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HERMITE-HADAMARD TYPE INEQUALITIES FOR PREINVEX FUNCTIONS WITH APPLICATIONS

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Abstract. In this article, we establish new Hermite-Hadamard Type inequalities for functions whose first derivative in absolute value are preinvex. Further, we give some application of our obtained results to some special means of real numbers. Moreover, we discuss several special cases of the results obtained in this paper.

1. Introduction

It is well known that convexity plays an important role in mathematical programming and optimization theory. An important generalization of convex functions is that of invex functions, which is introduced by Hanson [1]. The result of Hanson influenced many subsequent work, which has greatly expanded the role and applications of invexity in non-linear optimization and other branches of pure and applied science. The concept of preinvex functions was introduced by Ben-Israel and Mond [2]. It is well known that invex sets and preinvex functions may not be convex sets and convex functions. Weir and Mond [3] and further, Noor *et al.* [4] have studied the basic properties of the preinvex functions and their role in optimization see, [5].

The idea of integral inequality is a fascinating area for research within mathematical analysis. Hermite-Hadamard inequality [6, 7] plays most important role in the subject of convex analysis. This inequality is one of the most well established inequalities in the theory of convex function with a geometrical interpretation and many applications. Noor [9] introduced the Hermite-Hadamard inequalities for preinvex and log-preinvex function, which are the generalization of the classical Hermite-Hadamard inequality. Various refinements of the Hermite-Hadamard inequalities for the convex functions and their variants forms are being obtained in the literature by many researchers see, [10–17].

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Recent years, several well-known inequalities like the Hermite-Hadamard type inequality, the Ostrowski type inequalty and the Minkowski type inequality have been presented with the help of using interval analysis. Furthermore, mathematicians have recently begun to establish integral inequalities using interval-valued fractional operators. Srivastava *et al.* [18] introduced Hermite-Hadamard type inequalities for interval-Valued preinvex functions via fractional integral operators. Recently, Khan *et al.* [19] indroduced some new versions of integral inequalities for left and right preinvex functions in the interval-valued settings. Further, Lai *et al.* [20] introduced the concept of preinvex interval-valued function and obtained the new inclusion of fractional Hermite-Hadamard type inequalities of these functions. For more details of the connections between the different form of interval-valued functions and integral inequlities see, [21–24].

The paper is summarized as follows: In section 2, we recall some basic results that are necessary for our main results. In section 3, we prove Hermite-Hadamard-type inequalities for a differentiable preinvex functions. In section 4, we obtain the applications to some special means. In section 5, we present the conclusion and future direction of this study.

2. Preliminaries

In this section, we give some necessary definitions which are used throughout this paper. Let $\Omega : S \to \mathbb{R}$ and $\eta : S \times S \to \mathbb{R}$, where S in non-empty set in $\mathbb{R}^n$, be continuous functions.

Definition 2.1. [3] A set $S \subseteq \mathbb{R}^n$ is said to be invex with respect to the mapping $\eta : S \times S \to \mathbb{R}^n$, if

$$\tau + t\eta(\nu, \tau) \in S$$

for every $\nu, \tau \in S$ and $t \in [0, 1]$.

Notice that every convex set is invex with respect to the mapping $\eta(\nu, \tau) = \nu - \tau$, but the converse is not necessarily true see [28] and references therein.

Definition 2.2. [3] The function Ω defined on the invex set $S \subseteq \mathbb{R}^n$ is said to be preinvex with respect to η, if

$$\Omega(\tau + t\eta(\nu, \tau)) \leq (1 - t)\Omega(\tau) + t\Omega(\nu), \forall \nu, \tau \in S, t \in [0, 1].$$

The concept of preinvexity is more general then convexity since every convex function is preinvex with respect to the mapping $\eta(\nu, \tau) = \nu - \tau$, but the converse is not true.

Mohan and Neogy [8] introduced the following well-known condition C:

Condition C: Let $S \subseteq \mathbb{R}^n$ be an open invex subset with respect to $\eta : S \times S \to \mathbb{R}^n$. The function η satisfies the condition C if for any $\nu, \tau \in S$ and any $t \in [0, 1]$,

$$\eta(\tau, \tau + t\eta(\nu, \tau)) = -t\eta(\nu, \tau),$$
$$\eta(\nu, \tau + t\eta(\nu, \tau)) = (1 - t)\eta(\nu, \tau).$$

Note that for all $\nu, \tau \in S$ and $t \in [0, 1]$, then from condition C, we have

$$\eta(\tau + t_2\eta(\nu, \tau), \tau + t_1\eta(\nu, \tau)) = (t_2 - t_1)\eta(\nu, \tau).$$

In [9], Noor provided the following proof of the Hermite-Hadamard inequality for the preinvex functions:

THEOREM 2.3. *Let $\Omega : S = [\tau, \tau + \eta(\nu, \tau)] \to (0, \infty)$ be a preinvex function on the interval of real numbers S^0 (the interior of S) and $\tau, \nu \in S^0$ with $\tau < \tau + \eta(\nu, \tau)$. Then the following inequality holds:-*

$$\Omega\left(\frac{2\tau + \eta(\nu, \tau)}{2}\right) \leq \frac{1}{\eta(\nu, \tau)} \int_{\tau}^{\tau+\eta(\nu,\tau)} \Omega(x)dx \leq \frac{\Omega(\tau) + \Omega(\nu)}{2}. \tag{1}$$

Barani *et al.* [25], proved the following theorem:-

THEOREM 2.4. *Let $S \subseteq \mathbb{R}$ be an open invex subset with respect to $\eta : S \times S \to \mathbb{R}$. Suppose that $\Omega : S \to \mathbb{R}$ is a differential function. If $|\Omega'|$ is preinvex on S then, for every $\tau, \nu \in S$ with $\eta(\nu, \tau) \neq 0$ the following inequality holds:-*

$$\left|\frac{\Omega(\tau) + \Omega(\tau + \eta(\nu, \tau))}{2} - \frac{1}{\eta(\nu, \tau)} \int_{\tau}^{\tau+\eta(\nu,\tau)} \Omega(x)\, dx\right| \leq \frac{|\eta(\nu, \tau)|}{8} \left(|\Omega'(\tau)| + |\Omega'(\nu)|\right). \tag{2}$$

THEOREM 2.5. [26] Under the assumptions of the above theorem, the following inequality holds:

$$\left|\Omega\left(\frac{2\tau + \eta(\nu, \tau)}{2}\right) - \frac{1}{\eta(\nu, \tau)} \int_{\tau}^{\tau+\eta(\nu,\tau)} \Omega(x)\, dx\right| \leq \frac{|\eta(\nu, \tau)|}{8} \left(|\Omega'(\tau)| + |\Omega'(\nu)|\right). \tag{3}$$

3. Main Result

In this section, we introduce some generalizations of Hermite-Hadamard-type inequality for functions whose first derivatives absolute values are preinvex.

LEMMA 3.1. *Let Ω be an absolutely continuous function on an interval $[\tau, \tau+\eta(\nu, \tau)]$ and let $\Omega' \in L_1[\tau, \tau + \eta(\nu, \tau)]$ be its derivative. Then the following result holds true:-*

$$\begin{aligned}
&\frac{1}{3}\left[\Omega(\tau) + \Omega(\tau + \eta(\nu, \tau)) + \Omega\left[\frac{(2\tau + \eta(\nu, \tau))}{2}\right]\right] - \frac{1}{\eta(\nu, \eta)} \int_{\tau}^{\tau+\eta(\nu,\tau)} \Omega(x)dx \\
&= \eta(\nu, \tau)\left[\int_{0}^{1/2} \left(x - \frac{1}{3}\right)\Omega'(\tau + x\eta(\nu, \tau))dx + \int_{1/2}^{1} \left(x - \frac{2}{3}\right)\Omega'(\tau + x\eta(\nu, \tau))dx\right].
\end{aligned} \tag{4}$$

Proof. Let us solve the subsequent integral by integrating by parts,

$$\begin{aligned}
I_1 &= \int_{0}^{\frac{1}{2}} \left(x - \frac{1}{3}\right)\Omega'(\tau + x\eta(\nu, \tau))dx \\
&= \left[\left(x - \frac{1}{3}\right)\left[\frac{\Omega(\tau + x(\eta(\nu, \tau)))}{\eta(\nu, \tau)}\right]\right]_{0}^{\frac{1}{2}} - \int_{0}^{\frac{1}{2}} \frac{\Omega(\tau + x(\eta(\nu, \tau)))}{\eta(\nu, \tau)}dx \\
&= \frac{1}{6}\frac{1}{\eta(\nu, \tau)}\Omega\left(\frac{2\tau + \eta(\nu, \tau)}{2}\right) + \frac{1}{3}\frac{\Omega(\tau)}{\eta(\nu, \tau)} - \int_{0}^{\frac{1}{2}} \frac{\Omega(\tau + x(\eta(\nu, \tau)))}{\eta(\nu, \tau)}dx.
\end{aligned}$$

Similarly,

$$
\begin{aligned}
I_2 &= \int_{\frac{1}{2}}^{1}\left(x-\frac{2}{3}\right)\Omega'(\tau+x(\eta(\nu,\tau)))dx \\
&= \frac{1}{3}\frac{\Omega(\tau+\eta(\nu,\tau))}{\eta(\nu,\tau)} + \frac{1}{6\eta(\nu,\tau)}\Omega\left(\frac{2\tau+\eta(\nu,\tau)}{2}\right) - \int_{\frac{1}{2}}^{1}\frac{\Omega(\tau+x(\eta(\nu,\tau)))}{\eta(\nu,\tau)}dx.
\end{aligned}
$$

By adding both equalities I_1 and I_2, then multiplying by $\eta(\nu,\tau)$, we obtain

$$
\begin{aligned}
&\eta(\nu,\tau)[I_1+I_2] \\
&= \frac{1}{3}\left[\Omega(\tau)+\Omega(\tau+\eta(\nu,\tau))+\Omega\left[\frac{(2\tau+\eta(\nu,\tau))}{2}\right]\right] - \frac{1}{\eta(\nu,\tau)}\int_{\tau}^{\tau+\eta(\nu,\tau)}\Omega(x)dx.
\end{aligned}
$$

This completes the proof. □

THEOREM 3.2. *Let Ω be an absolutely continuous function on an interval $[\tau,\tau+\eta(\nu,\tau)]$ and $\Omega'\in L_1[\tau,\tau+\eta(\nu,\tau)]$ be its derivative . Suppose also that $|\Omega'|^q$ is preinvex on $[\tau,\tau+\eta(\nu,\tau)]$ for some $q\geq 1$. Then the following result holds true:*

$$
\begin{aligned}
&\left|\frac{1}{3}\left[\Omega(\tau)+\Omega(\tau+\eta(\nu,\tau))+\Omega\left(\frac{2\tau+\eta(\nu,\tau)}{2}\right)\right] - \frac{1}{\eta(\nu,\tau)}\int_{\tau}^{\tau+\eta(\nu,\tau)}\Omega(x)\,dx\right| \\
&\leq |\eta(\nu,\tau)|\left(\frac{5}{72}\right)^{(1-\frac{1}{q})}\left\{\left(\frac{111\,|\Omega'(\tau)|^q}{1944}+\frac{|\Omega'(\nu)|^q}{81}\right)^{\frac{1}{q}}\right. \qquad (5)\\
&\qquad \left. + \left(\frac{|\Omega'(\tau)|^q}{81}+\frac{111\,|\Omega'(\nu)|^q}{1944}\right)^{\frac{1}{q}}\right\}. \qquad (6)
\end{aligned}
$$

Proof. From Lemma 3.1, we have

$$
\begin{aligned}
&\left|\frac{1}{3}\left[\Omega(\tau)+\Omega(\tau+\eta(\nu,\tau))+\Omega\left(\frac{2\tau+\eta(\nu,\tau)}{2}\right)\right] - \frac{1}{\eta(\nu,\tau)}\int_{\tau}^{\tau+\eta(\nu,\tau)}\Omega(x)\,dx\right| \\
&= |\eta(\nu,\tau)|\left|\int_0^{1/2}\left(x-\frac{1}{3}\right)\Omega'(\tau+x\eta(\nu,\tau))\,dx + \int_{1/2}^{1}\left(x-\frac{2}{3}\right)\Omega'(\tau+x\eta(\nu,\tau))\,dx\right| \qquad (7)\\
&\leq |\eta(\nu,\tau)|\left\{\int_0^{\frac{1}{2}}\left|x-\frac{1}{3}\right|\left|\Omega'(\tau+x\eta(\nu,\tau))\right|\,dx + \int_{\frac{1}{2}}^{1}\left|x-\frac{2}{3}\right|\left|\Omega'(\tau+x\eta(\nu,\tau))\right|\,dx\right\}.
\end{aligned}
$$

Firstly, we assume that q=1, and by using the fact that the function $|\Omega'|^q$ is preinvex on $[\tau, \tau+\eta(\nu,\tau)]$, we derive the following inequality:-

$$
\begin{aligned}
&\int_0^{\frac{1}{2}}\left|x-\frac{1}{3}\right||\Omega'(\tau+x\eta(\nu,\tau))|\,dx+\int_{\frac{1}{2}}^1\left|x-\frac{2}{3}\right||\Omega'(\tau+x\eta(\nu,\tau))|\,dx\\
&\le\int_0^{\frac{1}{2}}\left|x-\frac{1}{3}\right|[(1-x)|\Omega'(\tau)|+x|\Omega'(\nu)|]\,dx\\
&\quad+\int_{\frac{1}{2}}^1\left|x-\frac{2}{3}\right|[(1-x)|\Omega'(\tau)|+x|\Omega'(\nu)|]\,dx\\
&\le|\Omega'(\tau)|\left(\int_0^{\frac{1}{2}}(1-x)\left|x-\frac{1}{3}\right|dx+\int_{\frac{1}{2}}^1(1-x)\left|x-\frac{2}{3}\right|dx\right)\\
&\quad+|\Omega'(\nu)|\left(\int_0^{\frac{1}{2}}x\left|x-\frac{1}{3}\right|dx+\int_{\frac{1}{2}}^1x\left|x-\frac{2}{3}\right|dx\right)\\
&=\frac{5\left(|\Omega'(\tau)|+|\Omega'(\nu)|\right)}{72}. \qquad (8)
\end{aligned}
$$

As a result, the desired inequality asserted by Theorem 3.2 holds true when $q=1$. Let us we assume that $q>1$. In addition, we will use the Hölder integral inequality in the classical settings for L_p-L_q functions; About this inequality, see, e.g.,the monograph [27]. Thus, from the Hölder integral inequality with p=$\frac{q}{q-1}$, we obtain-

$$
\begin{aligned}
&\int_0^{\frac{1}{2}}\left|x-\frac{1}{3}\right||\Omega'(\tau+x\eta(\nu,\tau))|\,dx\\
&=\int_0^{\frac{1}{2}}\left|x-\frac{1}{3}\right|^{(1-\frac{1}{q})}\left(\left|x-\frac{1}{3}\right|^{\frac{1}{q}}\Omega'(\tau+x\eta(\nu,\tau))\right)dx\\
&\le\left(\int_0^{\frac{1}{2}}\left|x-\frac{1}{3}\right|\right)^{(1-\frac{1}{q})}\left(\int_0^{\frac{1}{2}}\left|x-\frac{1}{3}\right|\Omega'(\tau+x\eta(\nu,\tau))\,dx\right)^{\frac{1}{q}}\\
&\le\left(\frac{5}{72}\right)^{(1-\frac{1}{q})}\left(|\Omega'(\tau)|^q\int_0^{\frac{1}{2}}(1-x)\left|x-\frac{1}{3}\right|dx+|\Omega'(\nu)|^q\int_0^{\frac{1}{2}}x\left|x-\frac{1}{3}\right|dx\right)^{\frac{1}{q}}\\
&\le\left(\frac{5}{72}\right)^{(1-\frac{1}{q})}\left(\frac{111\,|\Omega'(\tau)|^q}{1944}+\frac{|\Omega'(\nu)|^q}{81}\right)^{\frac{1}{q}}. \qquad (9)
\end{aligned}
$$

similarly, we find that

$$
\int_{\frac{1}{2}}^1\left|x-\frac{2}{3}\right||\Omega'(\tau+x\eta(\nu,\tau))|\,dx\le\left(\frac{5}{72}\right)^{(1-\frac{1}{q})}\left(\frac{|\Omega'(\tau)|^q}{81}+\frac{111\,|\Omega'(\nu)|^q}{1944}\right)^{\frac{1}{q}}. \qquad (10)
$$

From (7), (9) and (10), we obtain the result (5) asserted by Theorem 3.2. □

Remark 3.3. If we choose $\eta(\nu,\tau)=\nu-\tau$, Theorem 3.2 reduces to [[12]: Theorem 3].

THEOREM 3.4. *Let Ω be an absolutely continuous function on an interval $[\tau, \tau + \eta(\nu, \tau)]$ and let its derivative $\Omega' \in L_1[\tau, \tau + \eta(\nu, \tau)]$. Suppose also that $|\Omega'|^q$ is preinvex on $[\tau, \tau + \eta(\nu, \tau)]$ for some $q > 1$. Then the following result holds true:*

$$\left| \frac{1}{3} \left[\Omega(\tau) + \Omega(\tau + \eta(\nu, \tau)) + \Omega\left(\frac{2\tau + \eta(\nu, \tau)}{2} \right) \right] - \frac{1}{\eta(\nu, \tau)} \int_{\tau}^{\tau + \eta(\nu, \tau)} \Omega(x)\, dx \right|$$

$$\leq \frac{|\eta(\nu, \tau)|}{12} \left(\frac{1 + 2^{(p+1)}}{3(p+1)} \right)^{\frac{1}{q}} \left[|\Omega'(\tau)|^q + |\Omega'(\nu)|^q \right]^{\frac{1}{q}}. \tag{11}$$

with $\frac{1}{p} + \frac{1}{q} = 1$.

Proof. As the function $|\Omega'|^q$ is convex on $[\tau, \tau + \eta(\nu, \tau)]$, we have-

$$\int_0^{\frac{1}{2}} |\Omega'(\tau + x\eta(\nu, \tau))|^q \, dx \leq \frac{3\,|\Omega'(\tau)|^q + |\Omega'(\nu)|^q}{8},$$

and

$$\int_{\frac{1}{2}}^{1} |\Omega'(\tau + x\eta(\nu, \tau))|^q \, dx \leq \frac{|\Omega'(\tau)|^q + 3\,|\Omega'(\nu)|^q}{8}.$$

straightforward calculation yields

$$\int_0^{\frac{1}{2}} \left| x - \frac{1}{3} \right|^p dx = \int_{\frac{1}{2}}^{1} \left| x - \frac{2}{3} \right|^p dx = \frac{1 + 2^{(p+1)}}{6^{(p+1)}(p+1)}. \tag{12}$$

Thus, by applying the Hölder integral inequality, we obtain

$$\int_0^{\frac{1}{2}} \left| x - \frac{1}{3} \right| |\Omega'(\tau + x\eta(\nu, \tau))| \, dx$$

$$\leq \left(\int_0^{\frac{1}{2}} \left| x - \frac{1}{3} \right|^p dx \right)^{\frac{1}{p}} \left(\int_{\frac{1}{2}}^{1} |\Omega'(\tau + x\eta(\nu, \tau))|^q \, dx \right)^{\frac{1}{q}} \tag{13}$$

$$\leq \left(\frac{1 + 2^{(p+1)}}{6^{(p+1)}(p+1)} \right)^{\frac{1}{p}} \left(\frac{3\,|\Omega'(\tau)|^q + |\Omega'(\nu)|^q}{8} \right)^{\frac{1}{q}}, \tag{14}$$

and

$$\int_{\frac{1}{2}}^{1} \left| x - \frac{2}{3} \right| |\Omega'(\tau + x\eta(\nu, \tau))| \, dx$$

$$\leq \left(\int_{\frac{1}{2}}^{1} \left| x - \frac{2}{3} \right|^p dx \right)^{\frac{1}{p}} \left(\int_{1}^{\frac{1}{2}} |\Omega'(\tau + x\eta(\nu, \tau))|^q \, dx \right)^{\frac{1}{q}} \tag{15}$$

$$\leq \left(\frac{1 + 2^{(p+1)}}{6^{(p+1)}(p+1)} \right)^{\frac{1}{p}} \left(\frac{|\Omega'(\tau)|^q + 3\,|\Omega'(\nu)|^q}{8} \right)^{\frac{1}{q}}. \tag{16}$$

Finally, by combining (7), (13), and (15) and making some basic simplifications, the asserted result (11) is obtained. □

REMARK 3.5. If we choose $\eta(\nu, \tau) = \nu - \tau$, Theorem 3.4 reduces to [[12]: Theorem 4].

4. Applications to Special Means

Our aim in this section is to derive some new inequalities involving combinations of special means and its powers. In fact, the logarithmic mean and a generalized logarithmic mean are special cases of the means introduced by Tibor Radó and also establish several important inequalities for them see, for details, [27]. The study of different types of means is fulfilled see, [30–33].

We consider some means for arbitrary positive real numbers s, t with $s < t$.

1. The arithmetic mean:

$M(s,t) = \frac{s+t}{2}$.

2. The logarithmic mean:

$L(s,t) = \frac{t-s}{log(t)-log(s)}, \quad s \neq t$.

3. The generalized logarithmic mean:

$L_r(s,t) = \left(\frac{(t)^{r+1}-(s)^{r+1}}{(r+1)(t-s)}\right)^{\frac{1}{r}}, r \in \mathbb{R}\backslash\{-1,0\}; s \neq t$.

Proposition 4.1. *Let $\tau, \tau+\eta(\nu,\tau) \in \mathbb{R}$, $0 < \tau < \tau+\eta(\nu,\tau)$. Then the following inequality holds true:*

$$\begin{aligned} &\left|\frac{2}{3}M\left((\tau)^r,(\tau+\eta(\nu,\tau))^r\right)+\frac{1}{3}M^r(\tau,\tau+\eta(\nu,\tau))-L_r^r(\tau,\tau+\eta(\nu,\tau))\right| \\ &\leq r\left|\eta(\nu,\tau)\right|\left(\frac{5}{72}\right)^{1-\frac{1}{q}}\left\{\left(\frac{111\tau^{q(r-1)}}{1944}+\frac{\nu^{q(r-1)}}{81}\right)^{\frac{1}{q}}\right. \\ &\quad \left.+\left(\frac{\tau^{q(r-1)}}{81}+\frac{111\nu^{q(r-1)}}{1944}\right)^{\frac{1}{q}}\right\} \end{aligned} \tag{17}$$

for all $q \geq 1$.

Proof. Proposition 4.1 follows from Theorem 3.2 upon setting $q = 1$, $\Omega(x) = x^r$. □

Proposition 4.2. *Under the assumption of Proposition 1, the following inequalities hold true:*

$$\begin{aligned} &\left|\frac{2}{3}M\left((\tau)^r,(\tau+\eta(\nu,\tau))^r\right)+\frac{1}{3}M^r(\tau,\tau+\eta(\nu,\tau))-L_r^r(\tau,\tau+\eta(\nu,\tau))\right| \\ &\leq \frac{r\left|\eta(\nu,\tau)\right|}{6}\left(\frac{1+2^{p+1}}{6(p+1)}\right)^{\frac{1}{p}}M^{\frac{1}{q}}(\tau^{q(r-1)},\nu^{q(r-1)}), \end{aligned} \tag{18}$$

where

$$\frac{1}{p}+\frac{1}{q}=1,\ p>1.$$

Proof. Proposition 4.2 follows from Theorem 3.4 by putting $\Omega(x) = x^r, r \in (0,1]$. □

5. Conclusion and Future Directions

In this paper, we have established new Hermite-Hadamard-type inequalities of function for differentiable preinvex functions. As Consequences of some of our main results, we have obtained some applications to special means of real numbers. These results can be viewed as refinement and significant improvements of the previously known results. The results obtained in this paper can be extended to interval-valued functions and the corresponding differential equations and optimization problems. It is expected that the ideas and techniques of this paper may motivate further research in field of inequality.

Declarations

Authors' contributions. All authors contributed equally to this work.

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