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T_D-SPACES IN GENERALIZED TOPOLOGICAL SPACES

Chul Kang

Abstract. In this paper, we introduce the concept of a generalized derived set in generalized topological spaces and we investigate its properties. Using these, we study T_D-spaces in generalized topological spaces.

1. Introduction

Császár([1]) introduced the notion of generalized topological spaces. He also introduced the notions of continuous functions and associated interior and closure operators on generalized topological spaces. We recall some notions and notations defined in ([1]). Let X be a nonempty set and τ a collection of subsets of X. Then τ is called *a generalized topology (simply GT)* on X if and only if $\emptyset \in \tau$ and $G_i \in \tau$ for $i \in I$ implies $\cup_{i \in I} G_i \in \tau$. We call the pair (X, τ) *a generalized topological space(simply GTS)* on X. The elements of τ are called *τ-open sets* and the complements are called *τ-closed sets.* The generalized-closure of a subset A of X, denoted by $c_\tau(A)$, is the intersection of generalized closed sets including A.

In this paper, we give a generalization of the concept of a derived set in generalized topological spaces and investigate its properties. Further, we investigate T_D-spaces in generalized topological spaces and prove that two GTSs (X, τ) and (Y, μ) are homeomorphic if and only if the set of τ-closed sets is meet isomorphic to the set of μ-closed sets.

2. Accumulations and envolutions

We recall that a mapping $\lambda : \mathcal{P}(X) \longrightarrow \mathcal{P}(X)$ is *an envelope operation* (or briefly *an envelope*) on a set X if

(1.1) $A \subseteq \lambda A$ for $A \subseteq X$,

(1.2) $A \subseteq B$ implies $\lambda A \subseteq \lambda B$ for $A \subseteq X$ and $B \subseteq X$, and

(1.3) $\lambda\lambda A = \lambda A$ for $A \subseteq X$.

(We write λA for $\lambda(A)$.) More generally, $\lambda : \mathcal{P}(X) \longrightarrow \mathcal{P}(X)$ is said to be a *weak envelope* on X if λ satisfies (1.1) and (1.2).

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REMARK 2.1 ([1]). Let $\lambda : \mathcal{P}(X) \longrightarrow \mathcal{P}(X)$ be an envelope. Then $F \subseteq X$ is called λ-*closed* if $\lambda F = F$ and a subset of X is said to be λ-*open* if its complement is λ-closed. Let $\mathcal{F}^\lambda$ be the set of all λ-closed sets. Then we have the following :
(i) for any $A \subseteq X$, $\lambda A = \cap\{F \in \mathcal{F}^\lambda \mid A \subseteq F\}$,
(ii) the set τ^λ of all λ-open sets is a GT on X, and
(iii) λA is τ^λ-closed.

For example, the closure operation in a topological space is an envelope([1]).
We introduce some generalized concept of drived sets in topological spaces.

DEFINITION 2.2. A mapping $\delta : \mathcal{P}(X) \longrightarrow \mathcal{P}(X)$ is an *accumulation* on a set X if
(a1) for any $A \subseteq X$, $\delta(A \cup \delta A) \subseteq A \cup \delta A$ and
(a2) $A \subseteq B \subseteq X$ implies $\delta A \subseteq \delta B$.

PROPOSITION 2.3. *Let $\delta : \mathcal{P}(X) \longrightarrow \mathcal{P}(X)$ be an accumulation on a set X. Then the mapping $\lambda_\delta : \mathcal{P}(X) \longrightarrow \mathcal{P}(X)$, defined by $\lambda_\delta A = A \cup \delta A$, is an envelope.*

Proof. By the definition of λ_δ and (a2), (1.1) and (1.2) hold. Let $A \subseteq X$. By (a1),

$$\lambda_\delta A \subseteq \lambda_\delta\lambda_\delta A = \lambda_\delta A \cup \delta\lambda_\delta A = A \cup \delta A \cup \delta(A \cup \delta A) = \lambda_\delta A.$$

Hence λ_δ satisfies (1.3) and thus λ_δ is an envelope. □

For any envelope $\lambda : \mathcal{P}(X) \longrightarrow \mathcal{P}(X)$ and $A \subseteq X$, let $\delta_\lambda A = \{x \in X \mid$ for any λ-open set G with $x \in G, (G - \{x\}) \cap A \neq \emptyset\}$.

PROPOSITION 2.4. *Let $\lambda : \mathcal{P}(X) \longrightarrow \mathcal{P}(X)$ be an envelope on X. Then we have the following :*
(1) *$x \in X - \delta_\lambda A$ if and only if there is a λ-closed set F such that $x \notin F$ and $A \subseteq F \cup \{x\}$,*
(2) *$y \in X - \delta_\lambda\{x\}$ if and only if there is a λ-closed set F such that $y \notin F$ and $\{x\} \subseteq F \cup \{y\}$,*
(3) *$\lambda A = A \cup \delta_\lambda A$, and*
(4) *δ_λ is an accumulation on X.*

Proof. (1) and (2) are trivial.
(3) Suppose that $x \in X - \lambda A$. Then by (i), there is a λ-closed set F such that $A \subseteq F$ and $x \notin F$. Hence $A \subseteq F \cup \{x\}$ and by (1), $x \notin \delta_\lambda A$. Thus $A \cup \delta_\lambda A \subseteq \lambda A$.
Suppose that $x \in X - (A \cup \delta_\lambda A)$. Since $x \notin \delta_\lambda A$, by (1), there is an $F \in \mathcal{F}^\lambda$ such that $x \notin F$, $A \subseteq F \cup \{x\}$ and since $x \notin A$, $A \subseteq F$. Hence $x \notin \lambda A$ and thus $\lambda A \subseteq A \cup \delta_\lambda A$.
(4) Clearly, δ_λ satiesfies (a2). Let $A \subseteq X$. By (1.3) and (3), $\lambda A = A \cup \delta_\lambda A$ is λ-closed, $\delta_\lambda(A \cup \delta_\lambda A) \subseteq \lambda(A \cup \delta_\lambda A) = A \cup \delta_\lambda A$. Thus δ_λ satiesfies (a1). □

A GTS (X, τ) is called *strong* if $X \in \tau$. Using the definition λ_δ and Proposition 2.4, we have the following corollary:

COROLLARY 2.5. *Let $\delta : \mathcal{P}(X) \longrightarrow \mathcal{P}(X)$ be an accumulation on X and $A \subseteq X$. Then the following are equivlalent :*
(1) *A is λ_δ-closed ,*
(2) *$\delta_{\lambda_\delta} A \subseteq A$, and*
(3) *$\delta A \subseteq A$.*

Using Proposition 2.4, we have the following proposition:

Proposition 2.6. *Let $\delta : \mathcal{P}(X) \longrightarrow \mathcal{P}(X)$ be an accumulation on X such that $x \notin \delta\{x\}$ for all $x \in X$. Then we have the following :*
(1) $\delta\emptyset = \emptyset$,
(2) $(X, \tau^{\lambda_\delta})$ *is a strong GTS,*
(3) $\delta_{\lambda_\delta}\{x\} = \delta\{x\}$ *for all* $x \in X$, *and*
(4) $\lambda_\delta\{z\} - \delta\{z\} = \lambda_\delta\{z\} - \delta_{\lambda_\delta}\{z\} = \{z\}$ *for all* $z \in X$.

Proof. (1) For any $x \in X$, $\emptyset \subseteq \{x\}$, by (a2), $\delta\emptyset \subseteq \delta\{x\}$ and since $x \notin \delta\{x\}$, $x \notin \delta\emptyset$. Hence we have the result.

(2) By (3) of Proposition 2.4 and (1), $\emptyset$ is λ_δ-closed and hence $(X, \tau^{\lambda_\delta})$ is strong.

(3) Let $x \in X$. Since $(X - \{x\}) \cap \{x\} = \emptyset$, by (2), $x \notin \delta_{\lambda_\delta}\{x\}$. By (3) of Proposition 2.4, $\lambda_\delta\{x\} = \{x\} \cup \delta_{\lambda_\delta}\{x\} = \{x\} \cup \delta\{x\}$ and since $x \notin \delta\{x\}$, $\delta_{\lambda_\delta}\{x\} \supseteq \delta\{x\}$. Similarly, since $x \notin \delta_{\lambda_\delta}\{x\}$, $\delta_{\lambda_\delta}\{x\} \subseteq \delta\{x\}$.

(4) Suppose that there is an $x \in \lambda_\delta\{z\} - \delta_{\lambda_\delta}\{z\}$ with $z \neq x$. Then there is a λ_δ-closed set F such that $x \notin F$ and $\{z\} \subseteq F \cup \{x\}$. Hence $\{z\} \subseteq F$ and so $\lambda_\delta\{z\} \subseteq F$. Since $x \notin F$, $x \notin \lambda_\delta\{z\}$ which is a contradiction. Since $z \notin \delta\{z\}$, by (3), one has the results. □

Example 2.7. Let $X = \{a, b, c\}$ and define a mapping $\delta : \mathcal{P}(X) \longrightarrow \mathcal{P}(X)$ by

$$\delta\emptyset = \delta\{b\} = \emptyset,\ \delta A = \{a\}\ if\ A \neq \emptyset\ and\ A \neq \{b\}.$$

Then δ is an accumulation on X,

$$\lambda_\delta\emptyset = \emptyset,\ \lambda_\delta\{a\} = \{a\},\ \lambda_\delta\{b\} = \{b\},\ \lambda_\delta\{c\} = \lambda_\delta\{a, c\} = \{a, c\},$$
$$\lambda_\delta\{a, b\} = \{a, b\},\ \lambda_\delta\{b, c\} = \lambda_\delta X = X,$$

and

$$\mathcal{F}^{\lambda_\delta} = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}, \{a, c\}\}.$$

Hence we have

$$\delta_{\lambda_\delta}\emptyset = \delta_{\lambda_\delta}\{a\} = \delta_{\lambda_\delta}\{b\} = \delta_{\lambda_\delta}\{a, b\} = \emptyset$$
$$\delta_{\lambda_\delta}\{c\} = \delta_{\lambda_\delta}\{a, c\} = \delta_{\lambda_\delta}\{b, c\} = \delta_{\lambda_\delta}X = \{a\}$$

and so (3) of Proposition 2.6 dose not hold.

3. T_D-space in GTS

In this section, we consider properties of T_D-spaces in GTS.

We recall that a strong GTS (X, τ) is called *a T_D-space* if for any $x \in X$, there are τ-open set G and τ-closed set F such that $\{x\} = G \cap F$([4]).

By ([7]), for any topological space X, the following statements are equivalent :

(i) X is a T_D-space,

(ii) for any $x \in X$, the drived set $\{x\}'$ of $\{x\}$ in X is closed in X, and

(iii) for any $x \in X$, there are an open set G and a closed set F in X such that $\{x\} = F \cap G$.

Theorem 3.1. *Let $\delta : \mathcal{P}(X) \longrightarrow \mathcal{P}(X)$ be an accumulation on X such that for any $x \in X$, $x \notin \delta\{x\}$. Then the following are equivalent*
(1) $(X, \tau_{\lambda_\delta})$ *is* T_D,
(2) *for any* $x \in X$, $\delta\{x\}$ *is* λ_δ*-closed, and*
(3) *for any* $x \in X$, $\delta_{\tau_{\lambda_\delta}}\{x\}$ *is a* τ_{λ_δ}*-closed set.*

Proof. By Proposition 2.6, (2) and (3) are equivalent.

(1) $\Rightarrow$ (3) Let $x \in X$. By (1), there are λ_δ-open set G and λ_δ-closed set F such that $\{x\} = F \cap G$. Now, we claim that $\delta_{\lambda_\delta}\delta_{\lambda_\delta}\{x\} \subseteq \delta_{\lambda_\delta}\{x\}$. Let $y \notin \delta_{\lambda_\delta}\{x\}$. Then there is a λ_δ-closed set H such that $y \notin H$ and $\{x\} \subseteq H \cup \{y\}$.

case 1 $y \neq x$

Then $\{x\} \subseteq H$, and since H is λ_δ-closed, $\delta_{\lambda_\delta}\delta_{\lambda_\delta}\{x\} \subseteq \delta_{\lambda_\delta}\delta_{\lambda_\delta}H \subseteq H$. Since $y \notin H$, $y \notin \delta_{\lambda_\delta}\delta_{\lambda_\delta}\{x\}$.

case 2 $y = x$

Since $x \notin \delta_{\lambda_\delta}\{x\}$, $F \cap G \cap \delta_{\lambda_\delta}\{x\} = \emptyset$, and since $\delta_{\lambda_\delta}\{x\} \subseteq F$, $G \cap \delta_{\lambda_\delta}\{x\} = \emptyset$. Since $x \in G$, $x \notin \delta_{\lambda_\delta}\delta_{\lambda_\delta}\{x\}$.

Hence $\delta_{\lambda_\delta}\delta_{\lambda_\delta}\{x\} \subseteq \delta_{\lambda_\delta}\{x\}$ and so $\delta_{\lambda_\delta}\{x\}$ is λ_δ-closed.

(3) $\Rightarrow$ (1) Let $x \in X$. Since $x \notin \delta\{x\}$ and $\delta\{x\}$ is λ_δ-closed, there is a λ_δ-open set G such that $x \in G$ and $G \cap \delta_{\lambda_\delta}\{x\} = \emptyset$. Then $G \cap [\{x\} \cup \delta_{\lambda_\delta}\{x\}] = G \cap \lambda_\delta\{x\} = \{x\}$ and hence (X, λ_δ) is T_D. □

Let (X, τ) be a GTS and $\mathcal{F}^\tau = \{F \subseteq X \mid X - F \in \tau\}$. Define λ^τ and δ^τ on $\mathcal{P}(X)$ as follows :

$$\lambda^\tau A = \cap\{F \in \mathcal{F}^\tau \mid A \subseteq F\}, \;\; \delta^\tau = \delta_{\lambda^\tau}.$$

Then λ^τ is an envelope on X, δ^τ is an accumulation on X, and for $A \subseteq X$, $c_\tau(A) = \lambda^\tau(A)$.

Sarsak [4] studied the separation axioms $\tau - T_0$ and $\tau - T_1$ in GTS, which appear in more general forms in [1].

Let (X, τ) be a GTS, let $M_\tau = \cup\{G \mid G \in \tau\}$. Then a GTS (X, τ) is called (i) $\tau - T_0$ ([6]) if for any pair of distinct points $x, y \in M_\tau$, there exists τ-open set containing precisely one of x and y and

(ii) τ-T_1 ([3]) if $x, y \in M_\tau$ and $x \neq y$ implies the existence of τ-open sets U and V such that $x \in U$, $y \notin U$ and $y \in V$, $x \notin V$.

Theorem 3.2 ([6]). *A GTS (X, τ) is $\tau - T_0$ if, for any pair of distinct points $x, y \in M_\tau$, $c_\tau\{x\} \neq c_\tau\{y\}$.*

The proof of Theorem 3.2 is obvious ([6]).

Proposition 3.3. *Every T_D-GTS is T_0.*

Proof. Let (X, τ) be a T_D-GTS and $x, y \in X$ with $x \neq y$. Suppose that $\lambda^\tau\{x\} = \lambda^\tau\{y\}$. Then $x \in \lambda^\tau\{y\}$ and $y \in \lambda^\tau\{x\}$. Since (X, τ) is T_D, there are a τ-open set G and τ-closed set F such that $\{x\} = G \cap F$ and hence $y \notin G \cap F$. If $y \notin G$, then $x \notin \lambda^\tau\{y\}$ which is a contradiction. Hence $y \in G$ and so $y \notin F$. By the definition of λ^τ, $y \notin \lambda^\tau\{x\}$ which is a contradiction. □

Let (X, τ) be a GTS. Then $(\mathcal{F}^\tau, \cap)$ is a complete lower semilattice, that is, for any $\mathcal{F}' \subseteq \mathcal{F}^\tau$, $\cap\mathcal{F}' \in \mathcal{F}^\tau$. Let $(L, \wedge)$ and $(M, \wedge')$ be lower semilattices and $h : L \longrightarrow M$ a mapping. Then h is called *a meet homomorphism* if, for any $a, b \in L$, $h(a \wedge b) = h(a) \wedge' h(b)$ and h is called *a meet isomomorphism* if it is 1-1, onto and h is a meet homomorphism.

Let (X, τ) and (Y, μ) be GTS and $f : X \longrightarrow Y$ a mapping. Then f is called (τ, η)-*continuous* if for any $G \in \mu$, $f^{-1}(G) \in \tau$ and f is a homeomorphism if f is one-to-one, onto and (τ, η)-continuous and f^{-1} is (η, τ)-continuous.

Lemma 3.4. *Let (X, τ) be a GTS. Then there is no λ-closed C such that $\delta_{\lambda^\tau}\{z\} \subset C \subset \lambda^\tau\{z\}$, where $A \subset B$ means that A is a proper subset of B .*

Proof. Suppose that there is a τ-closed set C such that $\delta_{\lambda^\tau}\{z\} \subset C \subset \lambda^\tau\{z\}$. If $z \in C$, then $\lambda^\tau\{z\} = C$ which is a contradiction. Hence $z \notin C$ and, since $C \subset \lambda^\tau\{z\} = \{z\} \cup \delta_{\lambda^\tau}\{z\}$, $C = \delta_{\lambda^\tau}\{z\}$. This is a contradiction. □

THEOREM 3.5. *Let $\lambda : \mathcal{P}(X) \longrightarrow \mathcal{P}(X)$ and $\eta : \mathcal{P}(Y) \longrightarrow \mathcal{P}(Y)$ be envelopes such that $\lambda\emptyset = \emptyset = \eta\emptyset$ and (X, λ), (Y, η) are both T_D. Then (X, λ), (Y, η) are homeomorphic if and only if there is a meet isomorphim $\phi : \mathcal{F}^\lambda \longrightarrow \mathcal{F}^\eta$.*

Proof. ($\Rightarrow$) It is trivial.

($\Leftarrow$) Let $\phi : \mathcal{F}^\tau \longrightarrow \mathcal{F}^\eta$ be a meet isomorphism and $x \in X$.

Since (X, λ) is T_D, $\delta_\lambda\{x\}$ is λ-closed and so $\phi(\delta_\lambda\{x\})$ and $\phi(\lambda\{x\})$ are η-closed. Let $A = \phi(\delta_\lambda\{x\})$ and $B = \phi(\lambda\{x\})$. Since $\lambda\emptyset = \emptyset$, (X, λ) is strong and, so by Propoistion 2.6, $\{x\} = \lambda_{\delta_\lambda}\{x\} - \delta_\lambda\{x\}$. By Proposition 2.4 and Proposition 2.6,

$$\lambda_{\delta_\lambda}\{x\} = \{x\} \cup \delta_{\lambda_{\delta_\lambda}}\{x\} = \{x\} \cup \delta_\lambda\{x\} = \lambda\{x\}$$

and so $\{x\} = \lambda\{x\} - \delta_\lambda\{x\}$. Since ϕ is a meet isomorphism, $A \subset B$.

Now, we claim that $\mid A - B \mid= 1$. Suppsoe that there are $p, q \in B - A$ with $p \neq q$. Since (Y, η) is T_0, $q \notin \eta\{p\}$ or $p \notin \eta\{q\}$. We may assume that $q \notin \eta\{p\}$, that is, $\eta\{q\} \not\subseteq \eta\{p\}$. Since $\eta\{p\} \neq B$, $\phi^{-1}(\eta\{p\}) \subset \lambda\{x\}$. Let $y \in \phi^{-1}(\eta\{p\})$. Suppose that $y \notin \delta_\lambda\{x\}$. Since $\phi^{-1}(\eta\{p\}) \subset \lambda\{x\}$, by Propoistion 2.6, $y \in \lambda\{x\} - \delta_\lambda\{x\} = \{x\}$ and so $y = x$. Since $\lambda\{y\} \subseteq \phi^{-1}(\eta\{p\})$, $\phi^{-1}(\eta\{p\}) = \lambda\{x\}$ which is a contradiction. Hence $y \in \delta_\lambda\{x\}$ and so $\phi^{-1}(\eta\{p\}) \subseteq \delta_\lambda\{x\}$. Hence $\eta\{p\} \subseteq A$, which is a contradiction to $p \notin A$, and thus $\mid A - B \mid= 1$.

Let $\{y_x\} = B - A$. Since B is η-closed, $\eta\{y_x\} \subseteq B$ and $\phi^{-1}(\eta\{y_x\}) \subseteq \lambda\{x\}$. Suppose that $\phi^{-1}(\eta\{y_x\}) \subset \lambda\{x\}$. If $x \in \phi^{-1}(\eta\{y_x\})$, then $\lambda\{x\} = \phi^{-1}(\eta\{y_x\})$, which is a contradiction. Hence $x \notin \phi^{-1}(\eta\{y_x\})$ and, since $\phi^{-1}(\eta\{y_x\}) \subseteq \lambda\{x\}$, $\phi^{-1}(\eta\{y_x\}) \subseteq \delta_\lambda\{x\}$. Thus $\eta\{y_x\} \subseteq A$, which is a contradiction. Moreover, $\phi^{-1}(\eta\{y_x\}) = \lambda\{x\}$ and $\eta\{y_x\} = \phi(\lambda\{x\})$. Similarly, for any $y \in Y$, there is an x_y in X such that $\lambda\{x_y\} = \phi^{-1}(\eta\{y\})$.

Define a map $f : (X, \lambda) \longrightarrow (Y, \eta)$ by $f(x) = y_x$. Suppsoe that $f(a) = f(b)$. Then $\phi(\lambda\{a\}) = \eta\{f(a)\} = \eta\{f(b)\} = \phi(\lambda\{b\})$ and, since ϕ is one-to-one, $\lambda\{a\} = \lambda\{b\}$. Since (X, λ) is T_0, f is ono-to-one. Let $y \in Y$. Then $\phi^{-1}(\eta\{y\}) = \lambda\{x_y\}$, and so $\eta\{y\} = \phi(\lambda\{x_y\})$. By the definition of f, $f(x_y) = y$ and hence f is onto.

Let C be a λ-closed set and $x \in C$. Then $\lambda\{x\} \subseteq C$ and $\phi(\lambda\{x\}) = \eta\{f(x)\} \subseteq \phi(C)$. Hence $f(C) \subseteq \phi(C)$. Let $y \in \phi(C)$. Then $\eta\{y\} \subseteq \phi(C)$ and so $\phi^{-1}(\eta\{y\}) = \lambda\{f^{-1}(y)\} \subseteq C$. Since $f^{-1}(y) \in C$, $y \in f(C)$. Hence $\phi(C) \subseteq f(C)$ and so $\phi(C) = f(C)$. Thus f is a (λ, η)-closed map. Similarly, f^{-1} is a (η, λ)-closed map and thus f is a (λ, η)-homeomorphism. □

Csaszar [2] introduced the product of GTS as follows : Let $\{(X_i, \tau_i) \mid i \in I\}$ be a family of GTS, $\prod_{i\in I} X_i$ the Cartesian product, and $p_i : \prod_{i\in I} X_i \longrightarrow X_i$ the i-th projection. Let $\mathcal{B}$ denote the set of all sets of the form $\prod_{i\in I} G_i$, where $G_i \in \tau_i$ and, with the exception of a finite number of indices i, $G_i = M_{\tau_i}$. Let $\tau(\mathcal{B}) = \{\cup B' \mid \mathcal{B}' \subseteq \mathcal{B}\}$. Then $\tau(\mathcal{B})$ is a GT on $\prod_{i\in I} X_i$. We call $\tau(\mathcal{B})$ the *generalized product topology* of $\{\tau_i \mid i \in I\}$ and denote it by $\prod_{i\in I} \tau_i = \tau(\mathcal{B})$. Further, $(\prod_{i\in I} X_i, \prod_{i\in I} \tau_i)$ is called *the generalized product topological space* of $\{(X_i, \tau_i) \mid i \in I\}$ and denoted by $\prod_{i\in I} X_i$.

THEOREM 3.6 ([2]). *Let* $\{(X_i, \tau_i) \mid i \in I\}$ *be a family of GTS and* $A_i \subseteq X_i$. *Then* $c_\tau\Big(\prod_{i\in I} A_i\Big) = \prod_{i\in I} c_{\tau_i}(A_i)$, *where* $c_\tau\Big(\prod_{i\in I} A_i\Big)$ *is the closure of* $\prod_{i\in I} A_i$ *in* $\Big(\prod_{i\in I} X_i, \prod_{i\in I} \tau_i\Big)$ *and* $c_{\tau_i} A_i$ *is the* c_{τ_i}*-closure of* A_i *in* (X_i, τ_i).

THEOREM 3.7. *Let* (X, λ) *and* (Y, η) *be strong GTS. Then* $(X \times Y, \tau^\lambda \times \tau^\eta)$ *is* T_D *if and only if* (X, λ) *and* (Y, η) *are both* T_D.

Proof. $(\Rightarrow)$. Let $x \in X$. Since $(X \times Y, \tau^\lambda \times \tau^\eta)$ is T_D, $\delta^{\tau^\lambda\times\tau^\eta}\{(x, y)\}$ is $\lambda \times \eta$-closed for some $y \in Y$.

Now, we claim that $\delta^{\tau^\lambda\times\tau^\eta}\{(x, y)\} = \delta_\lambda\{x\} \times \delta_\eta\{y\}$. Let $A = \delta^{\tau^\lambda\times\tau^\eta}\{(x, y)\}$, $B = \delta_\lambda\{x\}$, and $C = \delta_\eta\{y\}$. Let $(p, q) \notin B \times C$. Then $p \notin B$ or $q \notin C$. We may assume that $p \notin B$. Then there is a λ-open set G such that $p \in G$ and $x \notin G$. Since Y is strong, $[(G \times Y) - \{(p.q)\}] \cap \{(x, y)\} = \emptyset$ and so $(p, q) \notin A$. Hence $A \subseteq B \times C$.

Suppose that $(p, q) \in B \times C$. Let G be a λ-open set and H an η-open set such that $(p, q) \in G \times H$. Then $(G - \{p\}) \cap \{x\} \neq \emptyset$ and $(H - \{q\}) \cap \{y\} \neq \emptyset$. Hence $[(G \times H) - \{(p, q)\}] \cap \{(x, y)\} \neq \emptyset$. Thus $(p, q) \in A$ and so $B \times C \subseteq A$.

Since $(X \times Y, \tau^\lambda \times \tau^\eta)$ is T_D, $B \times C$ is $\lambda \times \eta$-closed and by Theorem 3.6, $B \times C = c_{\lambda\times\eta}(B \times C) = \lambda B \times \eta C$. Hence B is λ-closed and thus (X, λ) is T_D. Similarly, (Y, η) is T_D.

$(\Leftarrow)$ Let $(x, y) \in X \times Y$. Since (X, λ) and (Y, η) are T_D, there are λ-open set G , λ-closed set F , η-open ste H, and η-closed set K such that $\{x\} = G \cap F$ and $\{y\} = H \cap K$. Then $\{(x, y)\} = (G \cap F) \times (H \cap K) = (G \times H) \cap (F \times K)$. Since $G \times H$ is $\lambda \times \eta$-open and $F \times K$ is $\lambda \times \eta$-closed, $(X \times Y, \tau^\lambda \times \tau^\eta)$ is T_D. □

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Chul Kang
School of Computer Engineering & Applied Mathematics,
Hankyong National University, Anseong, 17579, Korea
E-mail: stat@hknu.ac.kr