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# FABER POLYNOMIAL COEFFICIENT ESTIMATES FOR ANALYTIC BI-UNIVALENT FUNCTIONS ASSOCIATED WITH GREGORY COEFFICIENTS

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Abstract. In this work, we consider the function

$$\Psi(z)=\frac{z}{\ln(1+z)}=1+\sum_{n=1}^{\infty}G_n z^n$$

whose coefficients $G_n$ are the Gregory coefficients related to Stirling numbers of the first kind and introduce a new subclass $\mathcal{G}_{\Sigma}^{\lambda,\mu}(\Psi)$ of analytic bi-univalent functions subordinate to the function $\Psi$.

For functions belong to this class, we investigate the estimates for the general Taylor-Maclaurin coefficients by using the Faber polynomial expansions. In certain cases, our estimates improve some of those existing coefficient bounds.

## 1. Introduction

The generating equation of the Gregory coefficients $G_n$ (see [5, 24]) is given by

$$\frac{z}{\ln(1+z)}=1+\sum_{n=1}^{\infty}G_n z^n \tag{1}$$

$$\left(z\in\mathbb{U}:=\{z\in\mathbb{C}:|z|<1\},\quad G_0=1,\quad \ln 1=0\right),$$

where

$$G_n=\frac{1}{n!}\sum_{l=1}^{n}\frac{S_1(n,l)}{l+1}$$

and $S_1(n,l)$ is Stirling numbers of the first kind given by

$$S_1(n,l)=\begin{cases}\frac{(2n-l)!}{(l-1)!}\sum_{k=0}^{n-l}\frac{1}{(n+k)\,(n-l-k)!\,(n-l+k)!}\sum_{r=0}^{k}\frac{(-1)^r\,r^{n-l+k}}{r!\,(k-r)!} & ,\quad l\in[1,n]\\ 1 & ,\quad n=0,\ l=0\\ 0 & ,\quad \text{otherwise}\end{cases},$$

(see also [6]). In particular, we have:

$$\begin{aligned}S_1(1,1)&=1,\quad S_1(2,1)=-1,\quad S_1(2,2)=1,\\ S_1(3,1)&=2,\quad S_1(3,2)=-3,\quad S_1(3,3)=1.\end{aligned}$$

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Thus, initial values of $G_n$ for $n \in \mathbb{N} := \{1, 2, 3, \ldots\}$ are

$$G_1 = \frac{1}{2}, \quad G_2 = -\frac{1}{12}, \quad G_3 = \frac{1}{24}, \quad G_4 = -\frac{19}{720}, \quad G_5 = \frac{3}{160}. \tag{2}$$

Let $\mathcal{H}$ be the class of analytic functions in the open unit disc $\mathbb{U}$, and consider the classes $\mathcal{P}$, $\mathcal{A}$ and $\mathcal{S}$ defined by

$$\begin{aligned} \mathcal{P} &= \{p \in \mathcal{H} : p(0) = 1 \quad \text{and} \quad \Re(p(z)) > 0 \ (z \in \mathbb{U})\}, \\ \mathcal{A} &= \{f \in \mathcal{H} : f(0) = f'(0) - 1 = 0\}, \\ \mathcal{S} &= \{f \in \mathcal{A} : f \text{ is univalent in } \mathbb{U}\}, \end{aligned}$$

respectively.

For two functions $\mathfrak{f}, \mathfrak{g} \in \mathcal{H}$, we say that the function $\mathfrak{f}$ is subordinate to $\mathfrak{g}$ in $\mathbb{U}$, and write

$$\mathfrak{f}(z) \prec \mathfrak{g}(z), \qquad (z \in \mathbb{U}),$$

if there exists a Schwarz function

$$\omega \in \Omega := \{\omega \in \mathcal{H} : \omega(0) = 0 \quad \text{and} \quad |\omega(z)| < 1, \ (z \in \mathbb{U})\},$$

such that

$$\mathfrak{f}(z) = \mathfrak{g}(\omega(z)), \quad (z \in \mathbb{U}).$$

It is clear that the function $f \in \mathcal{A}$ can be expressed as

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \qquad (z \in \mathbb{U}). \tag{3}$$

By Koebe One-Quarter Theorem [12], every function $f \in \mathcal{S}$ has an inverse $f^{-1}$, which is defined by

$$f^{-1}(f(z)) = z \qquad (z \in \mathbb{U})$$

and

$$f\left(f^{-1}(w)\right) = w \qquad \left(|w| < r^*; \quad r^* \geq \frac{1}{4}\right).$$

For the inverse function $g := f^{-1}$, we obtain

$$\begin{aligned} g(w) &:= f^{-1}(w) \\ &= w - a_2 w^2 + \left(2a_2^2 - a_3\right) w^3 - \left(5a_2^3 - 5a_2 a_3 + a_4\right) w^4 + \cdots \\ &=: w + \sum_{n=2}^{\infty} A_n w^n. \end{aligned} \tag{4}$$

If the functions $f$ and $f^{-1}$ are univalent in $\mathbb{U}$, then $f$ is called bi-univalent function. Let $\Sigma$ denote the class of bi-univalent functions in $\mathbb{U}$ given by (3).

The class $\Sigma$ of analytic bi-univalent functions was first introduced by Lewin [20], where it was proved that $|a_2| < 1.51$. Brannan and Clunie [7] improved Lewin's result to $|a_2| \leq \sqrt{2}$ and later Netanyahu [22] proved that $\max_{f \in \Sigma} |a_2| = \frac{4}{3}$. For more details and examples of functions belong to the class $\Sigma$, see [28] (see also [11, 18, 27, 28, 31, 32]).

Brannan and Taha [8] and Taha [29] investigated certain subclasses of $\Sigma$ and found non-sharp estimates on $|a_2|$ and $|a_3|$. Furthermore, Sivasubramanian et al. [26] verified Brannan and Clunie's conjecture $|a_2| \leq \sqrt{2}$ for some subclasses of $\Sigma$. But the bounds on the general coefficient $|a_n|$ for $n > 3$ are not much known. In this study, we use the Faber polynomial expansions introduced by Faber [13] to determine general coefficient bound $|a_n|$ of analytic bi-univalent functions whose coefficients are related to Gregory coefficients. Some works investigated the general coefficient bounds $|a_n|$ using Faber polynomial expansions can be found in [3, 4, 9, 15, 16, 19, 25, 33], (see also [10, 14, 17]).

## 2. The Class $\mathcal{G}_{\Sigma}^{\lambda,\mu}(\Psi)$

Throughout this paper, we assume that $\lambda \geq 1$ and $\mu \geq 0$. By considering the function $\Psi$,

$$\Psi(z) = \frac{z}{\ln(1+z)}$$

given by $(1)$, we introduce a new subclass of analytic bi-univalent functions as follows.

Definition 2.1. A function $f \in \Sigma$ given by (3) is said to be in the class $\mathcal{G}_{\Sigma}^{\lambda,\mu}(\Psi)$ if the following conditions are satisfied:

$$(1-\lambda)\left(\frac{f(z)}{z}\right)^{\mu} + \lambda f'(z)\left(\frac{f(z)}{z}\right)^{\mu-1} \prec \Psi(z) \tag{5}$$

and

$$(1-\lambda)\left(\frac{g(w)}{w}\right)^{\mu} + \lambda g'(w)\left(\frac{g(w)}{w}\right)^{\mu-1} \prec \Psi(w) \tag{6}$$

where $z, w \in \mathbb{U}$ and $g = f^{-1}$ is defined by $(4)$.

Remark 2.2. We obtain the following bi-univalent function classes which consists of functions $f \in \Sigma$ for the special values of $\lambda$ and $\mu$.
$(i)$ For $\mu = 1$ :

$$\mathcal{H}_{\Sigma}^{\lambda}(\Psi) = \left\{ f \in \Sigma : \ (1-\lambda)\frac{f(z)}{z} + \lambda f'(z) \prec \Psi(z) \text{ and } (1-\lambda)\frac{f(z)}{z} + \lambda f'(z) \prec \Psi(z) \right\}.$$

$(ii)$ For $\mu = 1$ and $\lambda = 1$ :

$$\mathcal{N}_{\Sigma}(\Psi) = \{ f \in \Sigma : \ f'(z) \prec \Psi(z) \qquad \text{and} \qquad g'(w) \prec \Psi(w) \}.$$

introduced by Murugusundaramoorthy et al. [21].
$(iii)$ For $\mu = 0$ and $\lambda = 1$ :

$$\mathcal{S}_{\Sigma}(\Psi) = \left\{ f \in \Sigma : \ \frac{zf'(z)}{f(z)} \prec \Psi(z) \qquad \text{and} \qquad \frac{wg'(w)}{g(w)} \prec \Psi(w) \right\}.$$

$(iv)$ For $\lambda = 1$ :

$$\mathcal{B}_{\Sigma}^{\mu}(\Psi) = \left\{ f \in \Sigma : \ \frac{z^{1-\mu} f'(z)}{(f(z))^{1-\mu}} \prec \Psi(z) \qquad \text{and} \qquad \frac{w^{1-\mu} g'(w)}{(g(w))^{1-\mu}} \prec \Psi(w) \right\}.$$

Remark 2.3. Since the function $\Psi$ is univalent (see [21]), the class $\mathcal{G}_{\Sigma}^{\lambda,\mu}(\Psi)$ is related to the class $\mathcal{N}(\mu)$ of non-Bazilević functions defined by Obradović [23] as follows:

$$\Re\left( f'(z)\left(\frac{z}{f(z)}\right)^{1+\mu} \right) > 0 \qquad (0 < \mu < 1, \ z \in \mathbb{U}).$$

## 3. Coefficient estimates

Using the Faber polynomial expansion of functions $f \in \mathcal{A}$ of the form (3), the coefficients of its inverse function $g = f^{-1}$ can be expressed as, [1]:

$$g(w) = f^{-1}(w) = w + \sum_{n=2}^{\infty} \frac{1}{n} K_{n-1}^{-n}(a_2, a_3, \ldots) w^n, \tag{7}$$

where

$$\begin{aligned} K_{n-1}^{-n} &= \frac{(-n)!}{(-2n+1)!\,(n-1)!} a_2^{n-1} + \frac{(-n)!}{(2(-n+1))!\,(n-3)!} a_2^{n-3} a_3 \\ &+ \frac{(-n)!}{(-2n+3)!\,(n-4)!} a_2^{n-4} a_4 \\ &+ \frac{(-n)!}{(2(-n+2))!\,(n-5)!} a_2^{n-5}\left[a_5 + (-n+2) a_3^2\right] \\ &+ \frac{(-n)!}{(-2n+5)!\,(n-6)!} a_2^{n-6}\left[a_6 + (-2n+5) a_3 a_4\right] + \sum_{j \geq 7} a_2^{n-j} V_j, \end{aligned} \tag{8}$$

such that $V_j$ with $7 \leq j \leq n$ is a homogeneous polynomial in the variables $a_2, a_3, \ldots, a_n$, [2]. For $n = 2, 3, 4$, we obtain

$$K_1^{-2} = -2a_2, \qquad K_2^{-3} = 3\left(2a_2^2 - a_3\right), \qquad K_3^{-4} = -4\left(5a_2^3 - 5a_2a_3 + a_4\right), \tag{9}$$

respectively. In general case, for any $p \in \mathbb{N}$, $K_n^p$ is, [1],

$$K_n^p = pa_n + \frac{p(p-1)}{2} D_n^2 + \frac{p!}{(p-3)!\,3!} D_n^3 + \cdots + \frac{p!}{(p-n)!\,n!} D_n^n, \tag{10}$$

where

$$D_n^p = D_n^p(a_2, a_3, \ldots),$$

and by [30],

$$D_n^m(a_1, a_2, \ldots, a_n) = \sum \frac{m!}{i_1! \ldots i_n!} a_1^{i_1} \ldots a_n^{i_n}$$

while $a_1 = 1$, and the sum is taken over all non-negative integers $i_1, \ldots, i_n$ satisfying

$$\begin{cases} i_1 + i_2 + \cdots + i_n = m \\ i_1 + 2i_2 + \cdots + ni_n = n \end{cases}.$$

It is clear that

$$D_n^n(a_1, a_2, \ldots, a_n) = a_1^n.$$

Consequently, for functions $f \in \mathcal{A}$ of the form $(3)$, we can write

$$(1-\lambda)\left(\frac{f(z)}{z}\right)^{\mu} + \lambda f'(z)\left(\frac{f(z)}{z}\right)^{\mu-1} = 1 + \sum_{n=2}^{\infty} F_{n-1}(a_2, a_3, \ldots, a_n) z^{n-1}, \tag{11}$$

where

$$\begin{aligned} F_1 &= (\mu+\lambda) a_2, \\ F_2 &= (\mu+2\lambda)\left[\frac{\mu-1}{2} a_2^2 + a_3\right], \\ F_3 &= (\mu+3\lambda)\left[\frac{(\mu-1)(\mu-2)}{3!} a_2^3 + (\mu-1) a_2 a_3 + a_4\right]. \end{aligned}$$

In general,

$$F_{n-1}(a_2, a_3, \ldots, a_n) = [\mu + (n-1)\lambda] \times [(\mu-1)!] \times \sum \frac{a_2^{i_1} a_3^{i_2} \ldots a_n^{i_{n-1}}}{i_1! i_2! \ldots i_n! \left[\mu - (i_1 + i_2 + \cdots + i_{n-1})\right]!} \tag{12}$$

is a Faber polynomial of degree $(n-1)$.

Lemma 3.1. *[12] Let $p(z) = 1 + c_1 z + c_2 z^2 + \cdots \in \mathcal{P}$, then $|c_k| \le 2$ for $k \in \mathbb{N}$.*

Lemma 3.2. *[34] Let $k, l \in \mathbb{R}$ and $z_1, z_2 \in \mathbb{C}$. If $|z_1| < R$ and $|z_2| < R$, then*

$$|(k+l) z_1 + (k-l) z_2| \le \begin{cases} 2R\,|k| &, \quad |k| \ge |l| \\ 2R\,|l| & \quad |k| \le |l| \end{cases}.$$

Theorem 3.3. *Let the function $f \in \mathcal{G}_{\Sigma}^{\lambda,\mu}(\Psi)$ be given by $(3)$. If $a_k = 0$ $(2 \le k \le n-1)$, then*

$$|a_n| \le \frac{1}{2\left[\mu + (n-1)\lambda\right]} \qquad (n \ge 3).$$

*Proof.* Let $f \in \mathcal{G}_{\Sigma}^{\lambda,\mu}(\Psi)$ be of the form $(3)$. Then we have the expansion (11). For the inverse map $g = f^{-1}$ given by $(4)$, we get

$$(1-\lambda)\left(\frac{g(w)}{w}\right)^{\mu} + \lambda g'(w)\left(\frac{g(w)}{w}\right)^{\mu-1} = 1 + \sum_{n=2}^{\infty} F_{n-1}(A_2, A_3, \ldots, A_n)\, w^{n-1}, \tag{13}$$

with

$$A_n = \frac{1}{n} K_{n-1}^{-n}(a_2, a_3, \ldots, a_n). \tag{14}$$

On the other hand, since $f \in \mathcal{G}_{\Sigma}^{\lambda,\mu}(\Psi)$, by the subordination principle, there exist the Schwarz's function $\varkappa(z)$,

$$\varkappa \in \mathcal{H}: \ \varkappa(0) = 0 \quad \text{and} \quad |\varkappa(z)| < 1 \ (z \in \mathbb{U})$$

such that

$$(1-\lambda)\left(\frac{f(z)}{z}\right)^{\mu} + \lambda f'(z)\left(\frac{f(z)}{z}\right)^{\mu-1} = \Psi(\varkappa(z)) \qquad (z \in \mathbb{U}). \tag{15}$$

Since $\Psi$ is univalent in the open unit disk $\mathbb{U}$, by $(15)$, the function

$$p(z) := \frac{1+\varkappa(z)}{1-\varkappa(z)} = 1 + c_1 z + c_2 z^2 + c_3 z^3 + \cdots \tag{16}$$

belongs to the class $\mathcal{P}$. Solving $\varkappa(z)$ in terms of $p(z)$ in $(16)$, we obtain

$$\varkappa(z) = \frac{1}{2} c_1 z + \frac{1}{2}\left(c_2 - \frac{c_1^2}{2}\right) z^2 + \frac{1}{2}\left(c_3 - c_1 c_2 + \frac{c_1^3}{4}\right) z^3 + \frac{1}{2}\left(c_4 - c_1 c_3 + \frac{3}{4} c_1^2 c_2 - \frac{1}{2} c_2^2 - \frac{1}{8} c_1^4\right) z^4 + \cdots. \tag{17}$$

Using $(17)$ in $(1)$ and considering $(2)$, we find

$$\Psi(\varkappa(z)) = 1+\frac{1}{4}c_1z+\frac{1}{4}\left(c_2-\frac{7}{12}c_1^2\right)z^2+\frac{1}{4}\left(c_3-\frac{7}{6}c_1c_2+\frac{17}{48}c_1^3\right)z^3$$
$$+\frac{1}{4}\left(c_4-\frac{7}{6}c_1c_3+\frac{17}{16}c_1^2c_2-\frac{7}{12}c_2^2-\frac{649}{2880}c_1^4\right)z^4+\cdots. \tag{18}$$

Define

$$\varkappa(z):=\sum_{n=1}^{\infty}\varphi_n z^n.$$

Thus, we can write

$$\Psi(\varkappa(z))=1+\sum_{n=1}^{\infty}\sum_{j=1}^{n}G_jD_n^j\left(\varphi_1,\varphi_2,\ldots,\varphi_n\right)z^n. \tag{19}$$

Similarly, since $g\in\mathcal{G}_{\Sigma}^{\lambda,\mu}(\Psi)$, there exist the Schwarz's function $\tau(w)$,

$$\tau\in\mathcal{H}:\ \tau(0)=0 \quad \text{and} \quad |\tau(w)|<1\ (w\in\mathbb{U})$$

such that

$$(1-\lambda)\left(\frac{g(w)}{w}\right)^{\mu}+\lambda g'(w)\left(\frac{g(w)}{w}\right)^{\mu-1}=\Psi(\tau(w)) \qquad (w\in\mathbb{U}). \tag{20}$$

Since $\Psi$ is univalent in the open unit disk $\mathbb{U}$, by $(20)$, the function

$$q(w):=\frac{1+\tau(w)}{1-\tau(w)}=1+d_1w+d_2w^2+d_3w^3+\cdots \tag{21}$$

belongs to the class $\mathcal{P}$. Solving $\tau(w)$ in terms of $q(w)$ in $(21)$, we obtain

$$\tau(w) = \frac{1}{2}d_1w+\frac{1}{2}\left(d_2-\frac{d_1^2}{2}\right)w^2+\frac{1}{2}\left(d_3-d_1d_2+\frac{d_1^3}{4}\right)w^3$$
$$+\frac{1}{2}\left(d_4-d_1d_3+\frac{3}{4}d_1^2d_2-\frac{1}{2}d_2^2-\frac{1}{8}d_1^4\right)w^4+\cdots. \tag{22}$$

Using $(22)$ in $(1)$ and considering $(2)$, we find

$$\Psi(\tau(w)) = 1+\frac{1}{4}d_1w+\frac{1}{4}\left(d_2-\frac{7}{12}d_1^2\right)w^2+\frac{1}{4}\left(d_3-\frac{7}{6}d_1d_2+\frac{17}{48}d_1^3\right)w^3$$
$$+\frac{1}{4}\left(d_4-\frac{7}{6}d_1d_3+\frac{17}{16}d_1^2d_2-\frac{7}{12}d_2^2-\frac{649}{2880}d_1^4\right)w^4+\cdots. \tag{23}$$

Define

$$\tau(w):=\sum_{n=1}^{\infty}\psi_n w^n.$$

Thus, we can write

$$\Psi(\tau(w))=1+\sum_{n=1}^{\infty}\sum_{j=1}^{n}G_jD_n^j\left(\psi_1,\psi_2,\ldots,\psi_n\right)w^n. \tag{24}$$

Considering (11) and (19) in equality (15), for any $n\geq 2$, we get

$$F_{n-1}\left(a_2,a_3,\ldots,a_n\right)=\sum_{j=1}^{n-1}G_jD_{n-1}^j\left(\varphi_1,\varphi_2,\ldots,\varphi_{n-1}\right), \tag{25}$$

and similarly, (13) and (24) in equality (20), we obtain

$$F_{n-1}(A_2, A_3, \ldots, A_n) = \sum_{j=1}^{n-1} G_j D_{n-1}^{j}(\psi_1, \psi_2, \ldots, \psi_{n-1}). \tag{26}$$

By hypothesis, since $a_k = 0 \ (2 \le k \le n-1)$, we have

$$A_n = -a_n,$$

$$\varphi_1 = \cdots = \varphi_{n-2} = 0, \qquad \varphi_{n-1} = \frac{1}{2}c_{n-1}$$

and

$$\psi_1 = \cdots = \psi_{n-2} = 0, \qquad \psi_{n-1} = \frac{1}{2}d_{n-1}.$$

So (25) and (26) imply that

$$[\mu + (n-1)\lambda]\, a_n = G_1\varphi_{n-1} \qquad \text{and} \qquad -[\mu + (n-1)\lambda]\, a_n = G_1\psi_{n-1},$$

respectively. Using the fact that $G_1 = \frac{1}{2}$ and Lemma 3.1, we obtain

$$|a_n| = \frac{G_1|\varphi_{n-1}|}{\mu + (n-1)\lambda} = \frac{G_1|\psi_{n-1}|}{\mu + (n-1)\lambda} \le \frac{1}{2[\mu + (n-1)\lambda]},$$

which completes the proof of the Theorem 3.3. □

Theorem 3.4. *Let the function $f \in \mathcal{G}_{\Sigma}^{\lambda,\mu}(\Psi)$ be given by (3). Then*

$$|a_2| \le \sqrt{\frac{3}{3(\mu+2\lambda)(\mu+1) + 14(\mu+\lambda)^2}} \tag{27}$$

*and*

$$|a_3| \le \begin{cases} \frac{1}{2(\mu+2\lambda)} & , \ 3(\mu+2\lambda)(1-\mu) \le 14(\mu+\lambda)^2 \\ \frac{3}{3(\mu+2\lambda)(\mu+1)+14(\mu+\lambda)^2} & , \ 3(\mu+2\lambda)(1-\mu) \ge 14(\mu+\lambda)^2 \end{cases}. \tag{28}$$

*Proof.* If we set $n = 2$ and $n = 3$ in (25) and (26), respectively, we get

$$(\mu+\lambda)\, a_2 = G_1\varphi_1, \tag{29}$$

$$(\mu+2\lambda)\left[\frac{\mu-1}{2}a_2^2 + a_3\right] = G_1\varphi_2 + G_2\varphi_1^2, \tag{30}$$

$$-(\mu+\lambda)\, a_2 = G_1\psi_1, \tag{31}$$

$$(\mu+2\lambda)\left[\frac{\mu+3}{2}a_2^2 - a_3\right] = G_1\psi_2 + G_2\psi_1^2, \tag{32}$$

where

$$\varphi_1 = \frac{1}{2}c_1, \quad \varphi_2 = \frac{1}{2}\left(c_2 - \frac{c_1^2}{2}\right),$$
$$\psi_1 = \frac{1}{2}d_1, \quad \psi_2 = \frac{1}{2}\left(d_2 - \frac{d_1^2}{2}\right).$$

Note that

$$\varphi_1 = -\psi_1 \qquad \text{and} \qquad \varphi_1^2 + \psi_1^2 = 8(\mu+\lambda)^2 a_2^2$$

or equivalently

$$c_1 = -d_1 \qquad \text{and} \qquad c_1^2 + d_1^2 = 32\left(\mu+\lambda\right)^2 a_2^2. \tag{33}$$

From (29) and (31) , we find

$$|a_2| = \frac{G_1 |\varphi_1|}{\mu+\lambda} = \frac{G_1 |\psi_1|}{\mu+\lambda} \leq \frac{1}{2\left(\mu+\lambda\right)}. \tag{34}$$

Also from (30) and (32) , we obtain

$$\left(\mu+2\lambda\right)\left(\mu+1\right) a_2^2 = G_1\left(\varphi_2+\psi_2\right) + G_2\left(\varphi_1^2+\psi_1^2\right)$$

or equivalently

$$\left(\mu+2\lambda\right)\left(\mu+1\right) a_2^2 = \frac{1}{4}\left(c_2+d_2\right) - \frac{7}{48}\left(c_1^2+d_1^2\right). \tag{35}$$

Substituting the value of $c_1^2+d_1^2$ from (33) in the right hand side of the above equality, we deduce that

$$a_2^2 = \frac{3\left(c_2+d_2\right)}{4\left[3\left(\mu+2\lambda\right)\left(\mu+1\right)+14\left(\mu+\lambda\right)^2\right]}. \tag{36}$$

Using the Lemma 3.1, we get

$$|a_2| \leq \sqrt{\frac{3}{3\left(\mu+2\lambda\right)\left(\mu+1\right)+14\left(\mu+\lambda\right)^2}}, \tag{37}$$

and combining this with the inequality (34) , we obtain the desired estimate on the coefficient $|a_2|$ as asserted in (27) .

Next, in order to find the bound on the coefficient $|a_3|$ , we subtract (32) from (30) . We thus get

$$\left(\mu+2\lambda\right)\left(-2a_2^2+2a_3\right) = \frac{G_1}{2}\left(c_2-d_2\right)$$

or

$$a_3 = a_2^2 + \frac{c_2-d_2}{8\left(\mu+2\lambda\right)}. \tag{38}$$

Upon substituting the value of $a_2^2$ from (36) into (38) , it follows that

$$a_3 = \left(\Lambda\left(\lambda,\mu\right) + \frac{1}{8\left(\mu+2\lambda\right)}\right) c_2 + \left(\Lambda\left(\lambda,\mu\right) - \frac{1}{8\left(\mu+2\lambda\right)}\right) d_2,$$

where

$$\Lambda\left(\lambda,\mu\right) = \frac{3}{4\left[3\left(\mu+2\lambda\right)\left(\mu+1\right)+14\left(\mu+\lambda\right)^2\right]}.$$

By Lemma 3.2, we get the desired estimate on the coefficient $|a_3|$ as asserted in (28) . □

By setting $\lambda = 1$ and $\mu = 1$ in Theorem 3.4, we obtain the following consequence.

Corollary 3.5. *Let the function $f \in \mathcal{N}_\Sigma\left(\Psi\right)$ be given by* (3) . *Then*

$$|a_2| \leq \sqrt{\frac{3}{74}} \qquad \textit{and} \qquad |a_3| \leq \frac{1}{6}.$$

Remark 3.6. Note that the above corollary gives an improvement the result on $|a_3|$ given in [21, Theorem 1].

Theorem 3.7. *Let the function $f \in \mathcal{G}_{\Sigma}^{\lambda,\mu}(\Psi)$ be given by (3). Then for any $\delta \in \mathbb{R}$, we have*

$$\left|a_3 - \delta a_2^2\right| \leq \begin{cases} \frac{1}{2(\mu+2\lambda)} & , \quad \delta \in [\rho, \eta] \\ \frac{3|1-\delta|}{3(\mu+2\lambda)(\mu+1)+14(\mu+\lambda)^2} & , \quad \delta \in (-\infty, \rho] \cup [\eta, \infty) \end{cases},$$

*where*

$$\rho = \frac{3(\mu+2\lambda)(1-\mu) - 14(\mu+\lambda)^2}{6(\mu+2\lambda)}$$

*and*

$$\eta = \frac{3(\mu+2\lambda)(\mu+3) + 14(\mu+\lambda)^2}{6(\mu+2\lambda)}.$$

*Proof.* For the function $f \in \mathcal{G}_{\Sigma}^{\lambda,\mu}(\Psi)$ of the form (3), from (36) and (38) we have

$$a_3 - \delta a_2^2 = \left(\mathfrak{h}(\delta) + \frac{1}{8(\mu+2\lambda)}\right) c_2 + \left(\mathfrak{h}(\delta) - \frac{1}{8(\mu+2\lambda)}\right) d_2,$$

where

$$\mathfrak{h}(\delta) = \frac{3(1-\delta)}{4\left[3(\mu+2\lambda)(\mu+1) + 14(\mu+\lambda)^2\right]}.$$

Then by Lemma 3.1 and Lemma 3.2, we conclude that

$$\left|a_3 - \delta a_2^2\right| \leq \begin{cases} \frac{1}{2(\mu+2\lambda)} & , \quad |\mathfrak{h}(\delta)| \leq \frac{1}{8(\mu+2\lambda)} \\ 4|\mathfrak{h}(\delta)| & , \quad |\mathfrak{h}(\delta)| \geq \frac{1}{8(\mu+2\lambda)} \end{cases}.$$

□

By setting $\lambda = 1$ and $\mu = 1$ in Theorem 3.7, we obtain the following consequence.

Corollary 3.8. *[21] Let the function $f \in \mathcal{N}_{\Sigma}(\Psi)$ be given by (3). Then for any $\delta \in \mathbb{R}$, we have*

$$\left|a_3 - \delta a_2^2\right| \leq \begin{cases} \frac{1}{6} & , \quad \delta \in \left[-\frac{28}{9}, \frac{46}{9}\right] \\ \frac{3|1-\delta|}{74} & , \quad \delta \in \left(-\infty, -4\frac{28}{9}\right] \cup \left[\frac{46}{9}, \infty\right) \end{cases}.$$

## References


[1] H. Airault and A. Bouali, *Differential calculus on the Faber polynomials*, Bull. Sci. Math. **130** (3) (2006), 179–222.
https://dx.doi.org/10.1016/j.bulsci.2005.10.002

[2] H. Airault and J. Ren, *An algebra of differential operators and generating functions on the set of univalent functions*, Bull. Sci. Math. **126** (5) (2002), 343–367.
https://dx.doi.org/10.1016/S0007-4497(02)01115-6

[3] G. Akın and S. Sümer Eker, *Coefficient estimates for a certain class of analytic and bi-univalent functions defined by fractional derivative*, C. R., Math., Acad. Sci. Paris **352** (12) (2014), 1005–1010.
https://dx.doi.org/10.1016/j.crma.2014.09.022

[4] Ş. Altınkaya and S. Yalçın, *Faber polynomial coefficient bounds for a subclass of bi-univalent functions*, C. R., Math., Acad. Sci. Paris **353** (12) (2015), 1075–1080.
https://dx.doi.org/10.1016/j.crma.2015.09.003

[5] I.S. Berezin and N.P. Zhidkov, *Computing Methods*, Pergamon, North Atlantic Treaty Organization and London Mathematical Society, 1965. Zbl 0122.12903

[6] I.V. Blagouchine, *Two series expansions for the logarithm of the gamma function involving Stirling numbers and containing only rational coefficients for certain arguments related to* $\pi^{-1}$, Arxiv.
https://arxiv.org/abs/1408.3902v9

[7] D.A. Brannan and J. Clunie, *Aspects of contemporary complex analysis*. Academic Pr. (1980). Zbl 0483.00007

[8] D.A. Brannan and T.S. Taha, *On some classes of bi-univalent functions*, Studia Univ. Babeş-Bolyai Math. **31** (2) (1986), 70–77. Zbl 0614.30017

[9] S. Bulut, *Faber polynomial coefficient estimates for a comprehensive subclass of analytic bi-univalent functions*, C. R., Math., Acad. Sci. Paris **352** (6) (2014), 479–484.
https://dx.doi.org/10.1016/j.crma.2014.04.004

[10] S. Bulut, N. Magesh and V.K. Balaji, *Faber polynomial coefficient estimates for certain subclasses of meromorphic bi-univalent functions*, C. R., Math., Acad. Sci. Paris **353** (2) (2015), 113–116.
https://dx.doi.org/10.1016/j.crma.2014.10.019

[11] M. Çağlar, H. Orhan and N. Yağmur, *Coefficient bounds for new subclasses of bi-univalent functions*, Filomat **27** (7) (2013), 1165–1171.
https://dx.doi.org/10.2298/FIL1307165C

[12] P.L. Duren, *Univalent Functions*, Grundlehren der Mathematischen Wissenschaften, vol. 259, Springer, New York, 1983. Zbl 0514.30001

[13] G. Faber, *Über polynomische Entwicklungen*, Math. Ann. **57** (3) (1903) 389–408.
https://dx.doi.org/10.1007/BF01444293

[14] S.G. Hamidi, S.A. Halim and J.M. Jahangiri, *Coefficient estimates for a class of meromorphic bi-univalent functions*, C. R., Math., Acad. Sci. Paris **351** (9-10) (2013), 349–352.
https://dx.doi.org/10.1016/j.crma.2013.05.005

[15] S.G. Hamidi and J.M. Jahangiri, *Faber polynomial coefficient estimates for analytic bi-close-to-convex functions*, C. R., Math., Acad. Sci. Paris **352** (1) (2014), 17–20.
https://dx.doi.org/10.1016/j.crma.2013.11.005

[16] S.G. Hamidi and J.M. Jahangiri, *Faber polynomial coefficients of bi-subordinate functions*, C. R., Math., Acad. Sci. Paris **354** (4) (2016), 365–370.
https://dx.doi.org/10.1016/j.crma.2016.01.013

[17] S.G. Hamidi, T. Janani, G. Murugusundaramoorthy and J.M. Jahangiri, *Coefficient estimates for certain classes of meromorphic bi-univalent functions*, C. R., Math., Acad. Sci. Paris **352** (4) (2014), 277–282.
https://dx.doi.org/10.1016/j.crma.2014.01.010

[18] T. Hayami and S. Owa, *Coefficient bounds for bi-univalent functions*, Pan Amer. Math. J. **22** (4) (2012), 15–26. Zbl 1267.30040

[19] J.M. Jahangiri and S.G. Hamidi, *Advances on the coefficients of bi-prestarlike functions*, C. R., Math., Acad. Sci. Paris **354** (10) (2016), 980–985.
https://dx.doi.org/10.1016/j.crma.2016.08.009

[20] M. Lewin, *On a coefficient problem for bi-univalent functions*, Proc. Amer. Math. Soc. **18** (1967), 63–68.
https://dx.doi.org/10.1090/S0002-9939-1967-0206255-1

[21] G. Murugusundaramoorthy, K. Vijaya and T. Bulboacă, *Initial coefficients bounds for bi-univalent functions related to Gregory coefficients*, Mathematics 2023, 11(13), 2857. https://dx.doi.org/10.3390/math11132857

[22] E. Netanyahu, *The minimal distance of the image boundary from the origin and the second coefficient of a univalent function in* $|z| < 1$, Arch. Ration. Mech. Anal. **32** (1969), 100–112.
https://dx.doi.org/10.1007/BF00247676

[23] M. Obradović, *A class of univalent functions*, Hokkaido Math. J. **27** (2) (1998), 329–335.
https://dx.doi.org/10.14492/hokmj/1351001289

[24] G.M. Phillips, *Gregory's method for numerical integration*, Amer. Math. Monthly **79** (3) (1972), 270–274.
https://dx.doi.org/10.1080/00029890.1972.11993028

[25] S. Sivasubramanian, R. Sivakumar, T. Bulboacă and T.N. Shanmugam, *On the class of bi-univalent functions*, C. R., Math., Acad. Sci. Paris **352** (11) (2014), 895–900.
https://dx.doi.org/10.1016/j.crma.2014.09.015

[26] S. Sivasubramanian, R. Sivakumar, S. Kanas and Seong-A Kim, *Verification of Brannan and Clunie's conjecture for certain subclasses of bi-univalent functions*, Ann. Polon. Math. **113** (3) (2015), 295–304.

[27] H.M. Srivastava, S. Bulut, M. Çağlar and N. Yağmur, *Coefficient estimates for a general subclass of analytic and bi-univalent functions*, Filomat **27** (5) (2013), 831–842.
https://dx.doi.org/10.2298/FIL1305831S

[28] H.M. Srivastava, A.K. Mishra and P. Gochhayat, *Certain subclasses of analytic and bi-univalent functions*, Appl. Math. Lett. **23** (2010) 1188–1192.
https://dx.doi.org/10.1016/j.aml.2010.05.009

[29] T.S. Taha, *Topics in Univalent Function Theory*, Ph.D. Thesis, University of London, 1981.

[30] P.G. Todorov, *On the Faber polynomials of the univalent functions of class* $\Sigma$, J. Math. Anal. Appl. **162** (1) (1991), 268–276.
https://dx.doi.org/10.1016/0022-247X(91)90193-4

[31] Q.-H. Xu, Y.-C. Gui and H.M. Srivastava, *Coefficient estimates for a certain subclass of analytic and bi-univalent functions*, Appl. Math. Lett. **25** (2012), 990–994.
https://dx.doi.org/10.1016/j.aml.2011.11.013

[32] Q.-H. Xu, H.-G. Xiao and H.M. Srivastava, *A certain general subclass of analytic and bi-univalent functions and associated coefficient estimate problems*, Appl. Math. Comput. **218** (2012), 11461–11465.
https://dx.doi.org/10.1016/j.amc.2012.05.034

[33] Z.-G. Wang and S. Bulut, *A note on the coefficient estimates of bi-close-to-convex functions*, C. R. Acad. Sci. Paris, Ser. I **355** (2017).

[34] P. Zaprawa, *Estimates of initial coefficients for bi-univalent functions*, Abstr. Appl. Anal. 2014, Art. ID 357480, 6 pp.
https://dx.doi.org/10.1155/2014/357480

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