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ROUGH $\mathcal{I}$-CONVERGENCE OF SEQUENCES IN PROBABILISTIC NORMED SPACES

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Abstract. In this paper, we have studied the idea of rough $\mathcal{I}$-convergence in probabilistic normed spaces which is indeed a generalized version as compared to the notion of rough $\mathcal{I}$-convergence in normed linear spaces. On the other way, it is also a generalization of rough statistical convergence in probabilistic normed spaces. Furthermore, we have defined the notion of rough $\mathcal{I}$-cluster points and have proved some important results associated with the set of rough $\mathcal{I}$-limits of a sequence in the same space.

1. Introduction

In 1951, the idea of ordinary convergence of real sequences was extended to statistical convergence of real sequences based on the natural density of a set independently by Fast [9] and Steinhaus [29]. After long years, in 2000, Kostyrko et al. [13] developed a very useful generalization of statistical convergence which was named as $\mathcal{I}$-convergence based on the structure of an ideal of subsets of natural numbers. Since then this idea is still being carried out in various settings of related spaces [6, 7, 12, 16, 18, 20, 27, 32, 34, 35].

In 2001, Phu [24] initially introduced the concept of rough convergence of sequences in a finite dimensional normed linear space which is basically a generalization of usual convergence and in the same paper he showed that r-limit set is bounded, closed, convex and also established many more interesting results. Later on, this concept were extended to an infinite dimensional normed linear space [26]. Also, he [25] studied the notion of rough continuity of linear operators. Later, Ayter [2] extended this notion to rough statistical convergence based on natural density of a set and, in 2013, Pal et al. [28] introduced the concept of rough ideal convergence which is basically a generalization of rough statistical convergence in normed linear spaces. In this direction, works have been carried out in different spaces [4, 8, 11, 17, 22, 23].

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In 1942, Menger [19] first proposed the concept of statistical metric spaces, now called probabilistic metric spaces, which is an interesting and important generalization of the notion of metric spaces. This concept, later on, was studied by Schweizer and Sklar [31]. Combining the idea of statistical metric spaces and normed linear spaces, Šerstnev [30] introduced the idea of probabilistic normed spaces. In 1993 Alsina et al. [1] gave a new definition of probabilistic normed spaces which is basically a special case of the definition of Šerstnev. In recent times, Antal et al. [5] introduced the notion of rough statistical convergence of sequences in probabilistic normed spaces. In this paper, we have generalied this notion in ideal context. We have defined the notion of rough $\mathcal{I}$-cluster point and then have studied some interesting results associated with probabilistic normed spaces.

2. Preliminaries

Throughout the paper $\mathbb{N}$ and $\mathbb{R}$ denote the set of natural numbers and the set of reals respectively. First we recall some basic definitions and notations.

Definition 2.1. [13] A family $\mathcal{I} \subset 2^X$ of a non empty set X is said to be an ideal in X if the following conditions hold:

1. $\emptyset \in \mathcal{I}$;
2. $A, B \in \mathcal{I} \implies A \cup B \in \mathcal{I}$;
3. $A \in \mathcal{I}, B \subset A \implies B \in \mathcal{I}$.

If $X \in \mathcal{I}$ then $\mathcal{I} = 2^X$. Also $\{\emptyset\}$ is always an ideal. The ideals 2^X and $\emptyset$ are called trivial ideals. So $\mathcal{I}$ is non trivial if $X \notin \mathcal{I}$ and if $\mathcal{I} \neq \{\emptyset\}$. An ideal $\mathcal{I}$ in X is said to be an admissible ideal if $\{x\} \in \mathcal{I}$ for each $x \in X$.

If $\mathcal{I}$ is a non trivial proper ideal in X then the family of sets $\mathcal{F}(\mathcal{I}) = \{X \setminus A : A \in \mathcal{I}\}$ is clearly a filter on X which is called the filter associated with the ideal $\mathcal{I}$. Throughout the paper $\mathcal{I}$ will stand for a non trivial admissible ideal of $\mathbb{N}$ unless otherwise stated.

Definition 2.2. Let $K \subset \mathbb{N}$. Then the natural density $\delta(K)$ of K is defined by

$$\delta(K) = \lim_{n\to\infty} \frac{1}{n} |\{k \leq n : k \in K\}|,$$

provided the limit exists.

It is clear that if K is finite then $\delta(K) = 0$.

Now we recall some basic definitions and notations which will be useful in the sequal.

Definition 2.3. [31] A triangular norm, briefly t-norm, is a binary operation on $[0, 1]$ which is continuous, commutative, associative, non-decreasing and has 1 as identity element, i.e., it is the continuous mapping $\star : [0, 1] \times [0, 1] \to [0, 1]$ such that for all $a, b, c, d \in [0, 1]$:

1. $a \star 1 = a$;
2. $a \star b = b \star a$;
3. $a \star b \geq c \star d$ whenever $a \geq c$ and $b \geq d$;
4. $a \star (b \star c) = (a \star b) \star c$.

Definition 2.4. [31] A binary operation $\circ : [0, 1] \times [0, 1] \to [0, 1]$ is said to be a continuous t-conorm if the following conditions are satisfied.

1. $\circ$ is associative and commutative;
2. $\circ$ is continuous;
3. $x \circ 0 = x$ for all $x \in [0,1]$;
4. $x \circ y \leq z \circ w$ whenever $x \leq z$ and $y \leq w$ for each $x, y, z, w \in [0,1]$.

Example 2.5. [14] The following are the examples of t-norms:

1. $x \star y = min\{x, y\}$;
2. $x \star y = x.y$;
3. $x \star y = max\{x + y - 1, 0\}$. This t-norm is known as Lukasiewicz t-norm.

Lemma 2.6. [33] If $\star$ is a continuous t-norm, $\circ$ is a continuous t-conorm, $r_i \in (0,1)$ and $1 \leq i \leq 7$, then the following statements hold:

1. If $r_1 > r_2$, there are $r_3, r_4 \in (0,1)$ such that $r_1 \star r_3 \geq r_2$ and $r_1 \geq r_2 \circ r_4$
2. If $r_5 \in (0,1)$, there are $r_6, r_7 \in (0,1)$ such that $r_6 \star r_6 \geq r_5$ and $r_5 \geq r_7 \circ r_7$.

Definition 2.7. [10] A function $f : \mathbb{R} \to \mathbb{R}_0^+$ is said to be a distribution function if it is non decreasing and left continuous with $\inf_{t\in\mathbb{R}} f(t) = 0$ and $\sup_{t\in\mathbb{R}} f(t) = 1$. We denote D as the set of all distribution functions.

Definition 2.8. [10] Let X be a real vector space, ν be a mapping from X into D where $x \in X, t \in \mathbb{R}$, the value $\nu(x)(t)$ of the distribution function $\nu(x)$ at t is denoted by $\nu(x;t)$ and $\star$ be a t-norm satisfying the following conditions:

1. $\nu(x; 0) = 0$;
2. $\nu(x;t) = 1$, $\forall\ t > 0$ iff $x = \theta$, θ being the zero element of X;
3. $\nu(\alpha x;t) = \nu(x; \frac{t}{|\alpha|})$, $\forall\ \alpha \in \mathbb{R} \setminus \{0\}$ and $\forall\ t > 0$;
4. $\nu(x + y; s + t) \geq \nu(x;s) \star \nu(y;t)$, $\forall\ x, y \in X$ and $\forall\ s, t \in \mathbb{R}_0^+$.

Then the triplet $(X, \nu, \star)$ is called a probabilistic normed space (shortly PNS).

Example 2.9. [3] For a real normed space $(X, \|\cdot\|)$, we define the probabilistic norm ν for $x \in X, t \in \mathbb{R}$ as $\nu(x;t) = \frac{t}{t+\|x\|}$. Then $(X, \nu, \star)$ is a PNS under the t-norm $\star$ defined by $x \star y = \min\{x, y\}$.

Definition 2.10. [3] Let $(X, \nu, \star)$ be a PNS. For $r > 0$, the open ball $B(x, \lambda; r)$ with center $x \in X$ and radius $\lambda \in (0,1)$ is defined as

$$B(x, \lambda; r) = \{y \in X : \nu(y - x; r) > 1 - \lambda\}.$$

Similarly, the closed ball $\overline{B(x, \lambda; r)} = \{y \in X : \nu(y - x; r) \geq 1 - \lambda\}$

Definition 2.11. [15] Let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence in a PNS $(X, \nu, \star)$. Then $\{x_n\}_{n\in\mathbb{N}}$ is said to be convergent to $\xi \in X$ with respect to the probabilistic norm ν if for every $t > 0$ and $\varepsilon \in (0,1)$, there is a positive integer n_0 such that $\nu(x_n - \xi; t) > 1 - \varepsilon$ for all $n \geq n_0$. In this case we write $x_n \xrightarrow{\nu} \xi$ or $\nu\text{-}\lim_{n\to\infty} x_n = \xi$.

It is easy to check that in the PNS as in Example 2.9, $x_n \xrightarrow{\|\cdot\|} \xi$ if and only if $x_n \xrightarrow{\nu} \xi$.

Definition 2.12. [21] Let $\mathcal{I}$ be a non trivial ideal of $\mathbb{N}$ and $(X, \nu, \star)$ be a probabilistic normed space. A sequence $\{x_n\}_{n\in\mathbb{N}}$ of elements of X is said to be $\mathcal{I}$-convergent to $\xi \in X$ with respect to the probabilistic norm ν (or $\mathcal{I}_\nu$-convergent to ξ) if for each $\varepsilon > 0$ and $t > 0$, $\{n \in \mathbb{N} : \nu(x_n - \xi; t) \leq 1 - \varepsilon\} \in \mathcal{I}$. In this case we write $\mathcal{I}_\nu\text{-}\lim_{n\to\infty} x_n = \xi$ or $x_n \xrightarrow{\mathcal{I}_\nu} \xi$.

Definition 2.13. [21] Let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence in a PNS $(X,\nu,\star)$. An element $\xi\in X$ is said to be an $\mathcal{I}$-cluster point of $\{x_n\}_{n\in\mathbb{N}}$ with respect to the probabilistic norm ν (or $\mathcal{I}_\nu$-cluster point) if for each $\varepsilon>0$ and $t>0$, $K=\{n\in\mathbb{N}:\nu(x_n-\xi;t)>1-\varepsilon\}\notin\mathcal{I}$.

Definition 2.14. [5] Let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence in an PNS $(X,\nu,\star)$. Then $\{x_n\}_{n\in\mathbb{N}}$ is said to be rough convergent to $\xi\in X$ with respect to the probabilistic norm ν if for every $\varepsilon>0$, $\lambda\in(0,1)$ and some non negative number r there exists $n_0\in\mathbb{N}$ such that $\nu(x_n-\xi;r+\varepsilon)>1-\lambda$ for all $n>n_0$. In this case we write $r_\nu\text{-}\lim_{n\to\infty}x_n=\xi$ or $x_n\xrightarrow{r_\nu}\xi$ and ξ is called r_ν-limit of $\{x_n\}_{n\in\mathbb{N}}$.

Definition 2.15. [5] Let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence in an PNS $(X,\nu,\star)$. Then $\{x_n\}_{n\in\mathbb{N}}$ is said to be rough statistically convergent to $\xi\in X$ with respect to the probabilistic norm ν if for every $\varepsilon>0$ and $\lambda\in(0,1)$ and some non negative number r, $\delta(\{n\in\mathbb{N}:\nu(x_n-\xi;r+\varepsilon)\le 1-\lambda\})=0$. In this case we write $r\text{-}St_\nu\text{-}\lim_{n\to\infty}x_n=\xi$ or $x_n\xrightarrow{r-St_\nu}\xi$.

3. Main Results

Throughout the paper $\mathcal{I}$ stands for an admissible ideal. First we introduce the definition of rough $\mathcal{I}$-convergence in a PNS $(X,\nu,\star)$.

Definition 3.1. Let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence in a PNS $(X,\nu,\star)$ and r be a non negative real number. Then $\{x_n\}_{n\in\mathbb{N}}$ is said to be rough $\mathcal{I}$-convergent to $\xi\in X$ of roughness degree r with respect to the probabilistic norm ν if for every $\varepsilon>0$ and $\lambda\in(0,1)$, $\{n\in\mathbb{N}:\nu(x_n-\xi;r+\varepsilon)\le 1-\lambda\}\in\mathcal{I}$. In this case we write $r\text{-}\mathcal{I}_\nu\text{-}\lim_{n\to\infty}x_n=\xi$ or $x_n\xrightarrow{r-\mathcal{I}_\nu}\xi$ and ξ is called r-$\mathcal{I}_\nu$-limit of $\{x_n\}_{n\in\mathbb{N}}$.

Remark 3.2. (a) Suppose $\mathcal{I}_f$ is the class of all finite subsets of $\mathbb{N}$. Then clearly $\mathcal{I}_f$ is a non trivial admissible ideal. So, rough $\mathcal{I}_f$-convergence with respect to the probabilistic norm ν agrees with the rough convergence with respect to the probabilistic norm ν in a PNS $(X,\nu,\star)$.
(b) If we take $\mathcal{I}_\delta$ as a class of all subsets of $\mathbb{N}$ whose natural densities are zero. Then clearly $\mathcal{I}_\delta$ is a non trivial admissible ideal. In this case rough $\mathcal{I}_\delta$-convergence with respect to the probabilistic norm ν coincides with the rough statistical convergence with respect to the probabilistic norm ν in a PNS $(X,\nu,\star)$.
(c) If $r=0$, then the notion of rough $\mathcal{I}$-convergence with respect to the probabilistic norm ν coincides with $\mathcal{I}$-convergence with respect to the probabilistic norm ν in a PNS $(X,\nu,\star)$. So, our whole discussions is considered on the fact that $r>0$ unless otherwise stated.

From the Definition 3.1, it is clear that r-$\mathcal{I}_\nu$-limit of a sequence is not unique. So, throughout we use the notations $\mathcal{I}_\nu\text{-}LIM^r_{x_n}$ and $LIM^{r_\nu}_{x_n}$ to denote the set of all r-$\mathcal{I}_\nu$-limits and r_ν-limits of the sequence $\{x_n\}_{n\in\mathbb{N}}$. For an unbounded sequence, $LIM^{r_\nu}_{x_n}=\emptyset$ [5]. But, for such a sequence, $\mathcal{I}_\nu\text{-}LIM^r_{x_n}\neq\emptyset$ could happen as shown in the following example.

Example 3.3. Let $(X,\|\cdot\|)$ be a real normed linear space with the usual norm and let $\nu(x;t)=\frac{t}{t+\|x\|}$ for all $x\in X$ and $t>0$. Also, let $x\star y=min\{x,y\}$. Then $(X,\nu,\star)$ is a probabilistic normed space. Now, let us consider the ideal $\mathcal{I}$ consisting of all those subsets of $\mathbb{N}$ whose natural density are zero. Then $\mathcal{I}$ is a non trivial admissible ideal of $\mathbb{N}$. Let us define a sequence $\{x_n\}_{n\in\mathbb{N}}$ in X by $x_n = \begin{cases} (-1)^n, \text{ if } n \neq i^2 (i \in \mathbb{N}) \\ n, \text{ otherwise} \end{cases}$. Then for $r \geq 1$, $\mathcal{I}_\nu$-$LIM^r_{x_n} = [1-r, r-1]$, since for any $\xi \in [1-r, r-1]$, $\varepsilon > 0$ and $\lambda \in (0,1)$, we get $\{n \in \mathbb{N} : \nu(x_n - \xi; r+\varepsilon) \leq 1-\lambda\} \subset \{1^2, 2^2, 3^2, \ldots i^2, \ldots\}$. Since the later set of this inclusion has natural density zero, $\{n \in \mathbb{N} : \nu(x_n - \xi; r+\varepsilon) \leq 1-\lambda\} \in \mathcal{I}$. Also, we have $LIM^{r\nu}_{x_n} = \emptyset$ for any r [5].

Remark 3.4. From Example 3.3 we have $\mathcal{I}_\nu$-$LIM^r_{x_n} \neq \emptyset$ does not imply $LIM^{r\nu}_{x_n} \neq \emptyset$, but when $\mathcal{I}$ is an admissible ideal, $LIM^{r\nu}_{x_n} \neq \emptyset$ implies $\mathcal{I}_\nu$-$LIM^r_{x_n} \neq \emptyset$.

We are now giving an example of a sequence which is rough $\mathcal{I}$-convergent in a PNS but not rough $\mathcal{I}$-convergent in a normed linear space.

Example 3.5. Let $(X, \|\cdot\|)$ be a real normed space and $\nu(x;t) = \frac{t}{t+\|x\|}$ for all $x \in X$ and $t > 0$. Also, let $x \star y = \min\{x, y\}$. Then $(X, \nu, \star)$ is a probabilistic normed space. We take $\mathcal{I} = \mathcal{I}_\delta$ where $\mathcal{I}_\delta$ is a class of all subsets of $\mathbb{N}$ whose natural densities are zero. Then $\mathcal{I}$ is a non trivial admissible ideal of $\mathbb{N}$. Define a sequence $\{x_n\}_{n\in\mathbb{N}}$ as $x_n = \begin{cases} 0 \text{ if } n = i^2, i \in \mathbb{N} \\ n \text{ otherwise} \end{cases}$. Then the set of all rough $\mathcal{I}_\delta$-limits of $\{x_n\}_{n\in\mathbb{N}}$ with regards to ν is $[-r, r]$. But it is not rough $\mathcal{I}_\delta$-convergent to 0 with respect to the norm $\|\cdot\|$. Because for $\varepsilon > 0$ the set $\{n \in \mathbb{N} : \|x_n - 0\| \geq r+\varepsilon\} \notin \mathcal{I}_\delta$. For, if $\{n \in \mathbb{N} : \|x_n - 0\| \geq r+\varepsilon\} \in \mathcal{I}_\delta$ then, since $\{n : \|x_n - 0\| < r+\varepsilon\} = \{n : n = i^2\} \cup A \in \mathcal{I}_\delta$ where $A = \{n : n < r+\varepsilon,\ n \neq i^2\}$ is a finite set, the set $\mathbb{N} = \{n : \|x_n - 0\| \geq r+\varepsilon\} \cup \{n : \|x_n - 0\| < r+\varepsilon\} \in \mathcal{I}_\delta$ which is a contradiction. So $\{x_n\}_{n\in\mathbb{N}}$ is not rough $\mathcal{I}$-convergent to 0 with respect to the norm $\|\cdot\|$.

Definition 3.6. (cf. [5]) Let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence in a PNS $(X, \nu, \star)$. Then $\{x_n\}_{n\in\mathbb{N}}$ is said to be $\mathcal{I}$-bounded with respect to the probabilistic norm ν if for every $\lambda \in (0,1)$ there exists a positive real number M such that the set $\{n \in \mathbb{N} : \nu(x_n; M) \leq 1-\lambda\} \in \mathcal{I}$.

Theorem 3.7. *Let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence in a PNS $(X, \nu, \star)$. If $\{x_n\}_{n\in\mathbb{N}}$ is $\mathcal{I}$-bounded then $\mathcal{I}_\nu$-$LIM^r_{x_n} \neq \emptyset$ for some $r > 0$.*

Proof. First suppose that $\{x_n\}_{n\in\mathbb{N}}$ is an $\mathcal{I}$-bounded sequence. Then, for every $\lambda \in (0,1)$ there exists a positive real number G such that $\{n \in \mathbb{N} : \nu(x_n; G) \leq 1-\lambda\} \in \mathcal{I}$. Now, let $A = \{n \in \mathbb{N} : \nu(x_n; G) \leq 1-\lambda\}$ and θ be the zero element in X. Then for $k \in A^c$, $\nu(x_k; G) > 1-\lambda$, and so, $\nu(x_k - \theta; r+G) = \nu(x_k + \theta; r+G) \geq \nu(x_k; G) \star \nu(\theta; r) > (1-\lambda) \star 1 = 1-\lambda$. Therefore $\{k \in \mathbb{N} : \nu(x_k - \theta; r+G) \leq 1-\lambda\} \subset A$. Hence $\theta \in \mathcal{I}_\nu$-$LIM^r_{x_n}$. So, $\mathcal{I}_\nu$-$LIM^r_{x_n} \neq \emptyset$. □

We shall now show that rough $\mathcal{I}$-convergence of sequences in probabilistic normed spaces satisfies algebra of rough $\mathcal{I}$-limits.

Theorem 3.8. *Let $\{x_n\}_{n\in\mathbb{N}}$ and $\{y_n\}_{n\in\mathbb{N}}$ be two sequences in a PNS $(X, \nu, \star)$. Then, the following statements hold:*

1. *If $x_n \xrightarrow{r-\mathcal{I}_\nu} \xi$ and $y_n \xrightarrow{r-\mathcal{I}_\nu} \eta$ then $x_n + y_n \xrightarrow{r'-\mathcal{I}_\nu} \xi + \eta$, where $r' = 2r$.*
2. *If $x_n \xrightarrow{r-\mathcal{I}_\nu} \xi$ and $\alpha \in \mathbb{R}$ then $\alpha x_n \xrightarrow{r_1-\mathcal{I}_\nu} \alpha\xi$, where $r_1 = |\alpha| r$.*

Proof. 1. Let $\varepsilon > 0$. Now, for given $\lambda \in (0,1)$, choose $s \in (0,1)$ such that $(1-s) \star (1-s) > 1-\lambda$. Since $x_n \xrightarrow{r-\mathcal{I}_\nu} \xi$ and $y_n \xrightarrow{r-\mathcal{I}_\nu} \eta$, $A, B \in \mathcal{I}$, where $A = \{n \in \mathbb{N} : \nu(x_n - \xi; r + \frac{\varepsilon}{2}) \leq 1-s\}$ and $B = \{n \in \mathbb{N} : \nu(y_n - \eta; r + \frac{\varepsilon}{2}) \leq 1-s\}$. Now, for $n \in A^c \cap B^c \in \mathcal{F}(\mathcal{I})$ we have $\nu(x_n + y_n - (\xi + \eta); 2r + \varepsilon) \geq \nu(x_n - \xi; r + \frac{\varepsilon}{2}) \star \nu(x_n - \eta; r + \frac{\varepsilon}{2}) > (1-s) \star (1-s) > 1-\lambda$. Therefore $\{n \in \mathbb{N} : \nu(x_n + y_n - (\xi + \eta); r' + \varepsilon) \leq 1-\lambda\} \subseteq A \cup B \in \mathcal{I}$ where $r' = 2r$.

2. Since $x_n \xrightarrow{r-\mathcal{I}_\nu} \xi$, then for $\varepsilon > 0$ and $\lambda \in (0,1)$ the set $\{n \in \mathbb{N} : \nu(x_n - \xi; r + \frac{\varepsilon}{|\alpha|}) \leq 1-\lambda\} \in \mathcal{I}$. So, $\{n \in \mathbb{N} : \nu(\alpha x_n - \alpha\xi; r_1 + \varepsilon) \leq 1-\lambda\} \in \mathcal{I}$ where $r_1 = |\alpha| r$. □

We will discuss on some topological and geometrical properties of rough $\mathcal{I}$-limit set of a sequence in a PNS.

THEOREM 3.9. *Let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence in a PNS $(X, \nu, \star)$. Then the set $\mathcal{I}_\nu$-$LIM^r_{x_n}$ is a closed set.*

Proof. If $\mathcal{I}_\nu$-$LIM^r_{x_n} = \emptyset$, then we have nothing to prove. So let $\mathcal{I}_\nu$-$LIM^r_{x_n} \neq \emptyset$. Suppose that $\{y_n\}_{n\in\mathbb{N}}$ is a sequence in $\mathcal{I}_\nu$-$LIM^r_{x_n}$ such that ν-$\lim_{n\to\infty} y_n = \xi$. Now, for given $\lambda \in (0,1)$, choose $s \in (0,1)$ such that $(1-s) \star (1-s) > 1-\lambda$. Let $\varepsilon > 0$ be given. Then there exists a $n_0 \in \mathbb{N}$ such that $\nu(y_n - \xi; \frac{\varepsilon}{2}) > 1-s$ for all $n \geq n_0$. Suppose $y_k \in \mathcal{I}_\nu$-$LIM^r_{x_n}$ where $k > n_0$. Consequently the set $A = \{n \in \mathbb{N} : \nu(x_n - y_k; r + \frac{\varepsilon}{2}) \leq 1-s\} \in \mathcal{I}$. So, $M = \mathbb{N} \setminus A \in \mathcal{F}(\mathcal{I})$ and hence $M \neq \emptyset$. Let $i \in M$. Therefore, $\nu(x_i - y_k; r + \frac{\varepsilon}{2}) > 1-s$. Again we get, for $k > n_0$, $\nu(y_k - \xi; \frac{\varepsilon}{2}) > 1-s$. Now $\nu(x_i - \xi; r + \varepsilon) \geq \nu(x_i - y_k; r + \frac{\varepsilon}{2}) \star \nu(y_k - \xi; \frac{\varepsilon}{2}) > (1-s) \star (1-s) > 1-\lambda$. Therefore $M \subset \{n \in \mathbb{N} : \nu(x_n - \xi; r + \varepsilon) > 1-\lambda\}$. Consequently $\{n \in \mathbb{N} : \nu(x_n - \xi; r + \varepsilon) \leq 1-\lambda\} \in \mathcal{I}$. Hence $\xi \in \mathcal{I}_\nu$-$LIM^r_{x_n}$. Therefore $\mathcal{I}_\nu$-$LIM^r_{x_n}$ is closed. □

In the following theorem, likely in a normed linear space, it can be shown that rough $\mathcal{I}$-limit set in a PNS is convex.

THEOREM 3.10. *Let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence in a PNS $(X, \nu, \star)$. Then the set $\mathcal{I}_\nu$-$LIM^r_{x_n}$ is convex for some $r > 0$.*

Proof. Let $y_1, y_2 \in \mathcal{I}_\nu$-$LIM^r_{x_n}$ and $\beta \in (0,1)$. Suppose $\lambda \in (0,1)$. Choose $s \in (0,1)$ such that $(1-s) \star (1-s) > 1-\lambda$. Then for every $\varepsilon > 0$, the sets $A = \{n \in \mathbb{N} : \nu(x_n - y_1; r + \frac{\varepsilon}{2(1-\beta)}) \leq 1-s\} \in \mathcal{I}$ and $B = \{n \in \mathbb{N} : \nu(x_n - y_2; r + \frac{\varepsilon}{2\beta}) \leq 1-s\} \in \mathcal{I}$. Then $A^c \cap B^c \in \mathcal{F}(\mathcal{I})$. Now, for $j \in A^c \cap B^c$, we have

$$\begin{aligned}
&\nu(x_j - [(1-\beta)y_1 + \beta y_2]; r + \varepsilon) \\
&\geq \nu((1-\beta)(x_j - y_1); (1-\beta)r + \frac{\varepsilon}{2}) \star \nu(\beta(x_j - y_2); \beta r + \frac{\varepsilon}{2}) \\
&= \nu(x_j - y_1; \frac{(1-\beta)r}{1-\beta} + \frac{\varepsilon}{2(1-\beta)}) \star \nu(x_j - y_2; \frac{\beta r}{\beta} + \frac{\varepsilon}{2\beta}) \\
&= \nu(x_j - y_1; r + \frac{\varepsilon}{2(1-\beta)}) \star \nu(x_j - y_2; r + \frac{\varepsilon}{2\beta}) \\
&> (1-s) \star (1-s) > 1-\lambda.
\end{aligned}$$

This gives $A^c \cap B^c \subset \{n \in \mathbb{N} : \nu(x_n - [(1-\beta)y_1 + \beta y_2]; r + \varepsilon) > 1-\lambda\}$. Consequently $\{n \in \mathbb{N} : \nu(x_n - [(1-\beta)y_1 + \beta y_2]; r + \varepsilon) \leq 1-\lambda\} \in \mathcal{I}$. Therefore $(1-\beta)y_1 + \beta y_2 \in \mathcal{I}_\nu$-$LIM^r_{x_n}$ i.e. $\mathcal{I}_\nu$-$LIM^r_{x_n}$ is a convex set. This completes the proof. □

The following theorem gives a sufficient condition for a sequence $\{x_n\}_{n\in\mathbb{N}}$ to be rough $\mathcal{I}$-convergent in terms of a given $\mathcal{I}$-convergent sequence $\{y_n\}_{n\in\mathbb{N}}$.

THEOREM 3.11. *A sequence $\{x_n\}_{n\in\mathbb{N}}$ in a PNS $(X,\nu,\star)$ is rough $\mathcal{I}$-convergent to $\xi\in X$ with respect to the probabilistic norm ν for some $r>0$ if there exists a sequence $\{y_n\}_{n\in\mathbb{N}}$ in X such that $\mathcal{I}_\nu\text{-}\lim_{n\to\infty} y_n=\xi$ and for every $\lambda\in(0,1)$, $\nu(x_n-y_n;r)>1-\lambda$ for all $n\in\mathbb{N}$.*

Proof. Let $\varepsilon>0$ be given. For a given $\lambda\in(0,1)$, choose $s\in(0,1)$ such that $(1-s)\star(1-s)>1-\lambda$ Suppose that $\mathcal{I}_\nu\text{-}\lim_{n\to\infty} y_n=\xi$ and $\nu(x_n-y_n;r)>1-s$ for all $n\in\mathbb{N}$. Then the set $A=\{n\in\mathbb{N}:\nu(y_n-\xi;\varepsilon)\le 1-\lambda\}\in\mathcal{I}$. Then there exists a set $M\in\mathcal{F}(\mathcal{I})$ such that $M=\mathbb{N}\setminus A$. Now, for $n\in M$, we have $\nu(x_n-\xi;r+\varepsilon)\ge\nu(x_n-y_n;r)\star\nu(y_n-\xi;\varepsilon)>(1-s)\star(1-s)>1-\lambda$. Therefore $M\subset\{n\in\mathbb{N}:\nu(x_n-\xi;r+\varepsilon)>1-\lambda\}$. Consequently $\{n\in\mathbb{N}:\nu(x_n-\xi;r+\varepsilon)\le 1-\lambda\}\in\mathcal{I}$. Therefore $r\text{-}\mathcal{I}_\nu\text{-}\lim_{n\to\infty} x_n=\xi$. This completes the proof. □

If z is any point of the rough $\mathcal{I}$-limit set $\mathcal{I}_\nu\text{-}LIM^r_{x_n}$ of roughness degree r then any point $y\in\mathcal{I}_\nu\text{-}LIM^r_{x_n}$ belongs to an open ball centered at z of radius λ of roughness degree mr for $m>2$. This is shown in the following theorem.

THEOREM 3.12. *Let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence in a PNS $(X,\nu,\star)$. Then there does not exist $y,z\in\mathcal{I}_\nu\text{-}LIM^r_{x_n}$ for some $r>0$ and every $\lambda\in(0,1)$ such that $\nu(y-z;mr)\le 1-\lambda$ for $m(\in\mathbb{R})>2$.*

Proof. If possible, let there exist the elements $y,z\in\mathcal{I}_\nu\text{-}LIM^r_{x_n}$ for which

$$\nu(y-z;mr)\le 1-\lambda \text{ for } m(\in\mathbb{R})>2 \tag{1}$$

For a given $\lambda\in(0,1)$, choose $s\in(0,1)$ such that $(1-s)\star(1-s)>1-\lambda$. Now, since $y,z\in\mathcal{I}_\nu\text{-}LIM^r_{x_n}$, then for every $\varepsilon>0$ we have $A=\{n\in\mathbb{N}:\nu(x_n-y;r+\frac{\varepsilon}{2})\le 1-s\}\in\mathcal{I}$ and $B=\{n\in\mathbb{N}:\nu(x_n-z;r+\frac{\varepsilon}{2})\le 1-s\}\in\mathcal{I}$. Then $M=A^c\cap B^c\in\mathcal{F}(\mathcal{I})$. Now, for $n\in M$ we have $\nu(y-z;2r+\varepsilon)\ge\nu(x_n-y;r+\frac{\varepsilon}{2})\star\nu(x_n-z;r+\frac{\varepsilon}{2})>(1-s)\star(1-s)>1-\lambda$. Therefore

$$\nu(y-z;2r+\varepsilon)>1-\lambda \tag{2}$$

Now, if we choose $\varepsilon=mr-2r$, $m(\in\mathbb{R})>2$, then from 2 we get $\nu(y-z;mr)>1-\lambda$ for $m(\in\mathbb{R})>2$. This contradicts 1. This completes the proof. □

DEFINITION 3.13. Let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence in a PNS $(X,\nu,\star)$. Then a point $\zeta\in X$ is called rough $\mathcal{I}$-cluster point of $\{x_n\}_{n\in\mathbb{N}}$ with respect to the probabilistic norm ν if for every $\varepsilon>0$, $\lambda\in(0,1)$ and some non negative number r, $\{n\in\mathbb{N}:\nu(x_n-\zeta;r+\varepsilon)>1-\lambda\}\notin\mathcal{I}$. The set of all rough $\mathcal{I}$-cluster points with respect to the probabilistic norm ν of $\{x_n\}_{n\in\mathbb{N}}$ is denoted as $\Lambda^r_{(x_n)}(\mathcal{I}_\nu)$.

We denote by $\Lambda_{(x_n)}(\mathcal{I}_\nu)$ to mean the set of all ordinary $\mathcal{I}$-cluster points of $\{x_n\}_{n\in\mathbb{N}}$ with respect to the probabilistic norm ν. If $r=0$, then we have $\Lambda^r_{(x_n)}(\mathcal{I}_\nu)=\Lambda_{(x_n)}(\mathcal{I}_\nu)$.

THEOREM 3.14. *Let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence in a PNS $(X,\nu,\star)$. Then the set $\Lambda^r_{(x_n)}(\mathcal{I}_\nu)$ is closed for some $r>0$.*

Proof. If $\Lambda^r_{(x_n)}(\mathcal{I}_\nu)=\emptyset$, then we have nothing to prove. So, let $\Lambda^r_{(x_n)}(\mathcal{I}_\nu)\ne\emptyset$. Suppose that $\{y_n\}_{n\in\mathbb{N}}$ is a sequence in $\Lambda^r_{(x_n)}(\mathcal{I}_\nu)$ such that $\nu\text{-}\lim_{n\to\infty} y_n=\eta$, $\eta\in X$. Now for given $\lambda\in(0,1)$, choose $s\in(0,1)$ such that $(1-s)\star(1-s)>1-\lambda$. Let

$\varepsilon > 0$ be given. Then there exists a $n_0 \in \mathbb{N}$ such that $\nu(y_n - \eta; \frac{\varepsilon}{2}) > 1 - s$ for all $n \geq n_0$. Choose $m \in \mathbb{N}$ such that $m > n_0$. Then $\nu(y_m - \eta; \frac{\varepsilon}{2}) > 1 - s$. Consequently the set $\{n \in \mathbb{N} : \nu(x_n - y_m; r + \frac{\varepsilon}{2}) > 1 - s\} \notin \mathcal{I}$, since y_m is a rough $\mathcal{I}$-cluster point of $\{x_n\}_{n\in\mathbb{N}}$. Let $A = \{n \in \mathbb{N} : \nu(x_n - y_m; r + \frac{\varepsilon}{2}) > 1 - s\}$ and $k \in A$. Then we have $\nu(x_k - y_m; r + \frac{\varepsilon}{2}) > 1 - s$. Now we have $\nu(x_k - \eta; r + \varepsilon) \geq \nu(x_k - y_m; r + \frac{\varepsilon}{2}) \star \nu(y_m - \eta; \frac{\varepsilon}{2}) > (1 - s) \star (1 - s) > 1 - \lambda$. Therefore, $A \subset \{n \in \mathbb{N} : \nu(x_n - \eta; r + \varepsilon) > 1 - \lambda\}$. Clearly $\{n \in \mathbb{N} : \nu(x_n - \eta; r + \varepsilon) > 1 - \lambda\} \notin \mathcal{I}$, otherwise $A \in \mathcal{I}$, which leads to a contradiction. Hence $\eta \in \Lambda^r_{(x_n)}(\mathcal{I}_\nu)$. Therefore $\Lambda^r_{(x_n)}(\mathcal{I}_\nu)$ is closed. This completes the proof. □

We have found out a condition in the following theorem for a point β to be a rough $\mathcal{I}$-cluster point in a PNS.

THEOREM 3.15. *Let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence in a PNS $(X, \nu, \star)$. Then for an arbitrary $\zeta \in \Lambda_{(x_n)}(\mathcal{I}_\nu)$, $\lambda \in (0, 1)$ and for some $r > 0$ we have $\nu(\zeta - \beta; r) > 1 - \lambda$ implies $\beta \in \Lambda^r_{(x_n)}(\mathcal{I}_\nu)$.*

Proof. For $\lambda \in (0, 1)$, choose $s \in (0, 1)$ such that $(1 - s) \star (1 - s) > 1 - \lambda$. Since $\zeta \in \Lambda_{(x_n)}(\mathcal{I}_\nu)$, then for every $\varepsilon > 0$, the set $\{n \in \mathbb{N} : \nu(x_n - \zeta; \varepsilon) > 1 - s\} \notin \mathcal{I}$. Let $M = \{n \in \mathbb{N} : \nu(x_n - \zeta; \varepsilon) > 1 - s\}$. Now we prove that if $\beta \in X$ having the properties $\nu(\zeta - \beta; r) > 1 - s$, then $\beta \in \Lambda^r_{(x_n)}(\mathcal{I}_\nu)$. Clearly for $n \in M$, we have $\nu(x_n - \beta; r + \varepsilon) \geq \nu(x_n - \zeta; \varepsilon) \star \nu(\zeta - \beta; r) > (1 - s) \star (1 - s) > 1 - \lambda$. This shows that $M \subset \{n \in \mathbb{N} : \nu(x_n - \beta; r + \varepsilon) > 1 - \lambda\}$. Clearly $\{n \in \mathbb{N} : \nu(x_n - \beta; r + \varepsilon) > 1 - \lambda\} \notin \mathcal{I}$. If not, then M would belong to $\mathcal{I}$, which leads to a contradiction. Therefore $\beta \in \Lambda^r_{(x_n)}(\mathcal{I}_\nu)$. This completes the proof. □

The following theorem shows that $\Lambda^r_{(x_n)}(\mathcal{I}_\nu)$ contains a closed ball whose center is rough $\mathcal{I}$-cluster point of $\{x_n\}_{n\in\mathbb{N}}$.

THEOREM 3.16. *Let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence in a PNS $(X, \nu, \star)$. Then for some $r > 0$, $\lambda \in (0, 1)$ and fixed $c \in X$ we have*

$$\bigcup_{c\in\Lambda_{(x_n)}(\mathcal{I}_\nu)} \overline{B(c, \lambda; r)} \subset \Lambda^r_{(x_n)}(\mathcal{I}_\nu),$$

where bar denotes the closure of the open ball $B(c, \lambda; r)$.

Proof. Let $x_* \in \bigcup_{c\in\Lambda_{(x_n)}(\mathcal{I}_\nu)} \overline{B(c, \lambda; r)}$. Then there is $c \in \Lambda_{(x_n)}(\mathcal{I}_\nu)$ such that $x_* \in \overline{B(c, \lambda; r)}$. So, by definition $\nu(c - x_*; r) > 1 - \lambda$. Let $\varepsilon > 0$ be given. Since $c \in \Lambda_{(x_n)}(\mathcal{I}_\nu)$, then the set $B = \{n \in \mathbb{N} : \nu(x_n - c; \varepsilon) > 1 - \lambda\} \notin \mathcal{I}$. Now, for $i \in B$, we have $\nu(x_i - x_*; r + \varepsilon) \geq \nu(x_i - c; \varepsilon) \star (c - x_*; r) > (1 - \lambda) \star (1 - \lambda) > 1 - \lambda$. This shows that $B \subset \{n \in \mathbb{N} : \nu(x_n - x_*; r + \varepsilon) > 1 - \lambda\}$. Clearly $\{n \in \mathbb{N} : \nu(x_n - x_*; r + \varepsilon) > 1 - \lambda\} \notin \mathcal{I}$. Hence $x_* \in \Lambda^r_{(x_n)}(\mathcal{I}_\nu)$. Therefore $\bigcup_{c\in\Lambda_{(x_n)}(\mathcal{I}_\nu)} \overline{B(c, \lambda; r)} \subset \Lambda^r_{(x_n)}(\mathcal{I}_\nu)$. □

The rough $\mathcal{I}$-limit sets can be characterized in term of closed balls as shown in the following two theorems.

THEOREM 3.17. *Let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence in a PNS $(X, \nu, \star)$. Then for any $\lambda \in (0, 1)$, the following statements hold:*

1. *If $c \in \Lambda_{(x_n)}(\mathcal{I}_\nu)$ then $\mathcal{I}_\nu$-$LIM^r_{x_n} \subseteq \overline{B(c, \lambda; r)}$.*
2. *$\mathcal{I}_\nu$-$LIM^r_{x_n} \subseteq \bigcap_{c\in\Lambda_{(x_n)}(\mathcal{I}_\nu)} \overline{B(c, \lambda; r)} \subseteq \{y_0 \in X : \Lambda_{(x_n)}(\mathcal{I}_\nu) \subseteq \overline{B(y_0, \lambda; r)}\}$.*

Proof. 1. For given $\lambda \in (0,1)$, we choose $s \in (0,1)$ such that $(1-s)\star(1-s) > 1-\lambda$. Let $\eta \in \mathcal{I}_\nu\text{-}LIM^r_{x_n}$. Then for every $\varepsilon > 0$ and $s \in (0,1)$, we have $P = \{n \in \mathbb{N} : \nu(x_n - \eta; r+\varepsilon) \leq 1-s\} \in \mathcal{I}$ and since $c \in \Lambda_{(x_n)}(\mathcal{I}_\nu)$ $M = \{n \in \mathbb{N} : \nu(x_n - c; \varepsilon) > 1-s\} \notin \mathcal{I}$. Let $P^c = Q$. Then clearly $Q \cap M \neq \emptyset$. If not, let $Q \cap M = \emptyset$. So $M \subset \mathbb{N} \setminus Q$ i.e. $M \subset P$. Since $P \in \mathcal{I}$, $M \in \mathcal{I}$, a contradiction. Now, for $n \in Q \cap M$ we have $\nu(\eta - c; r) \geq \nu(x_n - \eta; r+\varepsilon)\star\nu(x_n - c; \varepsilon)\star\nu(\theta; -2\varepsilon) = \nu(x_n - \eta; r+\varepsilon)\star\nu(x_n - c; \varepsilon)\star 1 = \nu(x_n - \eta; r+\varepsilon)\star\nu(x_n - c; \varepsilon) > (1-s)\star(1-s) > 1-\lambda$. This shows that $\eta \in B(c,\lambda;r) \subset \overline{B(c,\lambda;r)}$. Therefore, $\mathcal{I}_\nu\text{-}LIM^r_{x_n} \subseteq \overline{B(c,\lambda;r)}$.

2. By (1), we have $\mathcal{I}_\nu\text{-}LIM^r_{x_n} \subseteq \bigcap_{c \in \Lambda_{(x_n)}(\mathcal{I}_\nu)} \overline{B(c,\lambda;r)}$. Let $y_0 \in \bigcap_{c \in \Lambda_{(x_n)}(\mathcal{I}_\nu)} \overline{B(c,\lambda;r)}$. Then we have $\nu(y_0 - c; r) \geq 1-\lambda$ for all $c \in \Lambda_{(x_n)}(\mathcal{I}_\nu) \implies \nu(c - y_0; r) \geq 1-\lambda\ \forall c \in \Lambda_{(x_n)}(\mathcal{I}_\nu)$. Hence $\Lambda_{(x_n)}(\mathcal{I}_\nu) \subseteq \overline{B(y_0,\lambda;r)}$, i.e., we have $\bigcap_{c \in \Lambda_{(x_n)}(\mathcal{I}_\nu)} \overline{B(c,\lambda;r)} \subseteq \{y_0 \in X : \Lambda_{(x_n)}(\mathcal{I}_\nu) \subseteq \overline{B(y_0,\lambda;r)}\}$. Therefore $\mathcal{I}_\nu\text{-}LIM^r_{x_n} \subseteq \bigcap_{c \in \Lambda_{(x_n)}(\mathcal{I}_\nu)} \overline{B(c,\lambda;r)} \subseteq \{y_0 \in X : \Lambda_{(x_n)}(\mathcal{I}_\nu) \subseteq \overline{B(y_0,\lambda;r)}\}$. This completes the proof.

□

THEOREM 3.18. *Let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence in a PNS $(X,\nu,\star)$ and $x_n \xrightarrow{\mathcal{I}_\nu} \beta$ then there exists $\lambda \in (0,1)$ such that $\mathcal{I}_\nu\text{-}LIM^r_{x_n} = \overline{B(\beta,\lambda;r)}$ for some $r > 0$.*

Proof. For given $\lambda \in (0,1)$ choose $s \in (0,1)$ such that $(1-s)\star(1-s) > 1-\lambda$. Since $x_n \xrightarrow{\mathcal{I}_\nu} \beta$, then for every $\varepsilon > 0$, the set $A = \{n \in \mathbb{N} : \nu(x_n - \beta; \varepsilon) \leq 1-s\} \in \mathcal{I}$. Let $\zeta \in \overline{B(\beta,\lambda;r)}$. So $\nu(\zeta-\beta; r) \geq 1-\lambda$ and hence $\nu(\beta-\zeta; r) \geq 1-\lambda$. Now for $n \in A^c$, we have $\nu(x_n - \zeta; r+\varepsilon) \geq \nu(x_n - \beta; \varepsilon)\star\nu(\beta-\zeta; r) > (1-s)\star(1-s) > 1-\lambda$, which shows that $\{n \in \mathbb{N} : \nu(x_n - \zeta; r+\varepsilon) \leq 1-\lambda\} \subset A$. Therefore $\{n \in \mathbb{N} : \nu(x_n - \zeta; r+\varepsilon) \leq 1-\lambda\} \in \mathcal{I}$. Hence $\zeta \in \mathcal{I}_\nu\text{-}LIM^r_{x_n}$. So, $\overline{B(\beta,\lambda;r)} \subseteq \mathcal{I}_\nu\text{-}LIM^r_{x_n}$. Again, since $x_n \xrightarrow{\mathcal{I}_\nu} \beta$, $\beta \in \Lambda_{(x_n)}(\mathcal{I}_\nu)$. Therefore, from Theorem 3.17, we have $\mathcal{I}_\nu\text{-}LIM^r_{x_n} \subseteq \overline{B(\beta,\lambda;r)}$. Therefore $\mathcal{I}_\nu\text{-}LIM^r_{x_n} = \overline{B(\beta,\lambda;r)}$. This completes the proof. □

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References

[1] C. Alsina, B. Schweizer and A. Sklar, *On the definition of a probabilistic normed space*, Aequ. Math., **46** (1993), 91–98.

[2] S. Aytar, *Rough statistical convergence*, Numer. Funct. Anal. Optim., **29** (3-4) (2008), 291–303. https://doi.org/10.1080/01630560802001064

[3] A. Aghajani and K. Nourouzi, *Convex sets in probabilistic normed spaces*, Chaos Solitons Fractals, **36** (2) (2008), 322–328. https://doi.org/10.1016/j.chaos.2006.06.051

[4] M. Arslan and E. Dündar, *Rough convergence in* 2*-normed spaces*, Bull. Math. Anal. Appl., **10** (3) (2018), 1–9.
[5] R. Antal, M. Chawla and V. Kumar, *Rough statistical convergence in probabilistic normed spaces*, Thai J. Math., **20** (4) (2022), 1707–1719.
[6] A. K. Banerjee and N. Hossain, *On $\mathcal{I}$-convergence of sequences of functions and uniform conjugacy*, Journal of Mathematical Analysis, **13** (5) (2022), 12–20.
[7] E. Dündar and C. Çakan, *Rough $\mathcal{I}$-convergence*, Demonstr. Math., **47** (3) (2014), 638–651. http://dx.doi.org/10.2478/dema-2014-0051
[8] E. Dündar, *On rough $\mathcal{I}_2$-convergence of double sequences*, Numer. Funct. Anal. Optim., **37** (4) (2016), 480–491. http://dx.doi.org/10.1080/01630563.2015.1136326
[9] H. Fast, *Sur la convergence statistique*, Colloq. Math., **2** (3-4) (1951), 241–244.
[10] M. J. Frank, *Probabilistic topological spaces*, J. Math. Anal. Appl., **34** (1) (1971), 67–81.
[11] S. Ghosal and M. Banerjee, *Effects on rough $\mathcal{I}$-lacunary statistical convergence to induce the weighted sequence*, Filomat, **32** (10) (2018), 3557–3568. http://dx.doi.org/10.2298/FIL1810557G
[12] N. Hossain and A. K. Banerjee, *Rough $\mathcal{I}$-convergence in intuitionistic fuzzy normed spaces*, Bull. Math. Anal. Appl., **14** (4) (2022), 1–10.
[13] P. Kostyrko, T. Šalát and W. Wilczyński, *$\mathcal{I}$-convergence*, Real Anal. Exchange, **26** (2) (2000/01), 669–685.
[14] E. P. Klement, R. Mesiar and E. Pap, *Triangular norms. Position paper I: basic analytical and algebraic properties*, Fuzzy Sets Syst., **143** (2004), 5–26. https://doi.org/10.1016/j.fss.2003.06.007
[15] S. Karakus, *Statistical convergence on probabilistic normed spaces*, Math. Commun., **12** (1) (2007), 11–23.
[16] V. Kumar, *On $\mathcal{I}$ and $\mathcal{I}^*$-convergence of double sequences*, Math. Commun., **12** (2) (2007), 171–181.
[17] Ö. Kişi and H. K. Ünal, *Rough $\Delta\mathcal{I}_2$-statistical convergence of double sequences in normed linear spaces*, Bull. Math. Anal. Appl., **12** (2) (2020), 1–11.
[18] B. K. Lahiri and P. Das, *$\mathcal{I}$ and $\mathcal{I}^*$-convergence in topological spaces*, Math. Bohem., **130** (2) (2005), 153–160. http://dx.doi.org/10.21136/MB.2005.134133
[19] K. Menger, *Statistical metrices*, Proc. Nat. Acad. Sci. USA, **28** (12) (1942), 535–537.
[20] M. Mursaleen and A. Alotaibi, *$\mathcal{I}$-convergence in random* 2*-normed spaces*, Math. Slovaca, **61** (6) (2011), 933–940.
[21] M. Mursaleen and S. A. Mohiuddine, *On ideal convergence in probabilistic normed spaces*, Math. Slovaca, **62** (1) (2012), 49–62. http://dx.doi.org/10.2478/s12175-011-0071-9
[22] P. Malik and M. Maity, *On rough convergence of double sequence in normed linear spaces*, Bull. Allah. Math. Soc., **28** (1) (2013), 89–99. http://dx.doi.org/10.1007/s13370-015-0332-9
[23] P. Malik and A. Ghosh, *Rough $\mathcal{I}$-statistical convergence of double sequences in normed linear spaces*, Malaya J. Mat., **7** (1) (2019), 55–61. https://doi.org/10.26637/MJM0701/0011
[24] H. X. Phu, *Rough convergence in normed linear spaces*, Numer. Funct. Anal. Optim., **22** (1-2) (2001), 199–222. http://dx.doi.org/10.1081/NFA-100103794
[25] H. X. Phu, *Rough continuity of linear operators*, Numer. Funct. Anal. Optim., **23**, (2002), 139–146.
[26] H. X. Phu, *Rough convergence in infinite dimensional normed spaces*, Numer. Funct. Anal. Optim., **24**, (2003), 285–301. http://dx.doi.org/10.1081/NFA-120022923
[27] S. K. Pal, E. Savaş and H. Čakallı, *$\mathcal{I}$-convergence on cone metric spaces*, Sarajevo J. Math., **9** (1) (2013), 85–93. http://dx.doi.org/10.5644/SJM.09.1.07

[28] S. K. Pal, D. Chandra and S. Dutta, *Rough ideal convergence*, Hacet. J. Math. Stat., **42** (6) (2013), 633–640.

[29] H. Steinhaus, *Sur la convergence ordinaire et la convergence asymptotique*, Colloq. Math., **2** (1) (1951), 73–74.

[30] A. N. Šerstnev, *Random normed spaces: Problems of completeness*, Kazan. Gos. Univ. Ucen. Zap., **122** (1962), 3–20.

[31] B. Schweizer and A. Sklar, *Probabilistic metric spaces*, North Holland, New York-Amsterdam-Oxford, 1983.

[32] T. Šalát, B. C. Tripathy and M. Ziman, *On some properties of $\mathcal{I}$-convergence*, Tatra Mt. Math. Publ., **28** (2) (2004), 274–286.

[33] R. Saadati and J.H. Park, *On the intuitionistic fuzzy topological spaces*, Chaos Solitons Fractals, **27** (2) (2006), 331–344.
https://doi.org/10.1016/j.chaos.2005.03.019

[34] B. C. Tripathy, M. Sen and S. Nath, *$\mathcal{I}$-convergence in probabilistic n-normed space*, Soft computing, **16** (6) (2012), 1021–1027.
https://doi.org/10.1007/s00500-011-0799-8

[35] Ö. Talo and Y. Sever, *On Kuratowski $\mathcal{I}$-convergence of sequences of closed sets*, Filomat, **31** (4) (2017), 899–912.
http://dx.doi.org/10.2298/FIL1704899T

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