Korean J. Math. **32** (2024), No. 2, pp. 329–348
https://dx.doi.org/10.11568/kjm.2024.32.2.329

SOME FIXED POINT RESULTS ON DOUBLE CONTROLLED CONE METRIC SPACES

A. Herminau Jothy*, P. S. Srinivasan, Laxmi Rathour, R. Theivaraman, and S. Thenmozhi

Abstract. In this text, we investigate some fixed point results in double-controlled cone metric spaces using several contraction mappings such as the B-contraction, the Hardy-Rogers contraction, and so on. Additionally, we prove the same fixed point results by using rational type contraction mappings, which were discussed by the authors Dass. B. K and Gupta. S. Also, a few examples are included to illustrate the results. Finally, we discuss some applications that support our main results in the field of applied mathematics.

1. Introduction

By using fixed point theory $(f.p.t)$, researchers from many fields have contributed to the progress of science and technology. Large scale problems requiring $f.p.t$ are highly esteemed for their lightning-fast solutions. As a result, in recent years, many scholars have focused on developing $f.p.t$ approaches and have provided various useful techniques for discovering $f.p$'s in complex issues[see [19], [23], [30] and [32]]. Authors have also proved interesting results on modified metric spaces such as partial metric and ordered metric, see [24] and [43] . These are currently crucial in many mathematics related areas and its applications, including economics, astronomy, dynamical systems, decision theory, and parameter estimation. The father of $f.p.t$, mathematician Brouwer [6], proposed $f.p$ theorems $(f.p.t's)$ for continuous mappings on finite dimensional spaces. In 1922, Banach [4] established and confirmed the renowned Banach contraction principle. Several authors used the Banach contraction principle in numerous ways and presented numerous fixed point results $(f.p.r's)$ [see, [5], [7], [8], [14], [15], [16], [17], [28] & [44]]. On the other hand, the concept of b-metric space was initiated by Bakhtin [3] in 1989, which was an interesting expansion of metric spaces. Recently, more and more extensions of b-metric spaces, such as $b_v(s)$-metric spaces and b-rectangular metric spaces, were introduced, and some $f.p.t's$ are shown on these spaces [refer, [23], [25], [29] & [31]]. Also, authors Mlaiki et al. [26] studied extension of the expanded b-metric spaces called controlled metric type spaces

Received September 21, 2023. Revised June 11, 2024. Accepted June 16, 2024.
2010 Mathematics Subject Classification: 47H10, 54H25.
Key words and phrases: fixed point, cone metric space, double controlled cone metric space, B-contraction, rational contraction mappings.

$*$ Corresponding author.

and proved some innovative $f.p.t's$. There are plenty of surveys available in the literature about b-metric spaces and their extended cone metric spaces, one can also see [37] and [39] which shows a very short survey on fixed point results in cone metric spaces obtained recently. See [12], [38] and [45] for more informations and results in cone metric space. Many results on approximating fixed point on suitable contractive conditions are proved in [33], [34], [35] and [42],. Subsequently, Abdeljawad et al. [2] proposed some $f.p.r's$ using contraction mappings including α-contraction [see, Theorem 1], Kannan contraction [see, Theorem 3] and their consequences in double controlled cone metric spaces [$DCCMS$]. In [36], which is also of recent, Shateri, T. L., proved many $f.p.r's$ on $DCCMS$. In particular, he showed the $f.p.r$ for Reich type contraction mapping (see, Theorem [2.5]). Researchers in the field of $f.p.t$ should consult manuscripts [11], [21], [22], [27], [40], [41] and the references therein for more relevant results on $DCCMS$.

On the other hand, the first researchers to investigate a generalization of the Banach $f.p$ theorem while simultaneously using a contraction condition of the rational type were Dass and Gupta [9]. Later, Jaggi [13], used a contraction condition of the rational type to prove a $f.p.t's$ in complete metric spaces. Moreover, rational contraction conditions have been heavily employed in both the $f.p$ and common $f.p$ locations. In this study, we first provide $f.p$ solutions utilising two contraction mappings, the B-contraction mapping [20] and the Hardy-Rogers [10] contraction mapping, and their associated consequents. Secondly, we demonstrate the $f.p.r's$ in the setting of rational contraction mappings, which were discussed mostly in [9] and [13].

The remaining parts of this manuscript are displayed as follows: In Section 2, some basic notions, such as notations, definitions and lemmas, are recalled from previous literature. In Section 3, we propose the main results of this work, where the existence and uniqueness of $f.p$'s are rigorously discussed in $DCCMS$. In Section 4, we extend these concepts to rational type contraction operators and prove some innovative $f.p.r's$. Finally, in Section 5, we present some conclusions.

2. Preliminaries

In this section, some notations and basic notions, such as definitions and lemmas, from earlier research are recalled. These are then employed throughout the remainder of the main findings of this manuscript.

Definition 2.1. [1] Let E be a real Banach space and P be a subset of E. P is called a **cone** if it satisfies the following conditions:
(C1) P is closed, non-empty and $P \neq \{0\}$;
(C2) $ax + by \in P$ for all $x, y \in P$ and non-negative real numbers a, b; and
(C3) $P \cap (-P) = \{0\}$.

Remark 2.2. [1] Consider a cone $P \subseteq E$. We can define a partial ordering $\preceq$ on E with respect to P by $x \preceq y$ if and only if $y - x \in P$. Here, $x \prec y$ indicate that $x \preceq y$ but $x \neq y$, while $x \ll$ stand for $y - x \in intP$, in which $intP \neq \emptyset$ and $\preceq$ is a partial ordering with respect to P.

Definition 2.3. [2] The cone P is called **normal** if

$$inf\{\|x + y\| : x, y \in P, \|x\| = \|y\| = 1\} > 0$$

or equivalently, there is a constant number $M > 0$ such that for all $x, y \in E$ where $0 \preceq x \preceq y$ implies $\|x\| \leq M\|y\|$. The least positive number satisfying above is called the normal constant of P.

DEFINITION 2.4. [2] Let T be a non-empty set. A mapping $d : T \times T \to E$ is said to be a **cone metric** (CM) on T if for all $p, q, z \in T$ the following hold:
(CM1) $0 \prec d(p,q)$ and $d(p,q) = 0$ if and only if $p = q$;
(CM2) $d(p,q) = d(q,p)$;
(CM3) $d(p,q) \preceq d(p,z) + d(z,q)$.
Then, the pair (T, d) is called **cone metric space** (CMS).

DEFINITION 2.5. [36] Let T be a non-empty set and let $\nu : T \times T \to [1, +\infty)$ be a mapping. A mapping $d : T \times T \to E$ is said to be a **controlled cone metric** (CCM) with ν if for all $p, q, z \in X$ the following hold:
(CCM1) $0 \prec d(p,q)$ and $d(p,q) = 0$ if and only if $p = q$;
(CCM2) $d(p,q) = d(q,p)$;
(CCM3) $d(p,q) \preceq \nu(p,z)d(p,z) + \nu(z,q)d(z,q)$.
Then the pair (T, d) is called **controlled cone metric space** $(CCMS)$. Note that each cone metric space is a controlled cone metric space with $\nu(p,q) = 1$.

DEFINITION 2.6. [36] Let T be a non-empty set and let $\nu, \mu : T \times T \to [1, +\infty)$ be non-comparable functions. A mapping $d : T \times T \to E$ is said to be a **double controlled cone metric** $(DCCM)$with respect to ν and μ if for all $p, q, z \in X$ the following hold:
(DCCM1) $0 \prec d(p,q)$ and $d(p,q) = 0$ if and only if $p = q$;
(DCCM2) $d(p,q) = d(q,p)$;
(DCCM3) $d(p,q) \preceq \nu(p,z)d(p,z) + \mu(z,q)d(z,q)$.
Then the pair (T, d) is called **double controlled cone metric space** $(DCCMS)$. It is clear that each $CCMS$ is a $DCCMS$. But there exists $DCCMS$ which are not $CCMS$.

EXAMPLE 2.7. Let $E = \mathbb{R}^2$, $P = \{(p,q) \in E : p, q \geq 0\}$, $T = [0, +\infty)$ and $d : T \times T \to E$ be defined by

$$d(p,q) = \begin{cases} (0,0) & \text{iff } p = q \\ (\frac{1}{p}, \frac{1}{3}) & \text{if } p \geq 1, q \in [0,1) \\ (\frac{1}{3}, \frac{1}{q}) & \text{if } q \geq 1, p \in [0,1) \\ (1,1) & \text{otherwise.} \end{cases}$$

Define $\nu, \mu : T \times T \to [1, +\infty)$ by:

$$\nu(p,q) = \begin{cases} p & \text{if } p, q \geq 1 \\ 1 & \text{otherwise.} \end{cases}$$

and

$$\nu(p,q) = \begin{cases} 1 & \text{if } p, q \leq 1 \\ max\{p,q\} & \text{otherwise.} \end{cases}$$

Then (T, d) is a double controlled cone metric space. Also, we have

$$d(0, \frac{1}{2}) = (1, 1) > (\frac{2}{3}, \frac{2}{3}) = \nu(0, 3)d(0, 3) + \nu(3, \frac{1}{2})d(3, \frac{1}{2}).$$

Which implies that d is not a controlled cone metric space when $\nu = \mu$.

DEFINITION 2.8. [36] Let (T, d) be a $DCCMS$ with respect to ν and μ.
(i) A sequence $\{p_n\}$ is **convergent** to some p in T, if for every $c \in E$ with $0 \ll c$ there is N such that for all $n > N$, $d(p_n, p) \ll c$, then $\{p_n\}$ is said to be convergent and $\{p_n\}$ converges to p, and p is the limit of $\{p_n\}$. It is written as $\lim_{n\to+\infty} p_n = p$.
(ii) A sequence $\{p_n\}$ is **Cauchy**, if for every $c \in E$ with $0 \ll c$ there is N such that for all $m, n > N, d(p_n, p) \ll c$.
(iii) (T, d) is said to be **complete** if every Cauchy sequence is convergent.

LEMMA 2.9. [36] *Let (T, d) be a $DCCMS$ with respect to ν and μ, P be a normal cone with normal constant M. Let $\{p_n\}$ be a sequence in T. Then $\{p_n\}$ converges to p if and only if $\lim_{n\to+\infty} d(p_n, p) = 0$.*

LEMMA 2.10. [36] *Let (T, d) be a $DCCMS$ with respect to ν and μ, P be a normal cone with normal constant M. Let $\{p_n\}$ be a sequence in T such that $\{p_n\}$ converges to p and q. If $\lim_{n\to+\infty} \nu(p_n, p)$ and $\lim_{n\to+\infty} \mu(p_n, q)$ exist and are finite, then $p = q$.*

3. Results for contraction mappings

In this section, we prove some $f.p.r's$ in $DCCMS$ using various contraction mappings. For that, assume (T, d) be a complete $DCCMS$ with respect to the functions $\nu, \mu : T \times T \to [1, +\infty)$ and P be a normal cone with normal constant M. Following is the B-contraction mapping considered here to prove our first $f.p$ theorem in $DCCMS$.

THEOREM 3.1. *Let $K : T \to T$ be a map satisfy the contraction condition*

$$d(Kp, Kq) \preceq rd(p, q) + s[d(p, Kp) + d(q, Kq)] + t[d(p, Kq) + d(q, Kp)] \tag{1}$$

for all $p, q \in T$, where $r, s, t \in (0, 1)$ with $r + 2s + 2t < 1$. For $p_0 \in T$, choose $p_n = K^n p_0$. Suppose that

$$\sup_{m\geq 1} \lim_{i\to+\infty} \frac{\nu(p_{i+1}, p_{i+2})}{\nu(p_i, p_{i+1})}\mu(p_{i+1}, p_m) < \frac{1 - s - t}{r + s + t}. \tag{2}$$

If for each $p \in T$,

$$\lim_{n\to+\infty} \nu(p, p_n) \text{ exists, is finite and } \lim_{n\to+\infty} \mu(p_n, p) < \frac{1}{s}. \tag{3}$$

then K has a $u.f.p$ in T.

Proof. Consider a sequence $\{p_n\}$ in T satisfies the hypothesis of the theorem. From (3.1), we obtain

$$
\begin{aligned}
&d(p_n, p_{n+1})\\
&\quad = d(Kp_{n-1}, Kp_n)\\
&\quad \preceq rd(p_{n-1}, p_n) + s[d(p_{n-1}, Kp_{n-1}) + d(p_n, Kp_n)] + t[d(p_{n-1}, Kp_n) + d(p_n, Kp_{n-1})]\\
&\quad \preceq rd(p_{n-1}, p_n) + s[d(p_{n-1}, p_n) + d(p_n, p_{n+1})] + t[d(p_{n-1}, p_{n+1}) + d(p_n, p_n)]\\
&\quad \preceq rd(p_{n-1}, p_n) + s[d(p_{n-1}, p_n) + d(p_n, p_{n+1})] + t[d(p_{n-1}, p_n) + d(p_n, p_{n+1})]\\
&\quad = \lambda d(p_{n-1}, p_n), \text{ where } \lambda = \frac{r+s+t}{1-s-t}\\
&\quad \preceq \lambda^2 d(p_{n-2}, p_{n-1})\\
&\quad \dots\\
&\quad \preceq \lambda^n d(p_0, p_1).
\end{aligned}
$$

That is, $d(p_n, p_{n+1}) = \lambda^n d(p_0, p_1)$, for all $n \geq 0$. Let m, n be integers such that $m > n$. Show that $\{p_n\}$ is a Cauchy sequence. Consider,

$$
\begin{aligned}
d(p_n, p_m) &\preceq \nu(p_n, p_{n+1})d(p_n, p_{n+1}) + \mu(p_{n+1}, p_m)d(p_{n+1}, p_m)\\
&\preceq \nu(p_n, p_{n+1})d(p_n, p_{n+1}) + \mu(p_{n+1}, p_m)\nu(p_{n+1}, p_{n+2})d(p_{n+1}, p_{n+2})\\
&\quad + \mu(p_{n+1}, p_m)\mu(p_{n+2}, p_m)d(p_{n+2}, p_m)\\
&\preceq \dots\\
&\preceq \nu(p_n, p_{n+1})d(p_n, p_{n+1}) + \sum_{i=n+1}^{m-2} (\prod_{j=n+1}^{i} \mu(p_j, p_m))\nu(p_i, p_{i+1})d(p_i, p_{i+1})\\
&\quad + \prod_{k=n+1}^{m-1} \mu(p_k, p_m)d(p_{m-1}, p_m)\\
&\preceq \nu(p_n, p_{n+1}) \left(\frac{r+s+t}{1-s-t}\right)^n d(p_0, p_1)\\
&\quad + \sum_{i=n+1}^{m-2} (\prod_{j=n+1}^{i} \mu(p_j, p_m))\nu(p_i, p_{i+1}) \left(\frac{r+s+t}{1-s-t}\right)^i d(p_0, p_1)\\
&\quad + \prod_{k=n+1}^{m-1} \mu(p_k, p_m) \left(\frac{r+s+t}{1-s-t}\right)^{m-1} d(p_0, p_1)\\
&\preceq \nu(p_n, p_{n+1}) \left(\frac{r+s+t}{1-s-t}\right)^n d(p_0, p_1)\\
&\quad + \sum_{i=n+1}^{m-1} (\prod_{j=n+1}^{i} \mu(p_j, p_m))\nu(p_i, p_{i+1}) \left(\frac{r+s+t}{1-s-t}\right)^i d(p_0, p_1)
\end{aligned}
$$

$$\preceq \nu(p_n, p_{n+1}) \left(\frac{r+s+t}{1-s-t}\right)^n d(p_0, p_1)$$
$$+ \sum_{i=n+1}^{m-1} (\prod_{j=0}^{i} \mu(p_j, p_m))\nu(p_i, p_{i+1}) \left(\frac{r+s+t}{1-s-t}\right)^i d(p_0, p_1).$$

This implies,

$$\|d(p_n, p_m)\| \leq M\Bigg\|\nu(p_n, p_{n+1}) \left(\frac{r+s+t}{1-s-t}\right)^n d(p_0, p_1)$$
$$+ \sum_{i=n+1}^{m-1} (\prod_{j=0}^{i} \mu(p_j, p_m))\nu(p_i, p_{i+1}) \left(\frac{r+s+t}{1-s-t}\right)^i d(p_0, p_1)\Bigg\|. \tag{4}$$

Choose $N_l = \sum_{i=0}^{l}(\prod_{j=0}^{l} \mu(p_j, p_m))\nu(p_i, p_{i+1}) \left(\frac{r+s+t}{1-s-t}\right)^i$, then we get

$$\|d(p_n, p_m)\| \leq M\Bigg\|d(p_0, p_1) \left[\left(\frac{r+s+t}{1-s-t}\right)^n \nu(p_n, p_{n+1}) + (N_{m-1} - N_n)\right]\Bigg\|.$$

Then (3.2) implies that the limit of the sequence $\{N_n\}$ exists and so $\{N_n\}$ is Cauchy. Letting $m, n \to +\infty$ in (3.3) gives $\lim_{m,n\to+\infty} d(p_n, p_m) = 0$, and so $\{p_n\}$ is a Cauchy sequence. By using the completeness of K, there exists $p \in K$ such that $\lim_{n\to+\infty} p_n = p$. We claim that $Kp = p$. It follows from (**DCCM3**) and (3.1) that

$$\begin{aligned}
0 &\prec d(p, Kp) \\
&\preceq \nu(p, p_{n+1})d(p, p_{n+1}) + \mu(p_{n+1}, Kp)d(p_{n+1}, Kp) \\
&\preceq \nu(p, p_{n+1})d(p, p_{n+1}) + \mu(p_{n+1}, Kp)d(Kp_n, Kp) \\
&\preceq \nu(p, p_{n+1})d(p, p_{n+1}) + \mu(p_{n+1}, Kp)[rd(p_n, p) + s[d(p_n, Kp_n) + d(p, Kp)] \\
&\quad + t[d(p_n, Kp) + d(p, Kp_n)]] \\
&\preceq \nu(p, p_{n+1})d(p, p_{n+1}) + \mu(p_{n+1}, Kp)[rd(p_n, p) + s[d(p_n, p_{n+1}) + d(p, Kp)] \\
&\quad + t[d(p_n, Kp) + d(p, p_{n+1})]]
\end{aligned}$$

and so

$$0 < \|d(p, Kp)\| \leq M\big\|\nu(p, p_{n+1})d(p, p_{n+1}) + \mu(p_{n+1}, Kp)[rd(p_n, p)$$
$$+ s[d(p_n, p_{n+1}) + d(p, Kp)] + t[d(p_n, Kp) + d(p, p_{n+1})]]\big\|. \tag{5}$$

Now, making use of the condition (3.3) and passing to the limit on (3.5) we get

$$0 < \|d(p, Kp)\| < b\|d(p, Kp)\|.$$

which is a contradiction, therefore $Kp = p$. Suppose that K has another $f.p.$(say, q), then

$$\begin{aligned}
d(p, q) &= d(Kp, Kq) \\
&\preceq rd(p, q) + s[d(p, Kp) + d(q, Kq)] + t[d(p, Kq) + d(q, Kp)] \\
&\preceq rd(p, q) + s[d(p, p) + d(q, q)] + t[d(p, q) + d(q, p)] \\
&\preceq (r + 2t)d(p, q).
\end{aligned}$$

But $r + 2t < 1$. Therefore, our supposition is wrong. Hence, K has a $u.f.p$ in T. □

EXAMPLE 3.2. Define a partial order relation $\preceq$ on E as follows:

$$p \preceq q \text{ if and only if } \|p\|_2 \leq \|q\|_2.$$

Let $E = l^2, X = l^2$ and $P = \{p \in E : p(i) \geq 0\}$, then P is a cone. Consider a map $d : X \times X \to E$ is defined by

$$d(p,q) = \begin{cases} z & \text{if } p \neq q \\ 0 & \text{if } p = q \end{cases}$$

where $z(i) = |p(i)| + |q(i)|$. Let $\nu, \mu : T \times T \to [1, \infty)$ be defined by $\nu(p,q) = 1$ and $\mu(p,q) = 1$, for all $(p,q) \in K \times K$. It can be easily verified that d is a $DCCM$. Define $K : l^2 \to l^2$ by $Kp = p/3$. Thus, $d(Kp, Kq) = d(p/3, q/3) = z/3$, where $z = |p(i)| + |q(i)|$. Choose $r = 1/2$; $s = t = 3/16$ then $r + 2(s+t) \leq 1$. Also,

$$\begin{aligned} rd(p,q) + s[d(p,Kp) + d(q,Kq)] + t[d(p,Kq) + d(q,Kp)] &= (r + \frac{4}{3}(s+t))|p(i)| + |q(i)| \\ &= |p(i)| + |q(i)| \\ &= z. \end{aligned}$$

Therefore, K is a B-contraction operator. Hence, K has a $f.p.$ More concretely, 0 is the only $f.p.$

Following is the Hardy-Rogers contraction mapping considered here to prove another $f.p$ theorem.

THEOREM 3.3. *Let $K : T \to T$ be a map satisfy the contraction condition*

$$d(Kp, Kq) \preceq rd(p,q) + sd(p,Kp) + td(q,Kq) + ud(p,Kq) + vd(q,Kp) \tag{6}$$

for all $p, q \in T$, where $r, s, t, u, v \in (0,1)$ with $r+s+t+u+v < 1$. For $p_0 \in T$, choose $p_n = K^n p_0$. Suppose that

$$\sup_{m \succeq 1} \lim_{i \to +\infty} \frac{\nu(p_{i+1}, p_{i+2})}{\nu(p_i, p_{i+1})} \mu(p_{i+1}, p_m) < \frac{1-t-u}{r+s+u}. \tag{7}$$

If for each $p \in T$,

$$\lim_{n \to +\infty} \nu(p, p_n) \text{ exists, is finite and } \lim_{n \to +\infty} \mu(p_n, p) < \frac{1}{t}. \tag{8}$$

Then K has a $u.f.p$ in T.

Proof. Consider a sequence $\{p_n\}$ in T satisfies the hypothesis of the theorem. From (3.6), we obtain

$$\begin{aligned}
&d(p_n, p_{n+1})\\
&\quad = d(Kp_{n-1}, Kp_n)\\
&\quad \preceq rd(p_{n-1}, p_n) + sd(p_{n-1}, Kp_{n-1}) + td(p_n, Kp_n) + ud(p_{n-1}, Kp_n) + vd(p_n, Kp_{n-1})\\
&\quad \preceq rd(p_{n-1}, p_n) + sd(p_{n-1}, p_n) + td(p_n, p_{n+1}) + ud(p_{n-1}, p_{n+1}) + vd(p_n, p_n)\\
&\quad \preceq rd(p_{n-1}, p_n) + sd(p_{n-1}, p_n) + td(p_n, p_{n+1}) + ud(p_{n-1}, p_n) + ud(p_n, p_{n+1})\\
&\quad = \lambda d(p_{n-1}, p_n), \text{ where } \lambda = \frac{r+s+u}{1-t-u}\\
&\quad \preceq \lambda^2 d(p_{n-2}, p_{n-1})\\
&\quad \dots\\
&\quad \preceq \lambda^n d(p_0, p_1).
\end{aligned}$$

That is, $d(p_n, p_{n+1}) = \lambda^n d(p_0, p_1)$, for all $n \geq 0$. Let m, n be integers such that $m > n$. Show that $\{p_n\}$ is a Cauchy sequence. Consider,

$$\begin{aligned}
d(p_n, p_m) &\preceq \nu(p_n, p_{n+1})d(p_n, p_{n+1}) + \mu(p_{n+1}, p_m)d(p_{n+1}, p_m)\\
&\preceq \nu(p_n, p_{n+1})d(p_n, p_{n+1}) + \mu(p_{n+1}, p_m)\nu(p_{n+1}, p_{n+2})d(p_{n+1}, p_{n+2})\\
&\quad + \mu(p_{n+1}, p_m)\mu(p_{n+2}, p_m)d(p_{n+2}, p_m)\\
&\preceq \dots\\
&\preceq \nu(p_n, p_{n+1})d(p_n, p_{n+1}) + \sum_{i=n+1}^{m-2}(\prod_{j=n+1}^{i}\mu(p_j, p_m))\nu(p_i, p_{i+1})d(p_i, p_{i+1})\\
&\quad + \prod_{k=n+1}^{m-1}\mu(p_k, p_m)d(p_{m-1}, p_m)
\end{aligned}$$

$$\begin{aligned}
&\preceq \nu(p_n, p_{n+1})\left(\frac{r+s+u}{1-t-u}\right)^n d(p_0, p_1)\\
&\quad + \sum_{i=n+1}^{m-2}(\prod_{j=n+1}^{i}\mu(p_j, p_m))\nu(p_i, p_{i+1})\left(\frac{r+s+u}{1-t-u}\right)^i d(p_0, p_1)\\
&\quad + \prod_{k=n+1}^{m-1}\mu(p_k, p_m)\left(\frac{r+s+u}{1-t-u}\right)^{m-1} d(p_0, p_1)\\
&\preceq \nu(p_n, p_{n+1})\left(\frac{r+s+u}{1-t-u}\right)^n d(p_0, p_1)\\
&\quad + \sum_{i=n+1}^{m-1}(\prod_{j=n+1}^{i}\mu(p_j, p_m))\nu(p_i, p_{i+1})\left(\frac{r+s+u}{1-t-u}\right)^i d(p_0, p_1)
\end{aligned}$$

$$\preceq \nu(p_n, p_{n+1}) \left(\frac{r+s+u}{1-t-u}\right)^n d(p_0, p_1)$$
$$+ \sum_{i=n+1}^{m-1} (\prod_{j=0}^{i} \mu(p_j, p_m))\nu(p_i, p_{i+1}) \left(\frac{r+s+u}{1-t-u}\right)^i d(p_0, p_1).$$

This implies,

$$\|d(p_n, p_m)\| \leq M \Bigg\| \nu(p_n, p_{n+1}) \left(\frac{r+s+u}{1-t-u}\right)^n d(p_0, p_1)$$
$$+ \sum_{i=n+1}^{m-1} (\prod_{j=0}^{i} \mu(p_j, p_m))\nu(p_i, p_{i+1}) \left(\frac{r+s+u}{1-t-u}\right)^i d(p_0, p_1) \Bigg\|. \tag{9}$$

Choose $N_l = \sum_{i=0}^{l} (\prod_{j=0}^{l} \mu(p_j, p_m))\nu(p_i, p_{i+1}) \left(\frac{r+s+u}{1-t-u}\right)^i$, then we get

$$\|d(p_n, p_m)\| \leq M \left\| d(p_0, p_1) \left[\left(\frac{r+s+u}{1-t-u}\right)^n \nu(p_n, p_{n+1}) + (N_{m-1} - N_n) \right] \right\|.$$

Then (3.2) implies that the limit of the sequence $\{N_n\}$ exists and so $\{N_n\}$ is Cauchy. Letting $m, n \to +\infty$ in (3.3) gives $\lim_{m,n\to+\infty} d(p_n, p_m) = 0$, and so $\{p_n\}$ is a Cauchy sequence. By using the completeness of K, there exists $p \in K$ such that $\lim_{n\to+\infty} p_n = p$. We claim that $Kp = p$. It follows from (**DCCM3**) and (3.6) that

$$\begin{aligned}
0 &\prec d(p, Kp) \\
&\preceq \nu(p, p_{n+1})d(p, p_{n+1}) + \mu(p_{n+1}, Kp)d(p_{n+1}, Kp) \\
&\preceq \nu(p, p_{n+1})d(p, p_{n+1}) + \mu(p_{n+1}, Kp)d(Kp_n, Kp) \\
&\preceq \nu(p, p_{n+1})d(p, p_{n+1}) + \mu(p_{n+1}, Kp)[rd(p_n, p) + sd(p_n, Kp_n) + td(p, Kp) \\
&\quad + ud(p_n, Kp) + vd(p, Kp_n)] \\
&\preceq \nu(p, p_{n+1})d(p, p_{n+1}) + \mu(p_{n+1}, Kp)[rd(p_n, p) + sd(p_n, p_{n+1}) + td(p, Kp) \\
&\quad + ud(p_n, Kp) + vd(p, p_{n+1})]
\end{aligned}$$

and so

$$\begin{aligned}
0 < \|d(p, Kp)\| &\leq M\|\nu(p, p_{n+1})d(p, p_{n+1}) + \mu(p_{n+1}, Kp)[rd(p_n, p) \\
&\quad + sd(p_n, p_{n+1}) + td(p, Kp) + ud(p_n, Kp) + vd(p, p_{n+1})\|.
\end{aligned} \tag{10}$$

Now, making use of the condition (3.3) and passing to the limit on (3.5), we get

$$0 < \|d(p, Kp)\| < (t+u)\|d(p, Kp)\|.$$

which is a contradiction, therefore $Kp = p$. Suppose that K has another $f.p$ (say, q), then

$$\begin{aligned}
d(p, q) &= d(Kp, Kq) \\
&\preceq rd(p, q) + sd(p, Kp) + td(q, Kq) + ud(p, Kq) + vd(q, Kp) \\
&\preceq rd(p, q) + sd(p, p) + td(q, q) + ud(p, q) + vd(q, p) \\
&\preceq (r + u + v)d(p, q).
\end{aligned}$$

But $r + u + v < 1$. Therefore, our supposition is wrong. Hence, K has a $u.f.p$ in T. $\square$

EXAMPLE 3.4. Example 3.2 satisfies the rational contraction equation (3.6) when $r = 1/5$ and $s = t = u = v = 3/20$. Therefore by Theorem 3.3, K has a $u.f.p.$ More concretely, 0 is the only $f.p.$

The following are some results that are derived from the above two theorems, which include the contraction, the Kannan contraction, the Chatterjee contraction, the Reich contraction mapping and so on.

COROLLARY 3.5. *Let $K : T \to T$ be a map satisfy the contraction condition*

$$d(Kp, Kq) \preceq rd(p, q) \tag{11}$$

for all $p, q \in T$, where $r \in (0, 1)$. For $p_0 \in T$, choose $p_n = K^n p_0$. Suppose that

$$\sup_{m\succeq 1} \lim_{i\to+\infty} \frac{\nu(p_{i+1}, p_{i+2})}{\nu(p_i, p_{i+1})} \mu(p_{i+1}, p_m) < \frac{1}{r}. \tag{12}$$

If for each $p \in T$, $\lim_{n\to+\infty} \nu(p, p_n)$ and $\lim_{n\to+\infty} \mu(p_n, p)$ exist and are finite, then K has a $u.f.p$ in T.

Proof. Substituting $s = t = u = v = 0$ in Theorem3.3 completes this corollary. □

COROLLARY 3.6. *Let $K : T \to T$ be a map satisfy the contraction condition*

$$d(Kp, Kq) \preceq rd(p, q) + sd(p, Kp) \tag{13}$$

for all $p, q \in T$, where $r, s \in (0, 1)$ with $r + s < 1$. For arbitrary $p_0 \in T$, choose $p_n = K^n p_0$. Suppose that

$$\sup_{m\succeq 1} \lim_{i\to+\infty} \frac{\nu(p_{i+1}, p_{i+2})}{\nu(p_i, p_{i+1})} \mu(p_{i+1}, p_m) < \frac{1-s}{r}. \tag{14}$$

If for each $p \in T$, $\lim_{n\to+\infty} \nu(p, p_n)$ is exists, finite and $\lim_{n\to+\infty} \mu(p_n, p) < \dfrac{1}{s}$ then K has a $u.f.p$ in T.

Proof. Substituting $t = u = v = 0$ in Theorem3.3 completes this corollary. □

COROLLARY 3.7. *Let $K : T \to T$ be a map satisfy the contraction condition*

$$d(Kp, Kq) \preceq rd(p, q) + sd(p, Kp) + td(q, Kq) \tag{15}$$

for all $p, q \in T$, where $r, s, t \in (0, 1)$ with $r + s + t < 1$. For arbitrary $p_0 \in T$, choose $p_n = K^n p_0$. Suppose that

$$\sup_{m\succeq 1} \lim_{i\to+\infty} \frac{\nu(p_{i+1}, p_{i+2})}{\nu(p_i, p_{i+1})} \mu(p_{i+1}, p_m) < \frac{1-t}{r+s}. \tag{16}$$

If for each $p \in T$, $\lim_{n\to+\infty} \nu(p, p_n)$ is exists, finite and $\lim_{n\to+\infty} \mu(p_n, p) < \dfrac{1}{t}$ then K has a $u.f.p$ in T.

Proof. Substituting $u = v = 0$ in Theorem3.3 completes this corollary. □

COROLLARY 3.8. *Let $K : T \to T$ be a map satisfy the contraction condition*

$$d(Kp, Kq) \preceq s[d(p, Kp) + d(q, Kq)] \tag{17}$$

for all $p, q \in T$, *where* $s \in (0,1)$ *with* $2s < 1$. *For arbitrary* $p_0 \in T$, *choose* $p_n = K^n p_0$. *Suppose that*

$$\sup_{m \succeq 1} \lim_{i \to +\infty} \frac{\nu(p_{i+1}, p_{i+2})}{\nu(p_i, p_{i+1})} \mu(p_{i+1}, p_m) < \frac{1-s}{s}. \tag{18}$$

If for each $p \in T$, $\lim_{n\to+\infty} \nu(p, p_n)$ *is exists, finite and* $\lim_{n\to+\infty} \mu(p_n, p) < \dfrac{1}{s}$ *then* K *has a u.f.p in* T.

Proof. Substituting $r = t = 0$ in Theorem3.1 completes this corollary. □

Corollary 3.9. *Let* $K : T \to T$ *be a map satisfy the contraction condition*

$$d(Kp, Kq) \preceq t[d(p, Kq) + d(q, Kp)] \tag{19}$$

for all $p, q \in T$, *where* $t \in (0,1)$ *with* $2t < 1$. *For arbitrary* $p_0 \in T$, *choose* $p_n = K^n p_0$. *Suppose that*

$$\sup_{m \succeq 1} \lim_{i \to +\infty} \frac{\nu(p_{i+1}, p_{i+2})}{\nu(p_i, p_{i+1})} \mu(p_{i+1}, p_m) < \frac{1-t}{t}. \tag{20}$$

If for each $p \in T$, $\lim_{n\to+\infty} \nu(p, p_n)$ *is exists, finite and* $\lim_{n\to+\infty} \mu(p_n, p) < \dfrac{1}{t}$ *then* K *has a u.f.p in* T.

Proof. Substituting $r = s = 0$ in Theorem3.1 completes this corollary. □

Corollary 3.10. *Let* $K : T \to T$ *be a map satisfy the contraction condition*

$$d(Kp, Kq) \preceq r[d(p, q) + d(p, Kp) + d(q, Kq)] \tag{21}$$

for all $p, q \in T$, *where* $t \in (0,1)$ *with* $3t < 1$. *For arbitrary* $p_0 \in T$, *choose* $p_n = K^n p_0$. *Suppose that*

$$\sup_{m \succeq 1} \lim_{i \to +\infty} \frac{\nu(p_{i+1}, p_{i+2})}{\nu(p_i, p_{i+1})} \mu(p_{i+1}, p_m) < \frac{1-t}{2t}. \tag{22}$$

If for each $p \in T$, $\lim_{n\to+\infty} \nu(p, p_n)$ *is exists, finite and* $\lim_{n\to+\infty} \mu(p_n, p) < \dfrac{1}{t}$ *then* K *has a u.f.p in* T.

Proof. Substituting $r = s$ and $t = 0$ in Theorem3.1 completes this corollary. □

4. Results for rational contraction mappings

In this section, we prove some $f.p.r's$ in $DCCMS$ by using rational contraction mappings which were discussed mainly in [9] and [13]. For that, assume (T, d) be a complete $DCCMS$ with respect to the functions $\nu, \mu : T \times T \to [1, +\infty)$ and P be a normal cone with normal constant M.

Theorem 4.1. *Let* $K : T \to T$ *be a map satisfy the rational contraction condition*

$$d(Kp, Kq) \preceq \frac{rd(q, Kq)[1 + d(p, Kp)]}{1 + d(p, q)} + sd(p, q) \tag{23}$$

for all $p, q \in T$, *with* $1 + d(p, q) \neq 0$ *where* $r, s \in (0, 1)$. *For* $p_0 \in T$, *choose* $p_n = K^n p_0$. *Suppose that*

$$\sup_{m \succeq 1} \lim_{i \to +\infty} \frac{\nu(p_{i+1}, p_{i+2})}{\nu(p_i, p_{i+1})} \mu(p_{i+1}, p_m) < \frac{1-r}{s}. \tag{24}$$

If for each $p \in T$,

$$\lim_{n \to +\infty} \nu(p, p_n) \text{ exists, is finite and } \lim_{n \to +\infty} \mu(p_n, p) < \frac{1}{r} \tag{25}$$

then K *has a u.f.p in* T.

Proof. Consider a sequence $\{p_n\}$ in T satisfies the hypothesis of the theorem. From (3.1), we obtain

$$\begin{aligned} d(p_n, p_{n+1}) &= d(Kp_{n-1}, Kp_n) \\ &\preceq \frac{rd(p_n, Kp_n)[1 + d(p_{n-1}, Kp_{n-1})]}{1 + d(p_{n-1}, p_n)} + sd(p_{n-1}, p_n) \\ &= \frac{rd(p_n, p_{n+1})[1 + d(p_{n-1}, p_n)]}{1 + d(p_{n-1}, p_n)} + sd(p_{n-1}, p_n) \\ &= rd(p_n, p_{n+1}) + sd(p_{n-1}, p_n). \end{aligned}$$

That is, $d(p_n, p_{n+1}) \preceq \frac{s}{1-r} d(p_{n-1}, p_n)$. By using induction on n, we get

$$d(p_n, p_{n+1}) \preceq \left(\frac{s}{1-r}\right)^n d(p_0, p_1), \text{ for all } n \geq 0. \tag{26}$$

Now, to prove $\{p_n\}$ is a Cauchy sequence. Using (**DCCM3**) and (4.1), for all $m, n \in \mathbb{N}$ implies that

$$\begin{aligned}
&d(p_n, p_m)\\
&\preceq \nu(p_n, p_{n+1})d(p_n, p_{n+1}) + \mu(p_{n+1}, p_m)d(p_{n+1}, p_m)\\
&\preceq \nu(p_n, p_{n+1})d(p_n, p_{n+1}) + \mu(p_{n+1}, p_m)\nu(p_{n+1}, p_{n+2})d(p_{n+1}, p_{n+2})\\
&\quad + \mu(p_{n+1}, p_m)\mu(p_{n+2}, p_m)d(p_{n+2}, p_m)\\
&\preceq \cdots\\
&\preceq \nu(p_n, p_{n+1})d(p_n, p_{n+1})\\
&\quad + \sum_{i=n+1}^{m-2}(\prod_{j=n+1}^{i}\mu(p_j, p_m))\nu(p_i, p_{i+1})d(p_i, p_{i+1}) + \prod_{k=n+1}^{m-1}\mu(p_k, p_m)d(p_{m-1}, p_m)\\
&\preceq \nu(p_n, p_{n+1})\left(\frac{s}{1-r}\right)^n d(p_0, p_1) + \sum_{i=n+1}^{m-2}(\prod_{j=n+1}^{i}\mu(p_j, p_m))\nu(p_i, p_{i+1})\left(\frac{s}{1-r}\right)^i d(p_0, p_1)\\
&\quad + \prod_{k=n+1}^{m-1}\mu(p_k, p_m)\left(\frac{s}{1-r}\right)^{m-1} d(p_0, p_1)\\
&\preceq \nu(p_n, p_{n+1})\left(\frac{s}{1-r}\right)^n d(p_0, p_1) + \sum_{i=n+1}^{m-1}(\prod_{j=n+1}^{i}\mu(p_j, p_m))\nu(p_i, p_{i+1})\left(\frac{s}{1-r}\right)^i d(p_0, p_1)\\
&\preceq \nu(p_n, p_{n+1})\left(\frac{s}{1-r}\right)^n d(p_0, p_1) + \sum_{i=n+1}^{m-1}(\prod_{j=0}^{i}\mu(p_j, p_m))\nu(p_i, p_{i+1})\left(\frac{s}{1-r}\right)^i d(p_0, p_1).
\end{aligned}$$

This implies,

$$\begin{aligned}
\|d(p_n, p_m)\| \leq M\|&\nu(p_n, p_{n+1})\left(\frac{s}{1-r}\right)^n d(p_0, p_1)\\
&+ \sum_{i=n+1}^{m-1}(\prod_{j=0}^{i}\mu(p_j, p_m))\nu(p_i, p_{i+1})\left(\frac{s}{1-r}\right)^i d(p_0, p_1)\|.
\end{aligned} \tag{27}$$

Choose $N_l = \sum_{i=0}^{l}(\prod_{j=0}^{l}\mu(p_j, p_m))\nu(p_i, p_{i+1})\left(\frac{s}{1-r}\right)^i$, then we get

$$\|d(p_n, p_m)\| \leq M\left\|d(p_0, p_1)\left[\left(\frac{s}{1-r}\right)^n \nu(p_n, p_{n+1}) + (N_{m-1} - N_n)\right]\right\|.$$

Since $\left(\frac{s}{1-r}\right) < 1$. Which implies that the limit of the sequence $\{N_n\}$ exists and so $\{N_n\}$ is Cauchy. Letting $m, n \to +\infty$ in (4.5) gives $\lim_{m,n\to+\infty} d(p_n, p_m) = 0$, and so $\{p_n\}$ is a Cauchy sequence. Using the completeness of K, there exists $p \in K$ such that $\lim_{n\to+\infty} p_n = p$. We claim that $Kp = p$. It follows from(**DCCM3**) and (4.1) that

$$\begin{aligned}0 &\prec d(p,Kp)\\ &\preceq \nu(p,p_{n+1})d(p,p_{n+1})+\mu(p_{n+1},Kp)d(p_{n+1},Kp)\\ &\preceq \nu(p,p_{n+1})d(p,p_{n+1})+\mu(p_{n+1},Kp)d(Kp_n,Kp)\\ &\preceq \nu(p,p_{n+1})d(p,p_{n+1})+\mu(p_{n+1},Kp)\left[\frac{rd(p,Kp)[1+d(p_n,p_{n+1})]}{1+d(p_n,p)}+sd(p_n,p)\right]\end{aligned}$$

and so

$$\begin{aligned}0 &< \|d(p,Kp)\|\\ &\le M\left\|\nu(p,p_{n+1})d(p,p_{n+1})+\mu(p_{n+1},Kp)\left[\frac{rd(p,Kp)[1+d(p_n,p_{n+1})]}{1+d(p_n,p)}+sd(p_n,p)\right]\right\|.\end{aligned}\tag{28}$$

Now, making use of the condition (4.3) and passing to the limit on (4.6) we get

$$0<\|d(p,Kp)\|<r\|d(p,Kp)\|$$

which is a contradiction, therefore $Kp=p$. Suppose that K has another fixed point (say, q), then

$$\begin{aligned}d(p,q) &= d(Kp,Kq)\\ &\preceq \frac{rd(q,Kq)[1+d(p,Kp)]}{1+d(p,q)}+sd(p,q)\\ &\preceq sd(p,q).\end{aligned}$$

But $s<1$. Therefore, our supposition is wrong. Hence, K has a $u.f.p$ in T. □

THEOREM 4.2. *Let $K:T\to T$ be a map satisfy the rational contraction condition*

$$d(Kp,Kq)\preceq \frac{rd(p,Kp)d(q,Kq)}{d(p,q)}+sd(p,q)\tag{29}$$

for all $p,q\in T$, and $d(p,q)\succ 0$ where $r,s\in(0,1)$ with $r+s<1$. For $p_0\in T$, choose $p_n=K^np_0$. Suppose that

$$\sup_{m\succeq 1}\lim_{i\to+\infty}\frac{\nu(p_{i+1},p_{i+2})}{\nu(p_i,p_{i+1})}\mu(p_{i+1},p_m)<\frac{1-r}{s}.\tag{30}$$

If for each $p\in T$,

$$\lim_{n\to+\infty}\nu(p,p_n)\ \textit{exists, is finite and}\ \lim_{n\to+\infty}\mu(p_n,p)<\frac{1}{r}\tag{31}$$

then K has a $u.f.p$ in T.

Proof. Let $\{p_n\}$ be a sequence satisfying the hypothesis of the theorem. From (4.7) and using the same procedure in the above Theorem 4.1, we have

$$d(p_n,p_{n+1})\preceq\left(\frac{s}{1-r}\right)^n d(p_0,p_1),\ \text{for all } n\ge 0.\tag{32}$$

Now, to prove $\{p_n\}$ is a Cauchy sequence. Using (**DCCM3**) and (4.7), for all $m, n \in \mathbb{N}$ implies that

$$
\begin{aligned}
&d(p_n, p_m)\\
&\preceq \nu(p_n, p_{n+1})\left(\frac{s}{1-r}\right)^n d(p_0, p_1) + \sum_{i=n+1}^{m-2}\Big(\prod_{j=n+1}^{i}\mu(p_j, p_m)\Big)\nu(p_i, p_{i+1})\left(\frac{s}{1-r}\right)^i d(p_0, p_1)\\
&\quad + \prod_{k=n+1}^{m-1}\mu(p_k, p_m)\left(\frac{s}{1-r}\right)^{m-1} d(p_0, p_1).
\end{aligned}
$$

Since $\left(\frac{s}{1-r}\right) < 1$, which gives $\|d(p_n, p_m)\| \to 0$ as $m, n \to +\infty$. Therefore the sequence $\{p_n\}$ is a Cauchy, and the completeness of K implies that there exists an element $p \in T$ such that $\{p_n\}$ converges to p. If $Kp \neq p$, we deduce that

$$
\begin{aligned}
0 &\prec d(p, Kp)\\
&\preceq \nu(p, p_{n+1})d(p, p_{n+1}) + \mu(p_{n+1}, Kp)\left[\frac{rd(p_n, Kp_n)d(p, Kp)}{d(p_n, p)} + sd(p_n, p)\right]
\end{aligned}
$$

and so

$$
\begin{aligned}
0 &< \|d(p, Kp)\|\\
&\leq M\left\|\nu(p, p_{n+1})d(p, p_{n+1}) + \mu(p_{n+1}, Kp)\left[\frac{rd(p_n, Kp_n)d(p, Kp)}{d(p_n, p)} + sd(p_n, p)\right]\right\|. \qquad (33)
\end{aligned}
$$

Now, making use of the condition (4.3) and passing to the limit on (4.6) we get

$$0 < \|d(p, Kp)\| < 0$$

which is a contradiction, therefore $Kp = p$. Suppose that K has another $f.p$ (say, q), then (4.1) becomes $d(p, q) = sd(p, q)$. But $s < 1$, therefore, our supposition is wrong. Hence, K has a $u.f.p$ in T. □

THEOREM 4.3. *Let $K : T \to T$ be a map satisfy the rational contraction condition*

$$d(Kp, Kq) \preceq \frac{rd(p, Kp)d(p, Kq)d(q, Kq)}{d(q, Kq) + d(p, q)} + sd(p, q) \qquad (34)$$

for all $p, q \in T$, and $d(q, Kq) + d(p, q) \succ 0$ where $r, s \in (0, 1)$ with $r + s < 1$. For $p_0 \in T$, choose $p_n = K^n p_0$. Suppose that

$$\sup_{m \succeq 1}\lim_{i \to +\infty}\frac{\nu(p_{i+1}, p_{i+2})}{\nu(p_i, p_{i+1})}\mu(p_{i+1}, p_m) < \frac{1}{s}. \qquad (35)$$

If for each $p \in T$,

$$\lim_{n \to +\infty}\nu(p, p_n) \text{ exists, is finite and } \lim_{n \to +\infty}\mu(p_n, p) < \frac{1}{r} \qquad (36)$$

then K has a $u.f.p$ in T.

Proof. Using the same procedure as in Theorem 4.1 completes this notion. □

THEOREM 4.4. *Let $K : T \to T$ be a map satisfy the rational contraction condition*

$$d(Kp, Kq) \preceq \frac{r[d(p, Kp)d(p, Kq) + d(q, Kq)d(q, Kp)]}{d(p, Kq) + d(q, Kp)} \qquad (37)$$

for all $p, q \in T$, *and* $d(p, Kq) + d(q, Kp) \succ 0$ *where* $r, s \in (0, 1)$ *with* $r + s < 1$. *For* $p_0 \in T$, *choose* $p_n = K^n p_0$. *Suppose that*

$$\sup_{m \succeq 1} \lim_{i \to +\infty} \frac{\nu(p_{i+1}, p_{i+2})}{\nu(p_i, p_{i+1})} \mu(p_{i+1}, p_m) < \frac{1}{r}. \tag{38}$$

If for each $p \in T$,

$$\lim_{n \to +\infty} \nu(p, p_n) \text{ exists, is finite and } \lim_{n \to +\infty} \mu(p_n, p) < \frac{1}{r} \tag{39}$$

then K *has a* $u.f.p$ *in* T.

Proof. Using the same procedure as in Theorem 4.1 completes this notion. □

Remark 4.5. All the results proved in the Section 3 and Section 4 hold in complete double controlled cone metric spaces, too.

5. Applications

The $f.p.t$ covers a wide range of applications in the field of mathematics, particularly differential geometry, numerical analysis, and so on. By reading [18] and references therein, one can find a variety of applications involving $f.p.r's$ in the field of applied mathematics. The examples below demonstrate how to apply $f.p$ findings in differential equations.

Example 5.1. Let $T = C([0, 1], \mathbb{R})$ and T is complete extended b-metric space defined by $d(p, q) = \sup_{t \in [0,1]} |p - q|^2$. Also, consider $y''(t) = 3y^2(t)/2$, $0 \le t \le 1$ and the initial conditions $y(0) = 4$, $y(1) = 1$. Here, the exact solution is $y(t) = 4/(1+t)^2$. We have, $y_0(t) = c_1 t + c_2$. By using the initial conditions, we get $y_0(t) = 4 - 3t$. Now, define the integral operator,

$$A(y) = y + \int_0^1 G(t, s)[y'' - f(s, y, y')]ds \tag{40}$$

where

$$G(t, s) = \begin{cases} s(1 - t) & 0 \le s \le t \\ t(1 - s) & t \le s \le 1 \end{cases}$$

Then, the equation (5.1) becomes

$$\begin{aligned} A(y) &= y(t) + \int_0^1 G(t, s) y''(s) ds - \int_0^1 G(t, s) f(s, y, y') ds \\ &= (4 - 3t) - \int_0^1 G(t, s)[-3/2y^2(s)]ds \\ &= 4 - 3t + \frac{3}{2} \left\{ \int_0^1 G(t, s) y^2(s) ds \right\}. \end{aligned}$$

Consider,

$$\begin{aligned}
d(Ap, Aq) &= \sup_{t\in[0,1]} |Ap - Aq|^2 \\
&= \sup_{t\in[0,1]} \left| \frac{3}{2} \int_0^1 G(t,s)p^2(s)ds - \frac{3}{2} \int_0^1 G(t,s)q^2(s)ds \right|^2 \\
&\leq \frac{9}{4} \left(\int_0^1 |G(t,s)|^2 ds \right) \left(\int_0^1 |p^2(s) - q^2(s)|^2 ds \right) \\
&\leq \frac{3}{4} \frac{t^2(1-t)^2}{3} \int_0^1 \left| p^2(s) - q^2(s) \right|^2 ds \\
&\leq \frac{3}{4} (\frac{1}{4})(\frac{1}{4}) \int_0^1 \left| p^2(s) - q^2(s) \right|^2 ds \\
&\leq \frac{3}{64} \sup_{t\in[0,1]} |p(s) - q(s)|^2 \\
&\leq \frac{3}{64} d(p,q).
\end{aligned}$$

Then, equation (3.1) gives

$$d(Ap, Aq) \leq (3/64)d(p,q) + e_2[d(p,Ap) + d(q,Aq)] + e_3[d(p,Aq) + d(q,Ap)].$$

Thus, $e_1 = 3/64$ and $e_2 = e_3 = 0$ satisfies all the conditions of Theorem3.1. Also, by Theorem3.1, A has $u.f.p$ in $T = C([0,1], \mathbb{R})$. Therefore, the given bounded value problem has $u.f.p$ in T.

6. Conclusion

This paper has introduced some new $f.p.r's$ that are applicable to both contraction and rational contraction operators on $DCCMS$. In particular, going in the same direction as [26] and [36], we provide the results in the setting of contraction mappings, namely the B-contraction type, the Hardy-Rogers contraction type, and their consequences. Additionally, we provide the $f.p.r's$ by using the rational contraction mappings, which were discussed mostly in [9] and [13]. In order to confirm the presence of the $f.p.r's$, alternative discoveries presented in the later can be demonstrated in a lower environment.

7. Acknowledgement

All the authors express gratitude to The Korean Journal of Mathematics for their assistance in getting this manuscript completed. We would like to express our gratitude to the editors and the reviewers for their thorough reading and giving the opportunity to reset the manuscript in a nice way.

References

[1] Abbas, M., Rhoades, B. E., *Fixed and periodic point results in cone metric space*, Appl. Math. Lett. **22** (4) (2009), 511–515.
https://doi.org/10.1016/j.aml.2008.07.001

[2] Abdeljawad, T., Mlaiki, N., Aydi, H., Souayah, N. *Double controlled metric type spaces and some fixed point results*, Mathematics. **6** (2018), 320.
https://doi.org/10.3390/math6120320

[3] Bakhtin, I. A., *The contraction mapping principle in almost metric spaces*, Funct. Anal., **30** (1989), 26–37.

[4] Banach, S., *Surles operations dans les ensembles abstract et leur application aux equation integrals*, Fund.Math., **3**(1922), 133–181.

[5] Bianchini, R. M. T., *Su un problema di S. Reich riguardante la teoria dei punti fissi*, Bolletino U.M.I., **4** (5) (1972), 103–106.

[6] Brouwer, F., *The fixed point theory of multiplicative mappings in topological vector spaces*, Mathematische Annalen., **177** (1968), 283–301.

[7] Chatterjea, S. K., *Fixed point theorems* , C. R. Acad. Bulg. Sci., **25** (1972), 727–730.

[8] Ciric, L. B., *Generalized contractions and fixed point theorems*, Publ. Inst. Math. (Bulgr). **12** (26) (1971), 19–26.

[9] Dass, B. K., Gupta, S., *An extension of Banach contraction principle through rational expression*, Communicated by F.C. Auluck, FNA.,1975.

[10] Hardy, G. E., Rogers, T.D., *A generalization of fixed point theorem of Reich*, Can. Math. Bull., **16** (1973), 201–206.

[11] Haung, L. G., Zhang, X., *Cone metric spaces and fixed point theorems of contractive mappings*, J. Math. Anal. Appl. **332** (4) (2007), 1468–1476.
https://doi.org/10.1016/j.jmaa.2005.03.087

[12] Huaping Huang, Stojan Radenović, Guantie Deng, *A sharp generalization on cone b-metric space over Banach algebra*, J. Nonlinear Sci. Appl. **10** (2017), 429–435.
http://dx.doi.org/10.22436/jnsa.010.02.09

[13] Jaggi, D. S., *Some unique fixed point theorems*, Indian Journal of Pure and Applied Mathematics, **8** (1977), 223–230.

[14] Kannan, R., *Some results on fixed points*, Bull Calcutta Math.Soc, **60** (1968), 71–76.

[15] Kannan, R., *Some results on fixed points II*, Am.Math.Mon. **76** (1969), 405–408.

[16] Karapinar, E., *A new non-unique fixed point theorem*, Ann. Funct. Annals. **2** (1) (2011), 51–58.

[17] Khan, M. S., *A fixed point theorems for metric spaces*, Rendiconti Dell 'istituto di mathematica dell' Universtia di tresti, **8** (1976), 69–72.

[18] Khuri, S. A., Louhichi, I., *A novel Ishikawa-Green's fixed point scheme for the solution of BVPs*, Appl. Math. Lett. **82** (2018), 50–57.
https://doi.org/10.1016/j.aml.2018.02.016

[19] Kumar. K, Rathour. L, Sharma. M. K, Mishra V. N. *Fixed point approximation for suzuki generalized nonexpansive mapping using* $B_{(\delta,\mu)}$ *condition*, Applied Mathematics **13** (2) (2022), 215–227.
https://doi.org/10.4236/am.2022.132017

[20] Marudai, M., Bright V. S., *Unique fixed point theorem weakly B-contractive mappings*, Far East journal of Mathematical Sciences (FJMS), **98** (7) (2015), 897–914.

[21] Mishra. L. N, Dewangan. V, Mishra. V. N, Karateke. S, *Best proximity points of admissible almost generalized weakly contractive mappings with rational expressions on b-metric spaces*, J. Math. Computer Sci. **22** (2) (2021), 97—109.
https://doi.org/10.22436/jmcs.022.02.01

[22] Mishra. L. N, Dewangan. V, Mishra. V. N, Amrulloh. H, *Coupled best proximity point theorems for mixed* g*-monotone mappings in partially ordered metric spaces*, J. Math. Comput. Sci. **11** (5) (2021), 6168–6192.
https://doi.org/10.28919/jmcs/6164

[23] Mishra. L. N, Mishra. V. N, Gautam. P, Negi. K, *Fixed point Theorems for Cyclic-Ćirić-Reich-Rus contraction mapping in Quasi-Partial b-metric spaces*, Scientific Publications of the State University of Novi Pazar Ser. A: Appl. Math. Inform. and Mech. **12** (1) (2020), 47–56.
http://dx.doi.org/10.5937/SPSUNP2001047M

[24] Mishra L. N, Tiwari. S. K, Mishra. V. N, *Fixed point theorems for generalized weakly S-contractive mappings in partial metric spaces*, Journal of Applied Analysis and Computation **5**

(4) (2015), 600–612.
https://doi.org/10.11948/2015047

[25] Mitrović, Z. D., Radenović, S., *The Banach and Reich contractions in $b_v(s)$-metric spaces*, J. Fixed Point Theory Appl. **19** (2017), 3087–3095.
http://dx.doi.org/10.1007/s11784-017-0469-2

[26] Mlaiki, N., Aydi, H., Souayah, N., Abdeljawad, T., *Controlled metric type spaces and related contraction principle*, Mathematics **6** (10) (2018), 194.
https://doi.org/10.3390/math6100194

[27] Mlaiki, N., *Double controlled metric-like spaces*, J. Inequal. Appl. **189** 2020.
https://doi.org/10.1186/s13660-020-02456-z

[28] Reich, S., *Some remarks connecting contraction mappings*, Can. Math. Bull. **14** (1971), 121–124.
https://doi.org/10.4153/CMB-1971-024-9

[29] Roshan, J. R., Parvanesh, V., Kadelburg, Z., Hussain, N., *New fixed point results in b-rectangular metric spaces*, Nonlinear Analalysis: Modelling and control **21** (5) (2016), 614–634.
http://dx.doi.org/10.15388/NA.2016.5.4

[30] Sanatee. A. G, Rathour. L, Mishra. V. N, Dewangan. V *Some fixed point theorems in regular modular metric spaces and application to Caratheodory's type anti-periodic boundary value problem*, The Journal of Analysis **31** (2023), 619–632.
https://doi.org/10.1007/s41478-022-00469-z

[31] Sanatee. A. G, Ranmanesh. M. Mishra. L. N, Mishra. V. N, *Generalized $2-$proximal $C-$contraction mappings in complete ordered $2-$metric space and their best proximity points*, Scientific Publications of the State University of Novi Pazar Ser. A: Appl. Math. Inform. and Mech, **12** (1) (2020), 1–11.
http://dx.doi.org/10.5937/SPSUNP2001001S

[32] Shahi P, Rathour L, Mishra. V. N *Expansive Fixed Point Theorems for tri-simulation functions*, The Journal of Engineering and Exact Sciences –jCEC **08** (3) (2022), 14303–01e.
https://doi.org/10.18540/jcecvl8iss3pp14303-01e

[33] Sharma. N, Mishra. L. N, Mishra. V. N, Almusawa. H, *Endpoint approximation of standard three-step multi-valued iteration algorithm for nonexpansive mappings*, Applied Mathematics and Information Sciences **15** (1) (2021), 73–81.
https://doi.org/10.18576/amis/150109

[34] Sharma. N, Mishra. L. N, Mishra. V. N, Pandey. S, *Solution of Delay Differential equation via N_1^v iteration algorithm*, European J. Pure Appl. Math. **13** (5) (2020), 1110–1130.
https://doi.org/10.29020/nybg.ejpam.v13i5.3756

[35] Sharma. N, Mishra. L. N, Mishra. S. N, Mishra. V. N, *Empirical study of new iterative algorithm for generalized nonexpansive operators*, Journal of Mathematics and Computer Science **25** (3) (2022), 284–295.
https://dx.doi.org/10.22436/jmcs.025.03.07

[36] Shateri, T. L., *Double controlled cone metric spaces and the related fixed point theorems*, J. Korean Soc. Math. Educ. Ser. B: Pure Appl. Math. **30** (1) (2023), 1–13.
https://doi.org/10.48550/arXiv.2208.06812

[37] Slobodanka Janković, Zoran Kadelburg, Stojan Radenović, *On cone metric spaces: A survey*, Nonlinear Analysis **74** (2011) 2591–2601.

[38] Stojan Radenović, *Common Fixed Points Under Contractive Condition in Cone Metric Spaces*, Computers and Mathematics with applycation **58** (2019),1273–1278.
https://doi.org/10.1016/j.camwa.2009.07.035

[39] Suzana Aleksić, Zoran Kadelburg, Zoran. D. Mitrović, Stojan Radenović, *A new survey: cone metric spaces*, Journal of the international Mathematical Vertiual Institute **9**(2019), 93-121.
https://api.semanticscholar.org/CorpusID:119572977

[40] Theivaraman. R, Srinivasan. P. S, Thenmozhi. S, Radenovic. S, *Some approximate fixed point results for various contraction type mappings*, **13** (9) (2023), 1–20.
https://doi.org/10.28919/afpt/8080

[41] Theivaraman. R, Srinivasan. P. S, Radenovic. S, Choonkil Park, *New Approximate Fixed Point Results for Various Cyclic Contraction Operators on E-Metric Space*, **27** (3) (2023), 160–179.
https://doi.org/10.12941/jksiam.2023.27.160

[42] Vishnu Narayanan P: B. Deshpande, V.N. Mishra, A. Handa, L.N. Mishra, *Coincidence Point Results for Generalized* (ψ, θ, ϕ)*-Contraction on Partially Ordered Metric Spaces*, Thai J. Math., **19** (1) (2021), 93–112.

[43] Vishnu Narayanan P, Mishra. L. N, Tiwari. S. K, Mishra. V. N, Khan. I. A; *Unique Fixed Point Theorems for Generalized Contractive Mappings in Partial Metric Spaces*, Journal of Function Spaces, **2015** (2021), Article ID 960827, 1–8.

[44] Zamfirescu, T., *Fixed point theorems in metric spaces*, Arch. Math. (Basel) **23**(1972), 292–298.

[45] Zoran Kadelburg, Stojan Radenović, Vladimir Rakočević, *A note on the equivalence of some metric and cone metric fixed point results*, Applied Mathematics Letters, **24** (2011), 370–374. https://doi.org/10.1016/j.aml.2010.10.030

A. Herminau Jothy
Department of Mathematics, Bharathidasan University, Trichy -620024, Tamilnadu, India.
E-mail: herminaujothy.a@bdu.ac.in

P. S. Srinivasan
Department of Mathematics, Bharathidasan University, Trichy -620024, Tamilnadu, India.
E-mail: pssrini@bdu.ac.in

Laxmi Rathour
Department of Mathematics, National Institute of Technology, Chaltlang, Aizawl-796012, Mizoram, India
E-mail: laxmirathour817@gmail.com

R. Theivaraman
Department of Mathematics, Bharathidasan University, Trichy -620024, Tamilnadu, India.
E-mail: deivaraman@gmail.com

S. Thenmozhi
Department of Mathematics, SRM Institute of Science and Technology, Kattankulathur, Chengalpattu, Tamilnadu-603203, India.
E-mail: sthenu85@gmail.com