Korean J. Math. **32** (2024), No. 1, pp. 59–72
https://dx.doi.org/10.11568/kjm.2024.32.1.59

APPLICATIONS OF FIXED POINT THEORY IN HILBERT SPACES

Kiran Dewangan

Abstract. In the presented paper, the first section contains strong convergence and demiclosedness property of a sequence generated by Karakaya et al. iteration scheme in a Hilbert space for quasi-nonexpansive mappings and also the comparison between the iteration scheme given by Karakaya et al. with well-known iteration schemes for the convergence rate. The second section contains some applications of the fixed point theory in solution of different mathematical problems.

1. Introduction

Fixed point theory plays an important role not only in the field of analysis, but also used to find out solutions of different problems like integral equations, differential equations, convex minimization problems, image recovery, signal processing (refer to [7, 8, 23]) etc.

There are lots of fixed point results in different spaces. One of the most important and fruitful result in a metric space was given by Banach [5] called "Banach Contraction Principle". This principle was further generalized and its several variants were studied by mathematicians over different spaces.

Let X be a Hilbert space and K be a non-empty subset of X. A point $x \in X$ is called a fixed point of a mapping $T : X \to X$ if $T(x) = x$. Through-out the literature, $F(T)$ denotes the set of fixed points of T, that is, $F(T) = \{x \in X : Tx = x\}$. Note that a mapping $T : K \to K$ is called

(i) Lipschitz if $||Tx - Ty|| \leq L||x - y||$, for all $x, y \in K$, where $L > 0$.
(ii) contraction if $||Tx - Ty|| \leq L||x - y||$, for all $x, y \in K$, where $0 < L < 1$.
(iii) nonexpansive if $||Tx - Ty|| \leq ||x - y||$, for all $x, y \in K$ and $L = 1$.

Note that the Banach contraction principle is no longer true for nonexpansive mappings. The study of fixed points of mappings with certain contraction condition attract many researcher, but nonexpansive mapping has also an important role in fixed point theory. The study of nonexpansive mappings were basically motivated by Browder's [6] work on relationship between monotone operators and nonexpansive

Received August 24, 2023. Revised September 13, 2023. Accepted January 14, 2024.
2010 Mathematics Subject Classification: 47H09, 47H10.
Key words and phrases: Hilbert space, fixed point, quasi-nonexpansive mappings, Fejer monotone.

mappings and the significance of the geometric properties of the norm for the existence of fixed point for nonexpansive mapping given by Kirk [19].

In 1967, Diaz and Metcalf [9] introduced the concept of quasi-nonexpansive mapping along with some related ideas. According to [9], a mapping $T : K \to K$ is called quasi-nonexpansive, provided that T has at least one fixed point in X, that is, $F(T) \neq \emptyset$ and if $z \in F(T)$, then $||Tx - z|| \leq ||x - z||$ for all $x \in X$. It is well known that the fixed point set of a quasi-nonexpansive mapping is closed and convex (refer to [13, 22]).

Example 1.1. Let $T : \mathbb{R} \to \mathbb{R}$ defined by $Tx = \begin{cases} \frac{x}{2}\sin(\frac{x}{2}), & x \neq 0, \\ 0, & x = 0. \end{cases}$

Clearly T is quasi-nonexpansive mapping.

Iterative techniques for approximating fixed points of quasi-nonexpansive mappings were studied by various authors. In 1974, Dotson [10] proved that " If X is uniformly convex Banach space, K is non-empty closed convex subset of X and T is quasi-nonexpansive mapping of K into itself, which satisfy condition (I), then the sequence $\{x_k\} \subset K$ generated by Mann iteration [20] converges to a point of T ". In 1978, Itoh and Takahashi [13] established the existence of common fixed points of a quasi-nonexpansive mapping by an elementary constructive method in a Hilbert space.

In 1992, Ghosh and Debnath [11] established convergence of Ishikawa iteration [12] to a unique fixed point of quasi-nonexpansive mapping in uniformly convex Banach space. They proved that " Let K be a closed convex subset of uniformly convex Banach space X and S, T be two quasi-nonexpansive mapping of K into itself. If T, S satisfy condition (C), then the sequence $\{x_k\} \subset K$ generated by Ishikawa iteration [12] converges to a common fixed point of S and T ".

In 2011, Tian and Jin [14] proved some fixed point results for quasi-non expansive mapping in a Hilbert space by using an iterative process involving Lipschitizian mapping. In 2013, Suantai [25] established fixed points of a finite family of multi-valued quasi-nonexpansive mappings in a uniformly convex Banach space. Several fixed point results have been established through different iterative scheme by mathematicians (refer to [2, 4, 15]).

In 2017, the following iteration scheme was introduced by Karakaya et al. [16] to approximate fixed point of nonlinear mappings in a Banach space. This iteration scheme is:
Let K be a non-empty subset of a normed space X and $T : K \to K$ be a nonlinear mapping. Then for each $x_1 \in K$, the sequence $\{x_k\}$ is defined by

$$(1) \qquad \begin{cases} z_k = Tx_k, \\ y_k = (1 - \alpha_k)z_k + \alpha_k Tz_k, \\ x_{k+1} = Ty_k, \ k \geq 1, \end{cases}$$

where $\{\alpha_k\}$ is a sequence in $(0, 1)$.

Karakaya et al. [16] proved stability and some convergence results of the iteration scheme (1) for contractive like operator in a Banach space and proved that their iterative scheme is faster than some well-known iterative schemes.

In this paper, the aim is to establish strong convergence of the iteration scheme defined by (1) for quasi-nonexpansive mappings in a Hilbert space, and comparison between Karakaya et al. iteration scheme (1) with some well-known iteration schemes. Some applications of the fixed point theory in solutions of different mathematical problems are discussed here.

2. Preliminaries

This section proceeds with some necessary concepts and includes some useful results.

Definition 2.1. [18] Let X be a uniformly convex Banach space. A sequence $\{x_k\}$ in X is said to be Fejer monotone with respect to subset K of X, if

$$||x_{k+1} - z|| \leq ||x_k - z||,$$

for all $z \in K$, $k \geq 1$.

Proposition 2.2. [18] *Let K be a non-empty subset of a uniformly convex Banach space X. Suppose that $\{x_k\}$ is a Fejer monotone sequence with respect to K. Then the following holds:*

(I) Sequence $\{x_k\}$ is bounded.
(II) For every $x \in K$, $\{||x_k - x||\}$ converges.

Definition 2.3. [10] Let X be a uniformly convex Banach space with the norm $||.||$ and K a convex subset of X. A mapping $T : K \to K$ with non-empty fixed point set $F(T)$ in K is said to satisfy Condition (I), if there is a non-decreasing function $f : [0,\infty) \to [0,\infty)$ with $f(0) = 0$ and $f(r) > 0$ for $r \in (0,\infty)$ such that $||x - Tx|| \geq f(d(x, F(T))$ for all $x \in K$, where $d(x, F(T)) = \inf\{||x - z|| : z \in F(T)\}$.

Lemma 2.4. [24] *Let X be a uniformly convex Banach space, and $\{\alpha_k\}$ be a sequence in $[\delta, 1-\delta]$ for some $\delta \in (0,1)$. Suppose that $\{x_k\}$ and $\{y_k\}$ are in X such that $\limsup_{k\to\infty} ||x_k|| \leq c$, $\limsup_{k\to\infty} ||y_k|| \leq c$, and $\limsup_{k\to\infty} ||\alpha_k x_k + (1-\alpha_k)y_k|| = c$ for some $c \geq 0$. Then $\lim_{k\to\infty} ||x_k - y_k|| = 0$.*

Theorem 2.5. [26] *Let X be a Hilbert space. Let $\{x_k\}$ be a sequence of X with $x_k \rightharpoonup x$. If $x \neq y$, hen*

$$\liminf_{k\to\infty} ||x_k - x|| < \liminf_{k\to\infty} ||x_k - y||.$$

Definition 2.6. [27] Let K be a non-empty closed convex subset of a Hilbert space X. For every point $x \in X$, there is a unique nearest point in K denoted by $P_K(x)$ such that

$$||x - P_K(x)|| \leq ||x - y||, \; for\ all\ y \in K.$$

P_K is called a metric projection of X onto K.

Lemma 2.7. [26] *Let K be a non-empty convex subset of a Hilbert space X and let $x \in H$ and $y \in K$. Then the following are equivalent:*

(i) $||x - y|| = d(x, K)$,

(ii) $< x - y, z - y > \geq 0, \ for \ all \ z \in K.$

3. Main results

Lemma 3.1. *Let X be a Hilbert space and K be a non-empty closed convex subset of X. Let $T : K \to K$ be a quasi-nonexpansive mapping with $F(T) \neq \emptyset$. Let $\{x_k\}$ be a sequence in K defined by (1). Then $\lim_{k\to\infty} ||x_k - z||$ exists.*

Proof. Let $z \in F(T)$. By quasi-nonexpansiveness of T, we have

$$\begin{aligned} ||z_k - z|| &= ||Tx_k - z|| \\ &\leq ||x_k - z||, \end{aligned}$$

$$\begin{aligned} ||y_k - z|| &= ||(1 - \alpha_k)z_k + \alpha_k T z_k - z|| \\ &\leq (1 - \alpha_k)||z_k - z|| + \alpha_k||Tz_k - z|| \\ &\leq (1 - \alpha_k)||x_k - z|| + \alpha_k||z_k - z|| \\ &\leq (1 - \alpha_k)||x_k - z|| + \alpha_k||x_k - z|| \\ &\leq ||x_k - z|| \end{aligned}$$

and

$$\begin{aligned} ||x_{k+1} - z|| &= ||Ty_k - z|| \\ &\leq ||y_k - z|| \\ &\leq ||x_k - z||. \end{aligned}$$

It follows that the sequence $\{x_k\}$ is Fejer momotone with respect to $F(T)$. Hence by the Proposition 2.2, sequence $\{x_k\}$ is bounded and $\{||x_k - z||\}$ converges, that is, $\lim_{k\to\infty} ||x_k - z||$ exists. □

Lemma 3.2. *Let X be a Hilbert space and K be a non-empty closed convex subset of X. Let $T : K \to K$ be a quasi-nonexpansive mapping with $F(T) \neq \emptyset$. Let $\{x_k\}$ be a sequence in K defined by (1) and $\{\gamma_k\}$ is sequence in $[\delta, 1 - \delta]$ for some $\delta \in (0, 1)$. Then $\lim_{k\to\infty} ||Tx_k - x_k|| = 0$.*

Proof. From Lemma 3.1, we have $\lim_{k\to\infty} ||x_k - z||$ exists, so suppose that $\lim_{k\to\infty} ||x_k - z|| = w$.
If $w = 0$, then by using quasi-nonexpansiveness of T, we have

$$\begin{aligned} ||Tx_k - x_k|| &\leq ||Tx_k - z|| + ||z - x_k|| \\ &\leq ||x_k - z|| + ||z - x_k||. \end{aligned}$$

Therefore, the result follows.
Suppose that $w > 0$. As $\lim_{k\to\infty} ||x_k - z|| = w$, it follows that $\limsup_{k\to\infty} ||x_k - z|| \leq$

w. Also $||Tx_k - z|| \leq ||x_k - z||$, this implies that $\limsup_{k\to\infty} ||Tx_k - z|| \leq w$. Note that

$$\limsup_{k\to\infty} ||\gamma_k(x_k - z) + (1-\gamma_k)(Tx_k - z)|| \leq \gamma_k \limsup_{k\to\infty} ||x_k - z|| + (1-\gamma_k) \limsup_{k\to\infty} ||x_k - z||.$$

Which gives us

$$\limsup_{k\to\infty} ||\gamma_k(x_k - z) + (1-\gamma_k)(Tx_k - z)|| \leq w.$$

Hence by Lemma 2.4, we have $\lim_{k\to\infty} ||(x_k - z) - (Tx_k - z)|| = 0$. Therefore $\lim_{k\to\infty} ||Tx_k - x_k|| = 0$. □

Following theorem shows the strong convergence of the iteration scheme defined by (1).

THEOREM 3.3. *Let X be a Hilbert space and K be a non-empty closed convex subset of X. Let $T : K \to K$ be a quasi-nonexpansive mapping with $F(T) \neq \emptyset$ and it satisfy Condition (I). Then the sequence $\{x_k\}$ defined by (1) converges strongly to a fixed point of T.*

Proof. Let $z \in F(T)$. From Lemma 3.1, we have

$$||x_{k+1} - z|| \leq ||x_k - z||,$$

it gives that

$$d(x_{k+1}, F(T)) \leq d(x_k, F(T)).$$

Thus $\lim_{k\to\infty} d(x_k, F(T))$ exists. Since T satisfy Condition (I) and from Lemma 3.2, we have $\lim_{k\to\infty} ||Tx_k - x_k|| = 0$, it follows that $\lim_{k\to\infty} f(d(x_k, F(T))) = 0$ and thus $\lim_{k\to\infty} d(x_k, F(T)) = 0$.

Next, we prove that $\{x_k\}$ is a Cauchy sequence in K. Since $\lim_{k\to\infty} d(x_k, F(T)) = 0$, for $\varepsilon' > 0$, there exists a constant k_0 such that for all $k \geq k_0$, we have

$$d(x_k, F(T)) < \frac{\varepsilon'}{4}.$$

Hence there must exists a point $p \in F(T)$ such that

$$||x_{k_0} - p|| < \frac{\varepsilon'}{2}.$$

Now for $m, n \geq k_0$, we have

$$\begin{aligned} ||x_{n+m} - x_n|| &\leq ||x_{n+m} - p|| + ||p - x_n|| \\ &\leq 2||x_{k_0} - p|| \\ &< \varepsilon'. \end{aligned}$$

It follows that $\{x_k\}$ is a Cauchy sequence in K. Since K is closed subset of Hilbert space X, so there exists a point say $x \in K$ such that $||x_k - x|| \to 0$ as $k \to \infty$. Next we prove that x is fixed point of T. For this,

$$\begin{aligned} 0 \le ||Tx - x|| &\le ||Tx - x_{k+1}|| + ||x_{k+1} - x|| \\ &\le ||Tx - Ty_k|| + ||x_{k+1} - x|| \\ &\le ||Tx - Tz|| + ||Tz - Ty_k|| + ||x_{k+1} - x|| \\ &\le ||x - z|| + ||z - y_k||||x_{k+1} - x|| \\ &\le ||x - z|| + ||x_k - z||||x_{k+1} - x|| \\ &= ||x - x_k|| + ||x_{k+1} - x|| \to \ 0 \ as \ k \to \infty. \end{aligned}$$

This shows that x is a fixed point of T. □

THEOREM 3.4. *Let X be a Hilbert space and K be a non-empty closed convex subset of X. Let $T : K \to K$ be a quasi-nonexpansive mapping with $F(T) \neq \emptyset$. Then the sequence $\{x_k\}$ defined by (1) converges strongly to a fixed point of T if and only if $\lim_{k\to\infty} d(x_k, F(T)) = 0$, where $d(x_k, F(T)) = \inf\{||x_k - z|| : z \in F(T)\}$.*

Proof. If the sequence $\{x_k\}$ converges strongly to a fixed point of T, then it is obvious that $\lim_{k\to\infty} d(x_k, F(T)) = 0$.
Conversely, suppose that $\lim_{k\to\infty} d(x_k, F(T)) = 0$. For $\varepsilon' > 0$, there exists $k_0 \in \mathbb{N}$, such that for all $k \ge k_0$,

$$d(x_k, F(T)) < \frac{\varepsilon'}{4}.$$

In particular, there exists a point $z' \in F(T)$ such that

$$||x_{k_0} - z'|| < \frac{\varepsilon'}{2}.$$

For $k, m \ge k_0$, we have

$$\begin{aligned} ||x_{k+m} - x_k|| &\le ||x_{k+m} - z'|| + ||z' - x_k|| \\ &= 2||x_{k_0} - z'|| < \varepsilon'. \end{aligned}$$

It follows that $\{x_k\}$ is a Cauchy sequence in K. Since K is closed subset of Hilbert space X, so there exists a point $x \in K$ such that $||x_k - x|| \to 0$ as $k \to \infty$. By our assumption $\lim_{k\to\infty} d(x_k, F(T)) = 0$, it gives that

$$d(x, F(T)) = 0 \Rightarrow x \in F(T).$$

□

THEOREM 3.5. *Let X be a Hilbert space and K be a non-empty compact convex subset of X. Let $T : K \to K$ be a quasi-nonexpansive mapping with $F(T) \neq \emptyset$. Then the sequence $\{x_k\}$ defined by (1) converges strongly to a fixed point of T.*

Proof. From Lemma 3.2, we have $\lim_{k\to\infty} ||Tx_k - x_k|| = 0$, and K is compact, there exists a subsequence $\{x_{k_p}\}$ of $\{x_k\}$ such that $x_{k_p} \to z$ for some $z \in K$. By qusi-nonexpansiveness of T, we have

$$\begin{aligned} ||x_{k_p} - Tz|| &\le ||x_{k_p} - Tx_{k_p}|| + ||Tx_{k_p} - Tz|| \\ &\le ||x_{k_p} - Tx_{k_p}|| + ||x_{k_p} - z||. \end{aligned}$$

This shows that $x_{k_p} \to Tz$ as $k \to \infty$. By uniqueness of limits, we have, $z = Tz$. Also by Lemma 3.1, $\lim_{k\to\infty} ||x_k - z||$ exists, thus z is the strong limit of the sequence $\{x_k\}$ itself . □

To prove demiclosedness property for quasi-nonexpansive mapping in a Hilbert space, note that a mapping $T: K \to K$ is demiclosed, if $x_k \rightharpoonup x \in K$ and $Tx_k \to y$, then $y = Tx$.

THEOREM 3.6. *Let K be a non-empty closed convex subset of a Hilbert space X. Let $\{x_k\}$ be a sequence of X with $x_k \rightharpoonup x$ and $T: K \to K$ be a quasi-nonexpansive mapping with $F(T) \neq \emptyset$. Then $I - T$ is demiclosed at zero.*

Proof. Let $z \in F(T)$ and $\{x_k\}$ be a sequence in K such that $x_k \rightharpoonup x$. From Lemma 3.2, we have $\lim_{k\to\infty} ||Tx_k - x_k|| = 0$. Let $\{x_{k_p}\}$ be a subsequence of $\{x_k\}$. We claim that $x = Tx$.
Suppose not. By quasi-nonexpansiveness of T and from Theorem 2.5, we have

$$\begin{aligned}
\liminf_{k\to\infty} ||x_{k_p} - x|| &< \liminf_{k\to\infty} ||x_{k_p} - Tx|| \\
&\leq \liminf_{k\to\infty} ||x_{k_p} - Tx_{k_p}|| + \liminf_{k\to\infty} ||Tx_{k_p} - Tx|| \\
&= \liminf_{k\to\infty} ||Tx_{k_p} - Tx|| \\
&\leq \liminf_{k\to\infty} ||Tx_{k_p} - z|| + \liminf_{k\to\infty} ||z - Tx|| \\
&\leq \liminf_{k\to\infty} ||x_{k_p} - z|| + \liminf_{k\to\infty} ||z - x|| \\
&= \liminf_{k\to\infty} ||x_{k_p} - x||.
\end{aligned}$$

Hence, we conclude that $\liminf_{k\to\infty} ||x_{k_p} - x|| < \liminf_{k\to\infty} ||x_{k_p} - x||$, which is a contradiction. Therefore $x = Tx$. $\square$

4. Numerical Examples

We illustrate our main results with the help of following example.

EXAMPLE 4.1. Let $X = \mathbb{R}$ be a Hilbert space, K be a closed subset of $\mathbb{R}$, and $T: \mathbb{R} \to \mathbb{R}$ is defined by

$$Tx = \begin{cases} 0, \ if \ x = 0, \\ x\sin(\frac{1}{x}), \ if \ x \neq 0. \end{cases} \tag{2}$$

Then T is a quasi-nonexpansive mapping, as 0 is a fixed point of T, we have

$$\begin{aligned}
||Tx - 0|| &= ||x\sin(\frac{1}{x}) - 0|| \\
&\leq ||x|| \\
&= ||x - 0||.
\end{aligned}$$

Also, T is not a nonexpansive mapping for $x = \frac{2}{\pi}$, $y = \frac{2}{3\pi}$, as we have $||Tx - Ty|| = \frac{8}{3\pi}$, while $||x - y|| = \frac{4}{3\pi}$.

Now, let $\{x_k\}$ be a sequence in K defined by (1). Since

$$\begin{aligned}
\lim_{k\to\infty} d(x_k, F(T)) &= \lim_{k\to\infty} \inf\{||x_k - z|| : z \in F(T)\} \\
&= \lim_{k\to\infty} \inf\{||x_k - 0|| : 0 \in F(T)\}.
\end{aligned}$$

Also from Lemma 3.1, $\lim_{k\to\infty} ||x_k - 0||$ exists for $0 \in F(T)$ and from Theorem 3.5, $\{x_k\}$ converges strongly to $0 \in F(T)$. Therefore from Theorem 3.4, we have $\lim_{k\to\infty} d(x_k, F(T)) = 0$.

Now with the help of Matlab software program, we compare the iteration scheme (1) with the iteration schemes given by Abbas [1], MP (modified Picard Mann) [17], Agrawal [3], and Noor [21].

Iteration	Karakaya (1)	Abbas (2)	MP (3)	Agrawal (4)	Noor (5)
0	0.50000000	0.50000000	0.50000000	0.50000000	0.50000000
1	-0.11446261	-0.12232388	0.53999888	0.07169060	-0.35915422
2	0.01725645	0.35804406	0.12780578	0.06755385	0.13118460
3	0.01171189	-0.07507173	0.12756071	0.04424888	0.12357188
4	-0.00081247	0.02744380	0.12748508	-0.04658929	-0.11580347
5	-0.00002573	-0.00834802	0.12744714	0.04023444	0.15443382
6	0.00001101	-0.00690044	0.12742407	0.01723836	-0.02521319
7	0.00000018	-0.00181367	0.12740848	0.01630729	0.02798271
8	0.00000001	-0.00142105	0.12739720	-0.01861953	0.00000529
9	0.00000000	0.00011591	0.12738864	0.00425234	0.00000279
10	0.00000000	-0.00001552	0.12738191	-0.00187809	0.00000003
11	0.00000000	-0.00000550	0.12737647	-0.00158199	-0.00000002
12	0.00000000	0.00000075	0.12737199	-0.00058866	0.00000001
13	0.00000000	0.00000016	0.12736822	0.00049998	0.00000000
14	0.00000000	0.00000006	0.12736501	0.00043898	0.00000000
15	0.00000000	0.00000000	0.12736224	0.00015496	0.00000000

TABLE 1. Strong convergence of Karakaya, Abbas, MP, Agrawal, Noor iterations to the fixed point $x = 0$ of T in Example 4.1

From the Table 1, it is clear that the iteration scheme (1) is faster than the Abbas [1], MP [17], Agrawal [3] and Noor [21] iteration schemes for quasi-nonexpansive mappings.

5. Applications of Fixed Point Theory

5.1. Application in Integral Equation. Consider the following nonlinear integral equation-

$$x(t) = \lambda \int_{\Omega} f(t, s, x(s))ds + y(t), \tag{3}$$

where $y : [a, b] \to \mathbb{R}$ and $f(s, t, x(s)) : [a, b] \times [a, b] \times [a, b] \to \mathbb{R}$ are continuous.

Consider the Hilbert space $X = \mathbb{R}$ and $K = [a, b]$ is subset of X. Define $||.|| : \mathbb{R} \to \mathbb{R}$ by

$$||x - y|| = \sup\{|x(s) - y(s)| : s \in [a, b]\},$$

for all $x \in \mathbb{R}$.

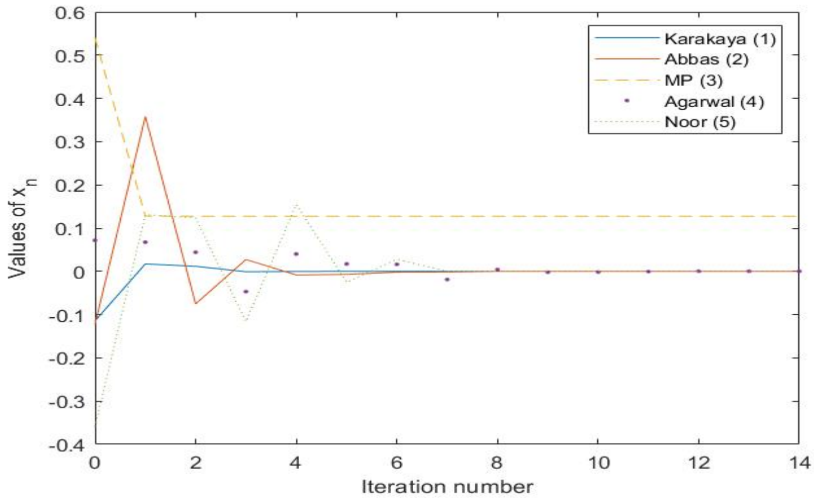

FIGURE 1. Behaviors of Karakaya iteration (cyan), Abbas iteration (carrot orange), Mp iteration (yellow), Agrawal iteration (purple) and Noor iteration (green) to the fixed point $x = 0$ of the mapping T in Example 4.1

THEOREM 5.1. *Let $X = \mathbb{R}$ and $K = [a, b]$ is subset of X and $T : K \to K$ is defined by*

$$Tx(t) = \lambda \int_a^b f(t, s, x(s))ds + y(t), \tag{4}$$

where $y : [a, b] \to \mathbb{R}$ and $f(s, t, x(s)) : [a, b] \times [a, b] \times [a, b] \to \mathbb{R}$ are continuous and $\lambda \in \mathbb{R}$. suppose that the following conditions are satisfied-

(i) *There exists a continuous mapping $F : X \times X \to [0, \infty)$ such that*

$$|f(t, s, x(s)) - f(t, s, y(s))| \le F(x, y)|x(s) - y(s)|,$$

for all $s \in [a, b]$, $x, y \in X$.

(ii) *There exists $\zeta \in (0, 1]$ such that $\int_a^b F(x, y) \le \zeta$.*

(iii) *$\lambda\zeta \le 1$.*

Let $\{x_k\}$ be a sequence in K defined by (1). Then the integral equation (4) has a solution.

Proof. Let $x \in [a,b]$ and $F(T) \neq \emptyset$ with $z \in F(T)$. Then

$$\begin{aligned}
|Tx(s) - z(s)| &= |Tx(s) - Tz(s)| \\
&= |\lambda \int_a^b f(t,s,x(s))dt - \lambda \int_a^b f(t,s,z(s))dt| \\
&= |\lambda \int_a^b (f(t,s,x(s)) - f(t,s,z(s)))dt| \\
&\leq \lambda| \int_a^b F(x,z)(x(s) - z(s))dt| \\
\sup|Tx(s) - z(s)| &\leq \sup|x(s) - z(s)|\lambda \int_a^b F(x,z)dt \\
&\leq ||x - z||\lambda\zeta \\
&\leq ||x - z||.
\end{aligned}$$

Hence, we conclude that T is a quasi-nonexpansive mapping. Now define

$$\mathbb{B} = \{x \in X : ||x|| \leq r\},$$

where r is sufficiently large. Clearly $T(\mathbb{B}) \subset \mathbb{B}$. Also $\mathbb{B}$ is compact subset of X, hence from the Theorem 3.5, T has a fixed point in $\mathbb{B}$ and this fixed point is solution of the integral equation (4). □

5.2. Application in Variational Inequality Problem.

THEOREM 5.2. *Let K be a non-empty compact convex subset of a Hilbert space X. Let $T : K \to K$ be a quasi-nonexpansive mapping and $\psi : K \to K$ be a contraction mapping with a contraction coefficient in $[0,1)$. Let $\{\alpha_k\}$ be a sequences in $(0,1)$. Then the sequence $\{x_k\}$ defined by (1) converges strongly to a fixed point $q \in F(T)$, which is also a unique solution of the following variational inequality*

$$< (I - \psi)q, x - q > \geq 0.$$

Proof. From Lemma 3.1, sequence $\{x_k\}$ is bounded and from Lemma 3.2, we have $\lim_{k\to\infty} ||Tx_k - x_k|| = 0$. We claim that

$$\limsup_{k\to\infty} < (I - \psi)q, x - q > \geq 0,$$

where $q \in F(T)$ is unique fixed point of ψ.
Since K is compact, there exists a subsequence $\{x_{k_p}\}$ of $\{x_k\}$ such that $x_{k_p} \to p$ for some $p \in K$. By using Lemma 2.7, we have

$$\begin{aligned}
\limsup_{k\to\infty} < \psi(q) - q, x_k - q > &= \limsup_{k\to\infty} < \psi(q) - q, x_{k_p} - q > \\
&=< (\psi(q) - q, p - q > \geq 0.
\end{aligned}$$

Now we claim that $x_k \to q \in F(T)$. By doing similar procedure as in the proof of the Theorem 3.5, $x_k \to q \in F(T)$. □

5.3. Application in Boundary Value Problem. Consider the following boundary value problem:

$$x^{''}(t) + \gamma(t, h(t)),\ x^{'}(0) = 0 = x(1), \tag{5}$$

where $t \in [0,1]$ and $\gamma : [0,1] \times \mathbb{R} \to \mathbb{R}$ is continous mapping. Let K be a compact subset of X. Then the solutions of (5) are the fixed points of the operator T defined on K by

$$F(t, h(t)) = \int_0^1 G(t,s)\gamma(t, h(t))dt,\ t, s \in [0,1],$$

where G is green function of (5) defined by

$$G(t,s) = \begin{cases} -s(\log s), 0 \leq t \leq s \leq 1, \\ -s(\log t), 0 \leq s \leq t \leq 1. \end{cases}$$

THEOREM 5.3. *Let K be a non-empty compact convex subset of a Hilbert space $X = \mathbb{R}$. Define $T : K \to K$ by*

$$T(h(t)) = \int_0^1 G(a,t)\gamma(t, h(t))dt,$$

for each $h(t) \in F(T)$. Now, choose $K = [0, e^{\frac{-1}{2}}]$. For fixed s, let

$$\int_0^1 G(t,s)ds \leq \int_0^1 G(0,s)ds = \frac{1}{4},\ 0 \leq t \leq 1$$

and

$$|\gamma(t, h_1(t)) - \gamma(t, h_2(t))| \leq \zeta((h_1(t), h_2(t))|h_1(t) - h_2(t)|,$$

where $\int_0^1 \zeta((h_1(t), h_2(t))dt \leq 1$.
Let $\{x_k\}$ be a sequence in K defined by (1). Then it converges to some solution of (5).

Proof. It is known that an element h of $F(T)$ is a solution of (5) if and only if it is a solution to the following integral equation:

$$h(a) = \int_0^1 G(a,t)f(t, h(t))dt.$$

Now, for $h_1, h_2 \in T$, $a \in [0,1]$, we have

$$\begin{aligned}
||T(h_1(t)) - T(h_2(t))|| &= || \int_0^1 G(a,t)\gamma(t,h_1(t))dt - \int_0^1 G(a,t)\gamma(t,h_2(t))dt|| \\
&= || \int_0^1 G(a,t)[\gamma(t,h_1(t)) - \gamma(t,h_2(t))]dt|| \\
&\leq || \int_0^1 (\gamma(t,h_1(t)) - \gamma(t,h_2)(t))dt||| \int_0^1 G(a,t)dt| \\
&\leq \frac{1}{4}|| \int_0^1 (\gamma(t,h_1(t)) - \gamma(t,h_2(t)))|| \\
&\leq \frac{1}{4}| \int_0^1 \zeta((h_1(t),h_2(t))||h_1(t) - h_2(t)||dt| \\
&\leq \frac{1}{4}||h_1(t) - h_2(t)|| \\
&\leq ||h_1(t) - h_2(t)||.
\end{aligned}$$

It conclude that T is a nonexpansive mapping with $F(T) \neq \emptyset$. Hence T is a quasi-nonexpansive mapping. Hence, From Theorem 3.5, a sequence $\{x_k\}$ in K defined by (1) converges strongly to a fixed point of T which is a solution of (5). □

References

[1] H. A. Abbas, A. A., Mebawondu and O. T. Mewomo, *Some results for a new three-step iteration scheme in Banach space*, Bull. Transilvania Uni., Brasov **11** (2) (2018), 1–18.

[2] H. A. Abass, A. A. Mebawondu, O. K. Narain and J. K. Kim, *Outer approximation method for zeros of sum of monotone operators and fixed point problems in Banach spaces*, Nonlnear Funct. Anal. Appl. **26** (3) (2021), 451–474.
https://doi.org/10.22771/nfaa.2021.26.03.02

[3] R. P. Agrawal, D. O'Regan and D. R. Sahu, *Iterative constructions of fixed points of nearly asymptotically nonexpansive mappings*, J. Convex Anal. **8**, (2007), 61–79.

[4] F. Akutsah, O. K. Narain and J. K. Kim, *Improved generalized M-iteration for quasi-nonexpansive multi-valued mappings with application in real Hilbert spaces*, Nonlinear Funct. Anal. Appl. **27** (1) (2022), 59–82.
http://dx.doi.org/10.22771/nfaa.2022.27.01.04

[5] S. Banach, *Sur les operations dans ensembles abstraits et leur application aux equations integrales*, Fundamenta Mathematicae **3** (1922), 133–181.
https://doi.org/10.4064/FM-3-1-133-181

[6] F. E. Browder, *Nonexpansive nonlinear operators in Banach space*, Proc. Nat. Sci., USA **54** (1965), 1041–1044.
https://doi.org/10.1073/pnas.54.4.1041

[7] F. E. Browder and W. V. Petryshyn, *Construction of fixed points of nonlinear mappings in Hilbert space*, J. Math. Anal. Appl. **20** (2) (1967), 197–228.
https://doi.org/10.1016/0022-247X(67)90085-6

[8] C. Byrne, *A unified treatment of some iterative algorithms in signal processing and image reconstruction*, Inverse Problems **20** (1) (2004), 103–120.
https://doi.org/10.1088/0266-5611/20/1/006

[9] J. B. Diaz and F.T. Metcalf, *On the structure of the set of subsequential limit points of successive approximations*, Bull. Amer. Math. Soc. **73** (1967), 516–519.
https://doi.org/10.1006/jmaa.1998.5944

[10] W. G. Dotson and H. F. Senter, *Approximating fixed points of nonexpansive mappings*, Proc. Amer. Math. Soc. **44** (1974), 375–380.

[11] M. K. Ghose and L. Debnath, *Approximation of the fixed points of quasi-nonexpansive mappings in a uniformly convex Banach space*, Appl. Math. Lett. **5** (3) (1992), 47–50.
https://doi.org/10.1016/0893-9659(92)90037-A

[12] S. Ishikawa, *Fixed points by a new iteration method*, Proc. Amer. Math. Soc. **44** (1974), 147–150.
https://doi.org/10.1090/S0002-9939-1974-0336469-5

[13] S. Itoh and W. Takahashi, *The common fixed point theory of single valued mappings and multi-valued mappings*, Pac. J. Math. **79** (1978), 493–508.
https://doi.org/10.2140/PJM.1978.79.493

[14] X. Jin and M. Tian, *Strong convergent result for quasi-nonexpansive mappings in Hilbert spaces*, Fixed Point Theory and Appl. **88** (2011), 1–8.
https://doi.org/10.1186/1687-1812-2011-88

[15] K. S. Kim, *Convergence theorems of mixed type implicit iteration for nonlinear mappings in convex metric spaces*, Nonlinear Funct. Anal. Appl. **27** (4), (2022), 903–920.
https://doi.org/10.22771/nfaa.2022.27.04.16

[16] V. Karakaya, V. Atalan and K. Dogan, *Some fixed point result for a new three-step iteration process in Banach space*, Fixed Point Theory and Appl. **18** (2) (2017), 625–640.
http://dx.doi.org/10.24193/fpt-ro.2017.2.50

[17] S. H. Khan, *A Picard-Mann hybrid iterative process*, Fixed Point Theory and Appl. **69** (2013), 1–10.
https://doi.org/10.1186/1687-1812-2013-69

[18] J. K. Kim, S. Dasputre and W. H. Lim, *Approximation of fixed points for multi-valued nonexpansive mappings in Banach spaces*, Global J. of Pure and Appl. Math. **12** (6) (2016), 4901–4912.

[19] W. A. Kirk and W. O. Ray, *Fixed point theorems for mappings define on unbounded sets in Banach spaces*, Studia mathematica **114** (1979), 127–138.
https://doi.org/10.4064/SM-64-2-127-138

[20] W. R. Mann, *Mean value methods in iteration*, Pro. Amer. Math. Soc. **4**, (1953), 506–510.
https://doi.org/10.1090/S0002-9939-1953-0054846-3

[21] M. A. Noor, *New approximation schemes for general variational inequalities*, J. Math. Anal. Appl. **251** (2000), 217–229.
https://doi.org/10.1006/jmaa.2000.7042

[22] P. Kocourek, W. Takahashi and J. C. Yao, *Fixed point theorems and weak convergence theorems for generalized hybrid mappings in Hilbert spaces*, Taiwan. J. Math. **14** (2010), 2497–2511.
https://doi.org/10.11650/twjm/1500406086

[23] C. I. Podilchuk and R. J. Mammone, *Image recovery by convex projections using a least-squares constraint*, J. Optical Soc. Amer. **7** (3) (1990), 517–521.
https://doi.org/10.1364/JOSAA.7.000517

[24] J. Schu, *Weak and strong convergence of fixed point of asymptotically nonexpansive mappings*, Bull. Aust. Math. Soc. **43** (1) (1991), 153–159.
https://doi.org/10.1017/S0004972700028884

[25] S. Suantai and A. Bunyawat, *Common fixed points of a finite family of multi-valued quasi-nonexpansive mappings in uniformly convex Banach spaces*, Bull. Ira. Math. Soc. **39** (6) (2013), 1125–1135.

[26] W. Takahashi, *Nonlinear functional analysis*, Yokohama publishers, Yokohama (2000).

[27] J. Tiammee, A. Kaewkhao and S. Suantai, *On Browder's convergence theorem and Halpern iteration process for $G-$ nonexpansive mappings in Hilbert spaces endowed with graphs*, Fixed Point Theory and Appl. **187** (2015), 1–12.
https://doi.org/10.1186/s13663-015-0436-9

[28] D. Thakur, B.S. Thakur and M. Postolache, *New iteration scheme for numerical reckoning fixed points of nonexpansive mappings*, J. Inequal. Appl. **2014** (2014), 1–15.
https://doi.org/10.1186/1029-242X-2014-328

Kiran Dewangan
Department of Mathematics,
Government Dudhadhari Bajrang Girls Postgraduate Autonomous College,
Raipur-492001, Chhattisgarh, India
E-mail: dewangan.kiran@gmail.com