(1-2\gamma)r^2}{1-r^2} \right| \le \frac{2(1-\gamma)r}{1-r^2}, \quad |z| = r.$$

2. $\mathcal{S}_L^*$-radius for classes $\mathcal{G}_i$

In this section, we will compute the upper bounds for the $\mathcal{S}_L^*$-radius for classes $\mathcal{G}_i$, $i = 1,2,3,4,5$. In fact, the radius constant in part (a) of the following theorem coincides with the result [5, Conjecture 2.2, p. 32] for the class $\mathcal{G}_1$.

THEOREM 2.1. *The upper bounds of $\mathcal{S}_L^*$-radius for the classes $\mathcal{G}_i$, $i = 1,2,3,4,5$ are:*

S.No.	*Class*	ψ_i	$\mathcal{R}_{\mathcal{S}_L^*}(\mathcal{G}_i) \le r_i$
(a)	$\mathcal{G}_1$	z	$r_1 = \frac{3}{2} + \frac{3}{2\sqrt{2}} - \frac{1}{2}\sqrt{\frac{27}{2} + 7\sqrt{2}} \approx 0.142009$
(b)	$\mathcal{G}_2$	$z + \frac{z^2}{2}$	$r_2 \approx 0.12209$
(c)	$\mathcal{G}_3$	$\frac{z}{1-z^2}$	$r_3 = \frac{\sqrt{17-4\sqrt{2}}-3}{2\sqrt{2}} \approx 0.130093$
(d)	$\mathcal{G}_4$	$\frac{z}{(1-z)^2}$	$r_4 = \frac{\sqrt{33-4\sqrt{2}}-5}{2\sqrt{2}} \approx 0.0809876$
(e)	$\mathcal{G}_5$	$\frac{z}{1+z}$	$r_5 = (\sqrt{7-2\sqrt{2}}-2)(\sqrt{2}+1) \approx 0.102466$

Here, r_2 is the smallest root in $(0,1)$ satisfying $(3-\sqrt{2})r^3 - (2\sqrt{2}-1)r^2 - (8-\sqrt{2})r + 2(\sqrt{2}-1) = 0$. Moreover, there exist functions $f_i \in \mathcal{G}_i$ such that $f_i(r_i z)/r_i \in \mathcal{S}_L^*$ and

$$\left| \left(\frac{z f_i'(z)}{f_i(z)} \right)^2 - 1 \right| = \sqrt{2}$$

for some $z_i = r_i e^{i\theta}$, $\theta \in [0, 2\pi)$ for each $i = 1,2,3,4,5$.

Proof. If f belongs to any of the $\mathcal{G}_i$'s, then $q(z) = g(z)/f(z) \in \mathcal{P}(1/2)$ for some $g \in \mathcal{A}$. The central idea behind the proof lies on the observation that $\operatorname{Re}(zf'(z)/f(z)) < \sqrt{2}$ is a necessary condition for $zf'(z)/f(z)$ to lie inside Ω_L. Also, all the computed radii $r_i < 1/3$ for $i = 1,2,3,4,5$.

(a) Let $f \in \mathcal{G}_1$. In this case, the function $p_1(z) = g(z)/z$ is a member of $\mathcal{P}$ and

$$\frac{zf'(z)}{f(z)} = 1 + \frac{zp_1'(z)}{p_1(z)} - \frac{zq'(z)}{q(z)}. \tag{1}$$

By using Lemmas 1.1 and 1.2 in (1), we have

$$\begin{aligned}\operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) &= 1+\operatorname{Re}\left(\frac{zp_1'(z)}{p_1(z)}\right)-\operatorname{Re}\left(\frac{zq_1'(z)}{q_1(z)}\right)\\ &\le 1+\frac{2r}{1-r^2}+\frac{r}{1+r} \quad \text{for} \quad |z|=r\le\frac{1}{3}\\ &= \frac{1+3r-2r^2}{1-r^2}<\sqrt{2}\end{aligned}$$

provided $r<r_1:=(3-\sqrt{25-12\sqrt{2}})/(2(2-\sqrt{2}))$. Thus $\mathcal{R}_{\mathcal{S}^*_L}(\mathcal{G}_1)\le r_1$. The radius r_1 is attained for the function

$$f_1(z)=\frac{z(1+z)^2}{1-z} \quad \text{with} \quad g_1(z)=\frac{z(1+z)}{1-z}.$$

Clearly, $f_1\in\mathcal{G}_1$. In order to show that $w=zf_1'(z)/f_1(z)\in\Omega_L$ for $|z|<r_1$, let us compute the expression for $|w^2-1|$. For $z=re^{it}$ and $u=\cos t$, a long and tedious calculation gives

$$\left|\left(\frac{zf_1'(z)}{f_1(z)}\right)^2-1\right|^2=\left|\left(\frac{1+3z-2z^2}{1-z^2}\right)^2-1\right|^2=\frac{a_1(r,u)}{b_1(r,u)},$$

where

$$a_1(r,u)=r^2(9+r^2-6ru)(4+21r^2+9r^4+12ru-18r^3u-24r^2u^2)$$

and

$$b_1(r,u)=(1+r^2-2ru)^2(1+r^2+2ru)^2.$$

Consequently, it is enough to show that the function $h_1(r,u)=b_1(r,u)-a_1(r,u)$ is positive for $r<r_1$. Observe that the roots of $h_1(r,u)=0$ in $(0,1)$ are decreasing as a function of $u\in[-1,1]$. Consequently, it follows that $h_1(r,u)>0$ for $-1\le u\le 1$ if and only if

$$h_1(r,1)=(1-3r-2r^2)^2(1-6r-9r^2+12r^3-2r^4)>0,$$

which gives $r<r_1$ (see Figure 1). Thus $f_1(r_1z)/r_1\in\mathcal{S}^*_L$.

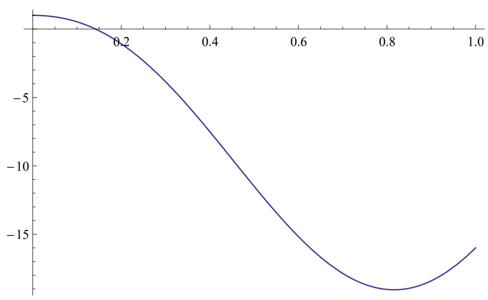


FIGURE 1. $h_1(r,1)$ for $r\in(0,1)$

(b) Let $f\in\mathcal{G}_2$. Then $p_2(z)=g(z)/(z+z^2/2)\in\mathcal{P}$ and $f(z)=p_2(z)(z+z^2/2)/q(z)$ which yields

$$\frac{zf'(z)}{f(z)}=\frac{zp_2'(z)}{p_2(z)}-\frac{zq'(z)}{q(z)}+\frac{2(1+z)}{2+z}. \tag{2}$$

Using Lemmas 1.1 and 1.2, (2) gives

$$\begin{aligned}\operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) &= \operatorname{Re}\left(\frac{zp_2'(z)}{p_2(z)}\right) - \operatorname{Re}\left(\frac{zq'(z)}{q(z)}\right) + \operatorname{Re}\left(\frac{2(1+z)}{2+z}\right)\\ &\leq \frac{2r}{1-r^2} + \frac{r}{1+r} + \frac{2(1+r)}{2+r} \quad \text{for} \quad |z| = r \leq \frac{1}{3}\\ &= \frac{2+8r-r^2-3r^3}{(1-r^2)(2+r)} < \sqrt{2},\end{aligned}$$

if $r < r_2$ where $r_2 \approx 0.12209$ is defined in the statement of the theorem. In order to show that this upper bound is achieved, consider the function

$$f_2(z) = \frac{(z+z^2/2)(1+z)^2}{1-z} \quad \text{with} \quad g_2(z) = \frac{(z+z^2/2)(1+z)}{1-z}.$$

Then $f_2 \in \mathcal{G}_2$. Now we will make use of the similar technique as carried out in part (a) to establish that $zf_2'(z)/f_2(z) \in \Omega_L$ for $|z| < r_2$. Observe that

$$\left|\left(\frac{zf_2'(z)}{f_2(z)}\right)^2 - 1\right|^2 = \left|\left(\frac{2+8z-z^2-3z^3}{(1-z^2)(2+z)}\right)^2 - 1\right|^2 = \frac{a_2(r,u)}{b_2(r,u)},$$

where

$$\begin{aligned}a_2(r,u) = r^2(&49+29r^2+4r^4+14ru-4r^3u-56r^2u^2)(16+105r^2+81r^4\\ &+16r^6+72ru+42r^3u+24r^5u-48r^2u^2-144r^4u^2-128r^3u^3)\end{aligned}$$

and

$$b_2(r,u) = (1+r^2-2ru)^2(1+r^2+2ru)^2(4+r^2+4ru)^2$$

for $z = re^{it}$ and $u = \cos t$. We now show that the function $h_2(r,u) = b_2(r,u) - a_2(r,u)$ is positive for $r < r_2$. It can be seen that the roots of $h_2(r,u) = 0$ in $(0,1)$ are decreasing as a function of $u \in [-1,1]$. Therefore, $h_2(r,u) > 0$ for $u \in [-1,1]$ if and only if

$$h_2(r,1) = (2+8r-r^2-3r^3)^2(4-24r-74r^2+12r^3+51r^4+2r^5-7r^6) > 0.$$

This gives $r < r_2$ (as illustrated in Figure 2). Hence $f_2(r_2z)/r_2 \in \mathcal{S}_L^*$.

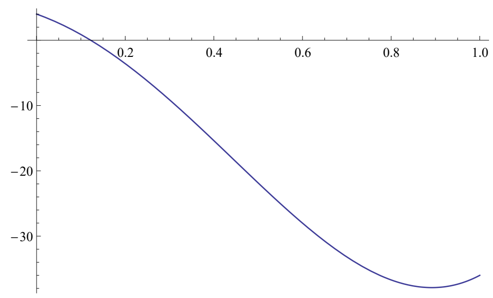


FIGURE 2. Graph of $h_2(r,1)$ for $r \in (0, 0.2)$

(c) If $f \in \mathcal{G}_3$ and $p_3(z) = g(z)(1-z^2)/z$, then $p_3 \in \mathcal{P}$ and $f(z) = zp_3(z)/(q(z)(1-z^2))$ which yields

$$\frac{zf'(z)}{f(z)} = \frac{zp_3'(z)}{p_3(z)} - \frac{zq'(z)}{q(z)} + \frac{1+z^2}{1-z^2}. \tag{3}$$

Equation (3) together with Lemmas 1.1 and 1.2 simplifies to

$$\begin{aligned}\operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) &= \operatorname{Re}\left(\frac{zp_3'(z)}{p_3(z)}\right) - \operatorname{Re}\left(\frac{zq'(z)}{q(z)}\right) + \operatorname{Re}\left(\frac{1+z^2}{1-z^2}\right)\\ &\leq \frac{2r}{1-r^2} + \frac{r}{1+r} + \frac{1+r^2}{1-r^2} \quad \text{for} \quad |z| = r \leq \frac{1}{3}\\ &= \frac{1+3r}{1-r^2}.\end{aligned}$$

If the above expression is less than $\sqrt{2}$, then it is easy to deduce that $\mathcal{R}_{\mathcal{S}_L^*}(\mathcal{G}_3) \leq r_3$ where $r_3 := (\sqrt{17-4\sqrt{2}}-3)/(2\sqrt{2})$. If we consider the function

$$f_3(z) = \frac{z(1+z)}{(1-z)^2} \quad \text{with} \quad g_3(z) = \frac{z}{(1-z)^2},$$

then $f_3 \in \mathcal{G}_3$ and for $z = re^{it}$ and $u = \cos t$, we have

$$\left|\left(\frac{zf_3'(z)}{f_3(z)}\right)^2 - 1\right|^2 = \left|\left(\frac{1+3z}{1-z^2}\right)^2 - 1\right|^2 = \frac{a_3(r,u)}{b_3(r,u)},$$

where $a_3(r,u) = r^2(9+r^2+6ru)(4+13r^2+r^4+12ru-6r^3u-8r^2u^2)$ and $b_3(r,u) = (1+r^2-2ru)^2(1+r^2+2ru)^2$. Since the roots of the equation $h_3(r,u) = b_3(r,u)-a_3(r,u) = 0$ in $(0,1)$ are decreasing as a function of $u \in [-1,1]$, therefore $h_3(r,u) > 0$ for $u \in [-1,1]$ if and only if

$$h_3(r,1) = (1+3r)^2(1-6r-13r^2+2r^4) > 0$$

which is possible if $r < r_3$ (Figure 3). Hence $zf_3'(z)/f_3(z) \in \Omega_L$ for $|z| < r_3$.

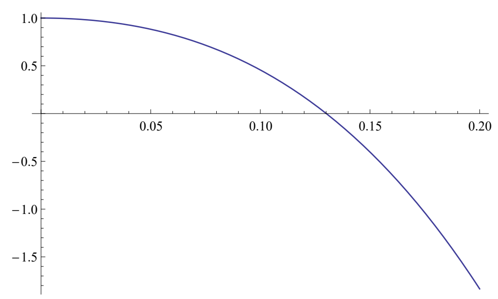


FIGURE 3. Graph of $h_3(r,1)$ for $r \in (0, 0.2)$

(d) Let $f \in \mathcal{G}_4$. Then $p_4(z) = g(z)(1-z)^2/z \in \mathcal{P}$ and $f(z) = zp_4(z)/(q(z)(1-z)^2)$. A straightforward calculation leads to

$$\frac{zf'(z)}{f(z)} = \frac{zp_4'(z)}{p_4(z)} - \frac{zq'(z)}{q(z)} + \frac{1+z}{1-z}. \tag{4}$$

In this case as well, the use of Lemmas 1.1 and 1.2 in (4) gives

$$\begin{aligned}\operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) &= \operatorname{Re}\left(\frac{zp_4'(z)}{p_4(z)}\right) - \operatorname{Re}\left(\frac{zq'(z)}{q(z)}\right) + \operatorname{Re}\left(\frac{1+z}{1-z}\right)\\ &\leq \frac{2r}{1-r^2} + \frac{r}{1+r} + \frac{1+r}{1-r} \quad \text{for} \quad |z| = r \leq \frac{1}{3}\\ &= \frac{1+5r}{1-r^2} < \sqrt{2},\end{aligned}$$

provided $r < r_4 := (\sqrt{33-4\sqrt{2}}-5)/(2\sqrt{2})$. This infers that $\mathcal{R}_{\mathcal{S}^*_L}(\mathcal{G}_4) \leq r_4$. The attainability of the bound r_4 can be seen by considering the function

$$f_4(z) = \frac{z(1+z)^2}{(1-z)^3} \quad \text{with} \quad g_4(z) = \frac{z(1+z)}{(1-z)^3}.$$

Clearly $f_4 \in \mathcal{G}_4$. We now show that $zf_4'(z)/f_4(z) \in \Omega_L$ for $|z| < r_4$. A lengthy calculation gives

$$\left|\left(\frac{zf_4'(z)}{f_4(z)}\right)^2 - 1\right|^2 = \left|\left(\frac{1+5z}{1-z^2}\right)^2 - 1\right|^2 = \frac{a_4(r,u)}{b_4(r,u)},$$

where $a_4(r,u) = r^2(25+r^2+10ru)(4+29r^2+r^4+20ru-10r^3u-8r^2u^2)$ and $b_4(r,u) = (1+r^2-2ru)^2(1+r^2+2ru)^2$ for $z = re^{it}$ and $u = \cos t$. Using the similar analysis executed in previous parts, $h_4(r,u) = b_4(r,u) - a_4(r,u) > 0$ for $u \in [-1,1]$ if and only if

$$h_4(r,1) = (1+5r)^2(1-10r-29r^2+2r^4) > 0$$

which leads to $r < r_4$ (illustrated in Figure 4). Hence $f_4(r_4z)/r_4 \in \mathcal{S}^*_L$.

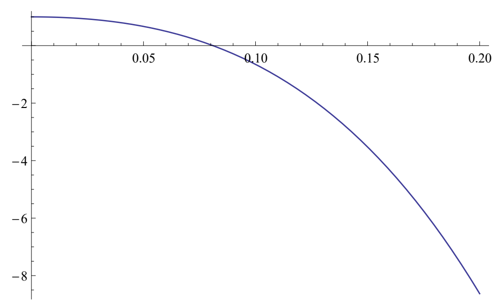


FIGURE 4. Graph of $h_4(r,1)$ for $r \in (0.0.2)$

(e) If $f \in \mathcal{G}_5$, then

$$\frac{zf'(z)}{f(z)} = \frac{zp_5'(z)}{p_5(z)} - \frac{zq'(z)}{q(z)} + \frac{1}{1+z} \tag{5}$$

where $p_5(z) = g(z)(1+z)/z \in \mathcal{P}$. By using Lemmas 1.1 and 1.2 in (5), we obtain

$$\begin{aligned}
\operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) &= \operatorname{Re}\left(\frac{zp_5'(z)}{p_5(z)}\right) - \operatorname{Re}\left(\frac{zq'(z)}{q(z)}\right) + \operatorname{Re}\left(\frac{1}{1+z}\right)\\
&\leq \frac{2r}{1-r^2} + \frac{r}{1+r} + \frac{1}{1-r} \quad \text{for} \quad |z| = r \leq \frac{1}{3}\\
&= \frac{1+4r-r^2}{1-r^2} < \sqrt{2},
\end{aligned}$$

if $r < r_5 := (\sqrt{7-2\sqrt{2}} - 2)(\sqrt{2}+1)$.

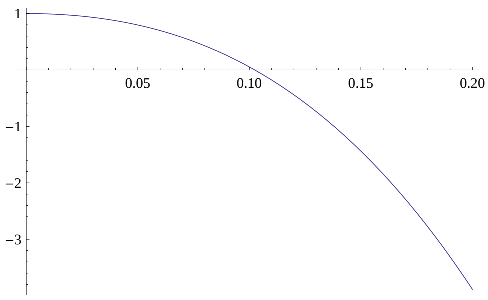


FIGURE 5. Graph of $h_5(r, -1)$ for $r \in (0, 0.2)$

The function

$$f_5(z) = \frac{z(1-z)^2}{(1+z)^2} \quad \text{with} \quad g_5(z) = \frac{z(1-z)}{(1+z)^2}$$

is in the class $\mathcal{G}_5$. For $z = re^{it}$ and $u = \cos t$, if we set $a_5(r,u) = 64r^2(1 + 6r^2 + r^4 - 4ru + 4r^3u - 4r^2u^2)$ and $b_5(r,u) = (1 + r^2 - 2ru)^2(1 + r^2 + 2ru)^2$, then

$$\left|\left(\frac{zf_5'(z)}{f_5(z)}\right)^2 - 1\right|^2 = \left|\left(\frac{1-4z-z^2}{1-z^2}\right)^2 - 1\right|^2 = \frac{a_5(r,u)}{b_5(r,u)}.$$

We now show that the function $h_5(r,u) = b_5(r,u) - a_5(r,u)$ is positive for $r < r_5$. Observe that the roots of $h_5(r,u) = 0$ in $(0,1)$ are increasing as a function of $u \in [-1,1]$. Therefore, $h_5(r,u) > 0$ for $u \in [-1,1]$ if and only if

$$h_5(r,-1) = (1 + 4r - r^2)^2(1 - 8r - 18r^2 + 8r^3 + r^4) > 0.$$

This gives $r < r_5$ (Figure 5) and hence $f_5(r_5 z)/r_5 \in \mathcal{S}_L^*$. □

3. $\mathcal{S}_L^*$-radius for classes $\mathcal{H}$ and $\mathcal{J}_\alpha$

The first theorem of this section calculates the upper bound of $\mathcal{S}_L^*$-radius for the class $\mathcal{H}$ which in turn matches with the result [5, Conjecture 2.3, p. 34].

THEOREM 3.1. *The upper bound for the radius $\mathcal{R}_{\mathcal{S}_L^*}(\mathcal{H})$ is*

$$\mathcal{R}_{\mathcal{S}_L^*}(\mathcal{H}) \leq -1 - \sqrt{2} + \sqrt{2(2+\sqrt{2})} \approx 0.198912.$$

Proof. Let $f \in \mathcal{H}$ with associated convex function g. By [22, Section 2.6, p. 56], $zg'/g \in \mathcal{P}(1/2)$. Also, $q = g/f$ is a member of $\mathcal{P}(1/2)$ and

$$\frac{zf'(z)}{f(z)} = \frac{zg'(z)}{g(z)} - \frac{zq'(z)}{q(z)}. \tag{6}$$

By Lemmas 1.1 and 1.3, (6) yields

$$\begin{aligned}\operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) &= \operatorname{Re}\left(\frac{zg'(z)}{g(z)}\right) - \operatorname{Re}\left(\frac{zq'(z)}{q(z)}\right)\\ &\leq \frac{1}{1-r} + \frac{r}{1+r} \quad \text{for} \quad |z| = r \leq \frac{1}{3}\\ &= \frac{1+2r-r^2}{1-r^2} < \sqrt{2},\end{aligned}$$

which simplifies to $r < r_{\mathcal{H}} := -1 - \sqrt{2} + \sqrt{2(2+\sqrt{2})}$. This infers that $\mathcal{R}_{\mathcal{S}^*_L}(\mathcal{H}) \leq r_{\mathcal{H}}$.

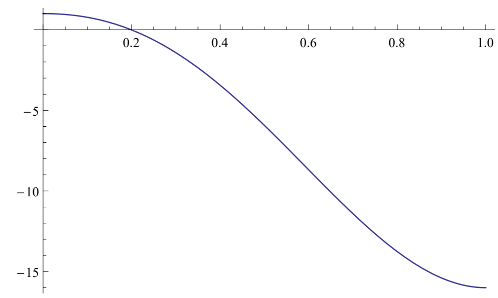


FIGURE 6. Graph of $k_1(r,1)$ for $r \in (0,1)$

The bound $r_{\mathcal{H}}$ is attained, as seen by the function

$$f_0(z) = \frac{z(1+z)}{1-z} \in \mathcal{H} \text{ with } g_0(z) = \frac{z}{1-z}.$$

If we write $z = re^{it}$ and $u = \cos t$, then

$$\left|\left(\frac{zf_0'(z)}{f_0(z)}\right)^2 - 1\right|^2 = \left|\left(\frac{1+2z-z^2}{1-z^2}\right)^2 - 1\right|^2 = \frac{c_1(r,u)}{d_1(r,u)}.$$

The roots of the equation $k_1(r,u) = d_1(r,u) - c_1(r,u) = 0$ in $(0,1)$ are decreasing as a function of $u \in [-1,1]$, where $c_1(r,u) = 16r^2(r^4 - 2r^3u + r^2(3-4u^2) + 2ru + 1)$ and $d_1(r,u) = (1+r^2-2ru)^2(1+r^2+2ru)^2$. Hence $k_1(r,u) > 0$ for $u \in [-1,1]$ if and only if

$$k_1(r,1) = (1+2r-r^2)^2(1-4r-6r^2+4r^3+r^4) > 0$$

which gives $r < r_{\mathcal{H}}$ (Figure 6). Therefore, $f_0(r_{\mathcal{H}}z)/r_{\mathcal{H}} \in \mathcal{S}^*_L$. □

The last result determines the upper bound of the radius $\mathcal{R}_{\mathcal{S}^*_L}(\mathcal{J}_\alpha)$, $\alpha \in [0,1)$.

THEOREM 3.2. *For* $0 \leq \alpha < 1$, *we have*

$$\mathcal{R}_{\mathcal{S}^*_L}(\mathcal{J}_\alpha) \leq \frac{2(\sqrt{2}-1)}{5 - 2\alpha + \sqrt{33 - 4\sqrt{2} - 12\alpha - 8\sqrt{2}\alpha + 4\alpha^2}}.$$

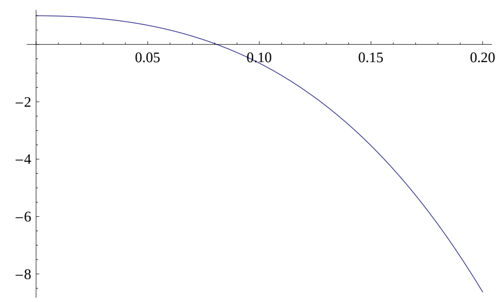
Proof. Let $f \in \mathcal{J}_\alpha$. Then $j_1 = g/f \in \mathcal{P}(1/2)$ and $j_2 = g/\zeta \in \mathcal{P}$ satisfy $f(z) = \zeta(z)j_2(z)/j_1(z)$. Note that

$$\frac{zf'(z)}{f(z)} = \frac{zj_2'(z)}{j_2(z)} - \frac{zj_1'(z)}{j_1(z)} + \frac{z\zeta'(z)}{\zeta(z)}. \tag{7}$$

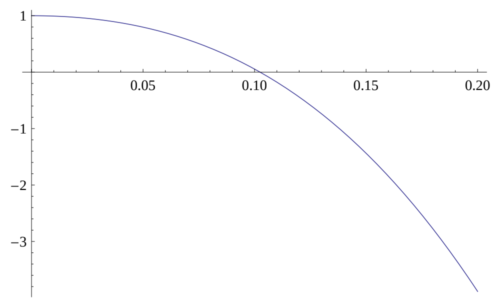
In accordance with Lemmas 1.1, 1.2 and 1.3 in (7), we obtain

$$\begin{aligned}
\operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) &= \operatorname{Re}\left(\frac{zj_2'(z)}{j_2(z)}\right) - \operatorname{Re}\left(\frac{zj_1'(z)}{j_1(z)}\right) + \operatorname{Re}\left(\frac{z\zeta'(z)}{\zeta(z)}\right)\\
&\le \frac{2r}{1-r^2} + \frac{r}{1+r} + \frac{1+r-2\alpha r}{1-r} \quad \text{for} \quad |z| = r \le \frac{1}{3}\\
&= \frac{1+(5-2\alpha)r - 2\alpha r^2}{1-r^2} < \sqrt{2},
\end{aligned}$$

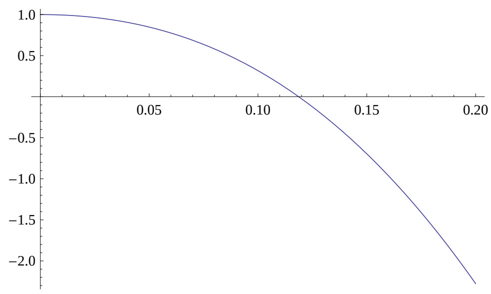
which yields $r < r_\alpha := 2(\sqrt{2}-1)/(5 - 2\alpha + \sqrt{33 - 4\sqrt{2} - 12\alpha - 8\sqrt{2}\alpha + 4\alpha^2})$.



(a) $\alpha = 0$



(b) $\alpha = 1/2$



(c) $\alpha = 3/4$

FIGURE 7. Graph of $k_2(r, 1, \alpha)$ for $r \in (0, 0.2)$

Consider the functions

$$f_\alpha(z) = \frac{z(1+z)^2}{(1-z)^{3-2\alpha}}, \quad g_\alpha(z) = \frac{z(1+z)}{(1-z)^{3-2\alpha}} \quad \text{and} \quad \zeta_\alpha(z) = \frac{z}{(1-z)^{2-2\alpha}}.$$

Then $f_\alpha \in \mathcal{J}_\alpha$. If $z = re^{it}$ and $u = \cos t$, then

$$\left|\left(\frac{zf_\alpha'(z)}{f_\alpha(z)}\right)^2 - 1\right|^2 = \left|\left(\frac{1-(2\alpha-5)z - 2\alpha z^2}{1-z^2}\right)^2 - 1\right|^2 = \frac{c_2(r,u,\alpha)}{d_2(r,u,\alpha)}.$$

We now show that the function $k_2(r,u,\alpha) = d_2(r,u,\alpha) - c_2(r,u,\alpha)$ is positive for $r < r_\alpha$ where $c_2(r,u,\alpha) = r^2(25 - 20\alpha + 4\alpha^2 + r^2 - 4\alpha r^2 + 4\alpha^2 r^2 + 10ru - 24\alpha ru +$

$8\alpha^2 ru)(4+29r^2-12\alpha r^2+4\alpha^2r^2+r^4+4\alpha r^4+4\alpha^2r^4+20ru-8\alpha ru-10r^3u-16\alpha r^3u+8\alpha^2r^3u-8r^2u^2-16\alpha r^2u^2)$ and $d_2(r,u,\alpha)=(1+r^2-2ru)^2(1+r^2+2ru)^2$. For each $0\leq\alpha<1$, the roots of $k_2(r,u,\alpha)=0$ in $(0,1)$ are decreasing as a function of $u\in[-1,1]$. Therefore, $k_2(r,u,\alpha)>0$ for $u\in[-1,1]$ if and only if $k_2(r,1,\alpha)=(1+(5-2\alpha)r-2\alpha r^2)^2(1-(10-4\alpha)r-(29-24\alpha+4\alpha^2)r^2+4\alpha(5-2\alpha)r^3+(2-4\alpha^2)r^4)>0$ which holds for $r<r_\alpha$ (Figure 7 depicts the graph of $k_2(r,1,\alpha)$ for specific values of α). This proves that $f_\alpha(r_\alpha z)/r_\alpha\in\mathcal{S}^*_L$. □

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