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FUZZY LATTICE ORDERED GROUP BASED ON FUZZY PARTIAL ORDERING RELATION

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Abstract. In this paper, we introduce the concept of a fuzzy lattice ordered group, which is based on a fuzzy lattice that Chon developed in his paper "Fuzzy Partial Order Relations and Fuzzy Lattice". We will also discuss fuzzy lattice-ordered groups in detail, provide several results that are analogous to the classical theory of lattice-ordered groups, and characterize the relationship between a fuzzy lattice-ordered group using its level set and support. Moreover, we define the concepts of fl-subgroups, quotients, and cosets of fl-groups and obtain some fundamental results for these fuzzy algebraic structures.

1. Introduction

The concept of ordering that led to the development of lattice theory is crucial for the study of mathematics and other areas of the natural sciences. One of the study in the theory of ordering is the idea of partially-ordered algebraic structures. Numerous experts in these fields have investigated various partially ordered algebraic structures, including lattice-ordered groups, rings, fields, and vector spaces. Lattice-ordered groups (also called l-groups) are one of the important classes of lattice-ordered algebraic structures. After Birkhoff introduced the study of l-groups [3], many scholars further studied on this ordered algebraic structure [2, 5–7, 10, 16].

On the other hand, the concepts of a fuzzy set and fuzzy relation were first introduced and studied by Zadeh in his original paper [18]. In [19], the author also introduced the concept of fuzzy partial ordering relation as a fuzzy relation satisfying the set of axioms given in Definition (2.19) and studied various properties of fuzzy ordering relations. These new generalizations provide motivation to other researchers for further development in fuzzy ordering algebraic structures. Chon in 2009 [4], considered a fuzzy partial order relation defined by Zadeh [19] and defined a fuzzy lattice as a fuzzy ordering relation, developed some basic properties of fuzzy lattices, and characterized a fuzzy lattice using its level set. Furthermore, he presented the

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ideas of distributive and modular fuzzy lattices and measured several undeveloped properties of fuzzy lattices.

Many authors have studied fuzzy-order algebraic structures with various approaches. Saibaba in 2008, considered a complete lattice L, and introduced the notions of the L-fuzzy lattice-ordered subgroup as a fuzzy set $\lambda : G \longrightarrow L$ on the l-group G satisfying some additional conditions given in Definition 2.1 of [13]. In [14], the author also introduced the concepts of L-fuzzy convex l-subgroups, L-fuzzy prime and maximal convex l-subgroups an l-group G. In [12] the author also extends his study on the set of all L-fuzzy prime convex l-subgroups of an l-group G. The concepts of fuzzy convex lattice-ordered subgroups were also studied by [1]. Moreover, the concept of fuzzy lattice-ordered G-modules and its properties were discussed by Paul et al. in [11]. There are also other studies of fuzzy ordering relations in different approaches. (See the references [15]). However, all the above investigations on the fuzzy lattice-ordered groups are not similar to the classical theory of lattice-ordered groups, as they are not defined based on fuzzy partially ordered relations.

However, Vimala J. in [17], began by introducing the concept of the fuzzy lattice ordered group as a fuzzy algebraic structure $(G, +, R)$ where $(G, +)$ is a group and (G, R) is a fuzzy lattice as defined by Chon in [4]. Two equivalent definitions (i.e., Definition 3.1 and Definition 3.2) were set and some of their properties developed. But we found the findings in the paper were incorrect due to the non-equivalence of these definitions. Therefore, this motivates us to study on fuzzy lattice ordered group, which is based on the fuzzy lattice defined by Chon and also our definition is slightly different from the definition given in [17], in order to gain a deeper understanding of fuzzy ordering structure and obtain analogous results from the classical theory of l-group.

The organization of this paper's structure is as follows: Some terms, definitions, notations, and significant findings on l-groups, fuzzy posets, and fuzzy lattices are reviewed in Section 2. In Section 3, by using Chon's definition of fuzzy lattice, a new kind of fuzzy algebraic structure, i.e., fuzzy lattice-ordered group, is introduced. We will also discuss fuzzy lattice-ordered groups in detail, exploring general properties such as the positive cone and absolute value in fuzzy lattice-ordered groups. The basic and fundamental properties of the fuzzy lattice-ordered group are presented. Finally, in Section 4, concepts of fl-subgroups, and quotients and cosets of fl-groups are presented.

2. Preliminaries

In this section, we will review and recall concepts, terminologies, and some important results of l-groups, fuzzy posets, and fuzzy lattices that will be useful in understanding the new ideas and related properties introduced and investigated in the sequel.

2.1. Lattice Ordered Group.

Definition 2.1. [16] A system $(G, +, \leq)$ is called a partially ordered group denoted by $po-group$ if

(a) $(G, +)$ is a group.
(b) $(G, \leq)$ is a poset.

(c) If $x \leq y$, then $a + x + b \leq a + y + b$ for all $x, y, a, b \in G$ (i.e., $\leq$ is translation invariant).

Definition 2.2. [16] Let $(G, +, \leq)$ be a partially ordered group. Then $(G, +, \leq)$ is called a lattice-ordered group denoted by l-group if $(G, \leq)$ is a lattice.

Definition 2.3. [16] Let $(G, +, \leq)$ be a fuzzy partially ordered group. Then $(G, +, \leq)$ is called a totally ordered group (or a linearly ordered group) if $(G, \leq)$ is a chain.

Theorem 2.4. [16] *A non-trivial* $po-group$ *is neither bounded above nor bounded below. Hence, not bounded.*

Definition 2.5. [16] Let $(G, +, \leq)$ be a $po - group$ with the identity element e.

(i) The positive cone of G, is denoted by G^+ is defined as,

$$G^+ = \{x \in G : e \leq x\}. \tag{1}$$

(ii) The negative cone of G, is denoted by G^- and is defined as,

$$G^- = \{x \in G : x \leq e\}. \tag{2}$$

Theorem 2.6. [16]

(i) Let $(G, +, \mu)$ *be a* $po - group$ *with a positive cone* $G^+ = P$*. Then,*
  - *(a)* $P + P \subseteq P$.
  - *(b)* $-g + P + g \subseteq P$ *for all* $g \in G$.
  - *(c)* $P \cap (-P) = \{e\}$.
  - *(d)* $x \leq y \Longleftrightarrow y - x \in P$ *where* $x, y \in G$.

(ii) Let $(G, +)$ *be a group. Let* $P \subseteq G$ *satisfy* $(a), (b), (c)$*. Define;* $\leq$ *on* G *by* (d)*. Then* $(G, +, \leq)$ *is a* $po - group$ *and* $G^+ = P$.

Definition 2.7. [16] Let $(G, +, \leq)$ be a l-group and $S \subseteq G$. Then $(S, +, \leq)$ is said to be an l-sub group of G if:

(i) S is sub group of G.
(ii) S is sublattice of G.

Definition 2.8. [16] Let $(G, \leq)$ be a poset. Let $C \subseteq G$. Then C is said to be a convex subset of G with respect to $\leq$; if $x, y \in C$ and $z \in G$ are such that $x \leq z \leq y$, then $z \in C$.

Definition 2.9. [16] Let $(G, +, \leq)$ be an l- group. Let $(C, +, \leq)$ be an l-subgroup of $(G, +, \leq)$. Then C is said to be a convex l-subgroup of G if C is a convex set in G.

Theorem 2.10. [16] *Let* G *be an* l*-group, and* S *be a subgroup of* G*. Then* S *is a* l*-subgroup of* G *if and only if* $x \in S \Rightarrow x \vee e \in S$.

Theorem 2.11. [16] *Let* $(G, +, \leq)$ *be a po group, and let* C *be a subgroup of* G*. Define the set* G/C *called the quotient of* G *by* C *by,*

$$G/C = \{x + C : x \in G\}$$

And define the relation $\leq'$ *on* G/C *by,*

$$x + C \leq' y + C \Leftrightarrow x \leq y + c \text{ for some } c \in C$$

Then $(G/C, \leq')$ *is a poset if and only if* C *is a convex set.*

Theorem 2.12. [16] *Let G be an l-group. Let C be a convex subgroup of G. Then the following are equivalent:*

(a) C is an l-subgroup of G.
(b) G/C is a lattice. Moreover, G/C is a distributive lattice, and the map $G \longrightarrow G/C$ defined by $x \longmapsto x + C$ is order preserving.

Theorem 2.13. [16]

(a) Let $(G, +, \leq)$ be a po-group and C be a convex normal subgroup of G. Then $(G/C, +, \leq')$ is po-group.
(b) Let $(G, +, \leq)$ be a l-group and C be a convex normal l-subgroup of G. Then $(G/C, +, \leq')$ is l-group.

2.2. Fuzzy Lattice.

Definition 2.14. [19] Let X and Y be two non-empty sets. Then any mapping $\mu : X \times Y \longrightarrow [0, 1]$ is called a fuzzy relation from X to Y.
If $X = Y$ then the mapping $\mu : X \times X \longrightarrow [0, 1]$ is called a fuzzy relation in X (or a binary fuzzy relation in X).

Definition 2.15. [19] Let μ be a fuzzy relation on X. For $\alpha \in [0, 1]$, the set (a crisp set set)

$$\mu_\alpha = \{(x, y) \in X \times X : \mu(x, y) \geq \alpha\} \tag{3}$$

is called the $\alpha - level$ (or $\alpha - cut$) subset of the fuzzy relation μ.

Definition 2.16. [19] Let X be a non-empty set and let $\mu : X \times X \longrightarrow [0, 1]$ be a fuzzy relation on X. Then the support of a fuzzy relation μ is denoted by $S(\mu)$ is defined to be the crisp relation of $X \times X$ over which $\mu(x, y) > 0$ for every $x, y \in X$. That is,

$$S(\mu) = \{(x, y) \in X \times X : \mu(x, y) > 0\}. \tag{4}$$

Definition 2.17. Let X be a non-empty set and let μ_1 and μ_2 be fuzzy relations on X . Then μ_1 is said to be contained in a fuzzy relation μ_2 and is denoted by $\mu_1 \subseteq \mu_2$ if and only if $\mu_1(x, y) \leq \mu_2(x, y)$ for all (x, y) in $X \times X$.

Definition 2.18. [19] Let X be a non-empty set and μ be a fuzzy relation in X, then μ is said to be:

(a) Reflexive: If and only if $\mu(x, x) = 1$, for every $x \in X$.
(b) Symmetric: If and only if $\mu(x, y) = \mu(y, x)$, for every $x, y \in X$.
(c) Anti-symmetric: If and only if $\mu(x, y) > 0$ and $\mu(y, x) > 0$ implies $x = y$ for every $x, y \in X$.
(d) Transitive: If and only if $\mu(x, z) \geq \sup_{y \in X} \min(\mu(x, y), \mu(y, z))$, for every $x, y, z \in X$.

Definition 2.19. [19] Let X be a non-empty set and μ be a fuzzy relation on X, then μ is said to be a fuzzy partial ordered relation on X if and only if μ is reflexive, anti-symmetric and transitive.

Definition 2.20. [4] Let X be a non-empty set and μ be a fuzzy partial ordered relation on X, the μ is said to be a fuzzy total order relation if and only if $\mu(x, y) > 0$ or $\mu(y, x) > 0$ for every $x, y \in X$.

Definition 2.21. [4] If μ is a fuzzy partial ordered relation on X, then the ordered pair (X, μ) is called a fuzzy partial ordered set or a fuzzy poset. Also, if μ is a fuzzy total order relation in X, then the ordered pair (X, μ) is called a fuzzy total ordered set or a fuzzy chain.

The following Theorems shows the characterization of fuzzy relations in terms of their level subsets.

Theorem 2.22. [4] *Let μ be a fuzzy relation in X. Then μ is a fuzzy partial order relation if and only if every α-cut, μ_α is a partial order relation in X for all α such that* $0 < \alpha \leq 1$.

Remark 2.23. In the above Theorem, if $\alpha = 0$ the $\alpha - level$ set becomes,

$$\mu_0 = \{(x, y) \in X \times X : \mu(x, y) \geq 0\} = X^2 \tag{5}$$

But, it is known that X^2; is not antisymmetric (Because X^2 is not partial ordering relation) for $|X| \geq 2$. Hence; μ_0 is not a partial ordering relation for $|X| \geq 2$.

Theorem 2.24. [4] *Let μ be a fuzzy relation in X and let $S(\mu) = \{(x, y) \in X \times X : \mu(x, y) > 0\}$ be a support of the fuzzy relation μ defined on X. If μ is a fuzzy partial order relation on X. Then $S(\mu)$ is a partial order relation on X.*

Definition 2.25. [4] Let (X, μ) be a fuzzy poset and let $B \subseteq X$.

1. An element $u \in X$ is said to be an upper bound for a subset B if and only if $\mu(b, u) > 0$ for all $b \in B$.
2. An upper bound u_0 for B is said to be the least upper bound (lub) of B if and only if $\mu(u_0, u) > 0$ for every upper bound u for B.
3. An element $v \in X$ is said to be a lower bound for a subset B if and only if $\mu(v, b) > 0$ for all $b \in B$.
4. A lower bound v_0 for B is said to be the greatest lower bound (glb) of B if and only if $\mu(v, v_0) > 0$ for every lower bound v for B.

Notation. We denote the least upper bound of the set $\{x, y\}$ if it exists by $x \vee_F y$ and we denote the greatest lower bound of the set $\{x, y\}$ if it exists by $x \wedge_F y$.

Definition 2.26. [4] Let (X, μ) be a fuzzy poset. Then (X, μ) is said to be a fuzzy lattice if and only if $x \vee_F y$ and $x \wedge_F y$ exist for all $x, y \in X$.

Example 2.27. [8] Let $L_1 = \{x, y, z, w\}$ defined as a fuzzy relation μ_1 on L_1 by the fuzzy relational matrix given in Figure 1.

μ_1	x	y	z	w
x	1.0	0.4	0.1	0.8
y	0.0	1.0	0.0	0.6
z	0.0	0.0	1.0	0.5
w	0.0	0.0	0.0	1.0

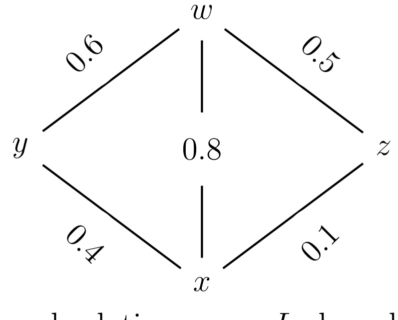

Figure 1. Representation of fuzzy ordered relation μ_1 on L_1 by relational matrix and Hasse diagram

From Figure 1, it is easy to verify that (L_1, μ_1) is fuzzy lattice.

Theorem 2.28 (Properties of Fuzzy Lattice). [4] *Let (X, μ) be a fuzzy lattice and let $x, y, z \in X$. Then,*

(i) $\mu(x, x \vee_F y) > 0$ *and* $\mu(y, x \vee_F y) > 0$.
(ii) $\mu(x \wedge_F y, x) > 0$, *and* $\mu(x \wedge_F y, y) > 0$.
(iii) If $\mu(x, z) > 0$ *and* $\mu(y, z) > 0$, *then* $\mu(x \vee_F y, z) > 0$.
(iv) If $\mu(z, x) > 0$ *and* $\mu(z, y) > 0$, *then* $\mu(z, x \wedge_F y) > 0$.
(v) If $\mu(y, z) > 0$ *Then* $\mu(x \wedge_F y, x \wedge_F z) > 0$ *and* $\mu(x \vee_F y, x \vee_F z) > 0$.

Theorem 2.29. [4] *Let (X, μ) be a fuzzy relation in X and let μ_α be α-level sub set of the fuzzy relation μ defined in X. If (X, μ_α) is a lattice for all $\alpha \in (0, 1]$, then (X, μ) is a fuzzy lattice.*

Remark 2.30. The converse of the above theorem is not necessarily true.

But we can find a lattice from a fuzzy lattice by specifying the α-cut as follows:

Theorem 2.31. [4] *Let (X, μ) be a fuzzy relation on X and μ_α be $\alpha - level$ sub set of the fuzzy relation μ defined on X. If (X, μ) is a fuzzy lattice, then (X, μ_α) is a lattice for some $\alpha \in (0, 1]$.*

Theorem 2.32. [4] *Let μ be a fuzzy relation in X and $S(\mu)$ be a support of the fuzzy relation μ defined on X. If (X, μ) is a fuzzy lattice, then $(X, S(\mu))$ is a lattice on X.*

Proposition 2.33. [9] *Let (X, μ) be a fuzzy lattice and $x, y \in X$, then $x \vee_F y$ and $x \wedge_F y$ coincides with $x \vee y$ and $x \wedge y$, respectively, in $(X, S(\mu))$.*

3. Fuzzy Lattice Ordered Groups

In this section, we introduce the concept of a fuzzy lattice ordered group in the fuzzy lattice and characterize it with its level sets and support. In addition, we present some basic and important properties of these systems.

Definition 3.1. A System $(G, +, \mu)$ is called fuzzy partial ordered group denoted by fpo-group if,

(a) $(G, +)$ is a group.
(b) (G, μ) is a fuzzy poset.
(c) $\mu(x, y) \leq \mu(a + x + b, a + y + b)$ for all $x, y, a, b \in G$ (i.e, μ is translation invariant).

Remark 3.2. From the above definition, it is clear that if $\mu(x, y) > 0$ then $\mu(a + x + b, a + y + c) > 0$ for all $x, y, a, b \in G$.

The following results are an immediate consequences of the above definition:

Proposition 3.3. *Let $(G, +, \mu)$ be an fpo-group. Then,*

(a) $\mu(x, y) = \mu(a + x + b, a + y + b)$ *for all* $x, y, a, b \in G$.
(b) $\mu(x, y) = \mu(-y, -x)$ *for all* $x, y \in G$.
(c) If $\mu(x_1, y_1) > 0$ *and* $\mu(x_2, y_2) > 0$ *then* $\mu(x_1 + x_2, y_1 + y_2) > 0$ *for all* $x_1, x_2, y_1, y_2 \in G$.
(d) If $\mu(x, y) > 0$ *then* $\mu(nx, ny) > 0$ *for* $n \in \mathbb{N}$ *and for all* $x, y \in G$.

Definition 3.4. Let $(G, +, \mu)$ be a fuzzy partial ordered (fpo) group. Then $(G, +, \mu)$ is called a fuzzy lattice-ordered group denoted by fl-group if (G, μ) is a fuzzy lattice.

Definition 3.5. Let $(G, +, \mu)$ be a fuzzy partial ordered group. Then $(G, +, \mu)$ is called a fuzzy totally ordered group (or a fuzzy linearly ordered group) if, $(G, \mu) = (G, \vee_F, \wedge_F)$. is a fuzzy chain.

Notation. Throughout the rest of this paper, $(G, +, \mu)$ stands for a fl-group $(G, +, \vee_F, \wedge_F)$.

Remark 3.6. [9] Let $\mathcal{L} = (X, \mu)$ be a fuzzy poset, then $\mathcal{L}$ is said to be:
(a) Bounded below if there exists $\perp \in X$ such that $\mu(\perp, x) > 0$ for any $x \in X$.
(b) Bounded above if there exists $\top \in X$ such that $\mu(x, \top) > 0$ for any $x \in X$.
(c) Bounded if it is bounded below and bounded above.

Example 3.7. Let $C(\mathbb{R})[0, 2]$ be the set of continuous real valued function defined on $[0, 2]$. Define the fuzzy relation $\mu : C(\mathbb{R})[0, 2] \times C(\mathbb{R})[0, 2] \longrightarrow [0, 2]$ by,

$$\mu(f, g) = \begin{cases} 1 & \text{if } \; f(x) \leq g(x) \\ 0 & \text{otherwise} \end{cases}$$

Then, it can be easily verify that $(C(\mathbb{R})[0, 2], +, \mu)$ is fuzzy lattice ordered group (fl-group).

Example 3.8. Let $D(\mathbb{R})[0, 2]$ be the set of differentiable real valued function defined on $[0, 2]$. Define the fuzzy relation $\mu : D(\mathbb{R})[0, 2] \times D(\mathbb{R})[0, 2] \longrightarrow [0, 1]$ by,

$$\mu(f, g) = \begin{cases} 1 & \text{if } \; f(x) \leq g(x) \\ 0 & \text{otherwise} \end{cases}$$

Then, it can be easily verify that $(D(\mathbb{R})[0, 2], +, \mu)$ is fuzzy partially ordered group (fpo-group). Now, let $f(x) = 1 - x$ and $g(x) = x - 1$. Clearly, $f(x), g(x) \in D(\mathbb{R})[0, 2]$. But,

$$l.u.b\{f(x), g(x)\} = \begin{cases} 1 - x & \text{if } \; x \leq 1 \\ x - 1 & \text{if } \; x \geq 1 \end{cases}$$

Then, $l.u.b\{f(x), g(x)\} \notin D(\mathbb{R})[0, 2]$. Hence, $(D(\mathbb{R})[0, 2], \mu)$ is not fuzzy lattice. Therefore, $(D(\mathbb{R})[0, 2], +, \mu)$ is not fl-group.

Proposition 3.9. *Let $(G, +, \mu)$ be the fl-group and define the relation $\leq$ on G by;*

$$a \leq b \Longleftrightarrow \mu(a, b) > 0$$

Then $(G, +, \leq)$ is a l-group.

Proof. (i) Clearly $(G, +)$ is a group.
(ii) To show $(G, \leq)$ is a lattice:
(a) Let $x \in G$. Since $\mu(x, x) = 1 > 0$ then $x \leq x$. Hence, $\leq$ is reflexive.
(b) Let $x, y \in G$ be such that $x \leq y$ and $y \leq x$. Then $\mu(x, y) > 0$ and $\mu(y, x) > 0$. By antisymmetricity of μ, we have $x = y$ and hence, $\leq$ is antisymmetric.

(c) $x, y, z \in G$ such that $x \leq y$ and $y \leq z$. Then $\mu(x, y) > 0$ and $\mu(y, z) > 0$. Hence,

$$\begin{aligned} \mu(x, z) &\geq \sup_{t \in G}\{\min(\mu(x, t), \mu(t, z))\} \\ &\geq \min(\mu(x, y), \mu(y, z)) \\ &> 0 \end{aligned}$$

Which implies $x \leq z$, and hence, $\leq$ is transitive. Therefore, $\leq$ is a partially ordering relation.

(d) Let $x, y \in G$. Assume $r = x \vee_F y$ with respect to the relation μ. Then, $\mu(x, r) > 0$ and $\mu(y, r) > 0$ and hence, by the definition of $\leq$ we have $x \leq r$ and $y \leq r$. This implies that r is an upper bound of the set $\{x, y\}$ with respect to the relation $\leq$. Again, let c be an upper bound of the set $\{x, y\}$ with respect to the relation $\leq$. Then, $x \leq c$ and $y \leq c$ and by the definition of μ we have $\mu(x, c) > 0$ and $\mu(y, c) > 0$ and since $r = x \vee_F y$ with respect to the relation μ, we get $\mu(r, c) > 0$. Then $r \leq c$. This shows that r is the least upper bound of the set $\{x, y\}$ with respect to the relation $\leq$. Similarly, if we let $s = x \wedge_F y$ with respect to the relation μ. Then, s is the greatest lower bound of the set $\{x, y\}$ with respect to the relation $\leq$. Therefore, $(G, \leq)$ is a lattice.

(iii) Suppose $x, y, a, b \in G$ are such that $x \leq y$. Then $\mu(x, y) > 0$. and hence $0 < \mu(x, y) \leq \mu(a+x+b, a+y+b)$. Thus, by definition, we have $a+x+b \leq a+y+b$. This implies $\leq$ is translation invariant.

Therefore, $(G, +, \leq)$ is an l group. □

Definition 3.10. Let (G, μ) be a fuzzy poset. Then G is said to be:

(a) Directed above, if for any $x, y \in G$, there exists $z \in G$ such that $\mu(x, z) > 0$ and $\mu(y, z) > 0$.
(b) Directed below, if for any $x, y \in G$, there exists $z \in G$ such that $\mu(z, x) > 0$ and $\mu(z, y) > 0$.
(c) Directed, if it is directed above and directed below.

Definition 3.11. Let $(G, +, \mu)$ be a *fpo*-group. Then, G is said to be a directed group if (G, μ) is a directed fuzzy poset.

Theorem 3.12. *A non-trivial $fpo-group$ is neither bounded above nor bounded below. Hence, not bounded.*

Proof. Let $(G, +, \mu)$ be a $fpo-group$ and assume $G \neq \{e\}$ where e is the identity element of the group G.
Assume G is bounded above, and say that $\top$ is the greatest element of G. Thus, $\mu(x, \top) > 0$ for all $x \in G$, and hence $\mu(e, \top) > 0$. Now, by translation invariant, we have $\mu(e + \top, \top + \top) > 0$. Therefore,

$$\mu(\top, \top + \top) > 0 \tag{6}$$

Again, since $\top$ is the greatest element of G we have,

$$\mu(\top + \top, \top) > 0 \tag{7}$$

By antisymmetric of μ, we have $\top + \top = \top$. And by cancellation law, we have $\top = e$. Let $a \in G$, then $-a \in G$. Since $\mu(a, \top) > 0$ and $\mu(-a, \top) > 0$, then $\mu(a, e) > 0$ and $\mu(-a, e) > 0$. From $\mu(-a, e) > 0$, we have $\mu(-a + a, a + e) > 0$. Then $\mu(a, e) > 0$ and $\mu(e, a) > 0$, and by antisymmetric of μ, we have $a = e$. This is a contradiction.

Therefore, G is not bounded above. Similarly, we can verify that G is not bounded below, and therefore, G is not bounded. □

REMARK 3.13. Since all finite lattices are bounded, it is clear that, non-trivial $fpo-group$ is infinite.

DEFINITION 3.14. Let $(G,+,\mu)$ be a $fpo-group$ with the identity element e.

(a) The positive cone of G is denoted by G^+ and is defined as,

$$G^+ = \{x \in G : \mu(e,x) > 0\} \tag{8}$$

(b) The negative cone of G is denoted by G^- and is defined as,

$$G^- = \{x \in G : \mu(x,e) > 0\} \tag{9}$$

THEOREM 3.15. *(i) Let $(G,+,\mu)$ be a $fpo-group$ with positive cone $G^+ = P$. Then*
(a) $P + P \subseteq P$
(b) $-g + P + g \subseteq P$ for all $g \in G$
(c) $P \cap (-P) = \{e\}$
(ii) Let $(G,+)$ be a group. Let $P \subseteq G$ satisfy $(a),(b)$, and (c). Define $\mu : G \times G \longrightarrow [0,1]$ by

$$\mu(x,y) = \begin{cases} 1 & \text{if } \ y - x \in P \\ 0 & \text{otherwise} \end{cases} \tag{10}$$

Then $(G,+,\mu)$ is a $fpo-group$ and $G^+ = P$.

Proof. (i) Suppose $(G,+,\mu)$ is a $fpo-group$ with a positive cone $G^+ = P$.
(a) Let $x, y \in P$. Then $\mu(e,x) > 0$, and $\mu(e,y) > 0$. Hence, $\mu(y,x+y) > 0$ and by transitivity of μ, we have $\mu(e,x+y) > 0$. Thus, $x + y \in P$, and therefore $P + P \subseteq P$.
(b) Let $g \in G$ and $x \in P$. Thus, $\mu(e,x) > 0$ and then $\mu(-g+e+g, -g+x+g) > 0$ for all $g \in G$. Thus, $\mu(e, -g+x+g) > 0$ and implies $-g+x+g \in P$. Therefore, $-g + P + g \subseteq P$ for all $g \in G$.
(c) Since $\mu(e,e) > 0$, we have $e \in P \cap (-P)$. Again, let $x \in P \cap (-P)$. Then $x \in P$, and $x \in (-P)$, and hence $\mu(e,x) > 0$ and $\mu(x,e) > 0$. Thus, by antisymmetric of μ, we have $x = e$

(ii) Let $(G,+)$ be a group. Let $P \subseteq G$ satisfy $(a),(b)$, and (c).
(a) Let $x \in G$. Since $x - x = e \in P$, then $\mu(x,x) = 1$, and hence μ is reflexive.
(b) Let $x, y \in G$ be such that $\mu(x,y) > 0$ and $\mu(y,x) > 0$. Then $\mu(x,y) = 1$ and $\mu(y,x) = 1$. Hence, $y - x \in P$ and $x - y \in P$. Then $y - x \in P \cap (-P)$ and by (c) above, we have $y - x = e$. Hence, $x = y$, and therefore, μ is antisymmetric.
(c) Let $x, y \in G$. If $\mu(x,y) = 1$, then $\mu(x,y) \geq \sup_{z \in G}(\min(\mu(x,z), \mu(z,y)))$. Suppose $\mu(x,y) = 0$, then $y - x \notin P$. Then, for all $z \in G$ we have $y - z \notin P$ or $z - x \notin P$. Thus, $\mu(z,y) = 0$ or $\mu(x,z) = 0$. Thus, $\mu(x,y) \geq \sup_{z \in G}(\min(\mu(x,z), \mu(z,y))$. Therefore, μ is transitive, and hence (G,μ) is a fuzzy poset.

(d) Let $x, y \in G$ be such that $\mu(x, y) > 0$. Then, $\mu(x, y) = 1$, and by the definition of μ, we have $y - x \in P$. Now for all $a, b \in G$,

$$\begin{aligned}(a+y+b)-(a+x+b) =& a+y+b-b-x-a\\ =& a+y-x-a\\ \in& a+P+-a \subseteq P\end{aligned}$$

Thus, $\mu(a+x+b, a+y+b) = 1$ which shows $\mu(x, y) \leq \mu(a+x+b, a+y+b)$ and hence $(G, +, \mu)$ is a fpo-group.

(e) To show $G^+ = P$, let $x \in G^+$. Then $\mu(e, x) > 0$, and hence, $\mu(e, x) = 1$. Then $x = x - e \in P$, and thus $G^+ \subseteq P$. Again, let $x \in P$, then $x - e \in P$. Therefore, $\mu(e, x) = 1 > 0$. Then $x \in G^+$, which implies $P \subseteq G^+$. Therefore, $G^+ = P$.

□

Now we give and prove some fundamental properties of an fl-group in the following theorem:

THEOREM 3.16. *Let G be $fl-group$ and $a, b, x, y \in G$. Then the following are true:*

(i) $a + (x \vee_F y) + b = (a + x + b) \vee_F (a + y + b)$
(ii) $a + (x \wedge_F y) + b = (a + x + b) \wedge_F (a + y + b)$
(iii) $-(x \wedge_F y) = -x \vee_F -y$
(iv) $-(x \vee_F y) = -x \wedge_F -y$

Proof. (i) Since $\mu(x, x \vee_F y) > 0$ and $\mu(y, x \vee_F y) > 0$, we have $\mu(a+x, a+(x \vee_F y)) > 0$ and $\mu(a+y, a+(x \vee_F y)) > 0$. This implies that $a+(x \vee_F y)$ is the upper bound of $\{a + x, a + y\}$. Now let z be upper bound of the set $\{a + x, a + y\}$. That is, $\mu(a+x, z) > 0$ and $\mu(a+y, z) > 0$, and we obtain $\mu(x, -a+z) > 0$ and $\mu(y, -a+z) > 0$, which implies that $-a+z$ is the upper bound of the set $\{x, y\}$. Thus, $\mu(x \vee_F y, -a + z) > 0$ and then $\mu(a + (x \vee_F y), z) > 0$, which shows that $a+(x \vee_F y)$ is the least upper bound of $\{a+x, a+y\}$. Therefore, $a+(x \vee_F y) = (a+x) \vee_F (a+y)$. Similarly, we can show that $(x \vee_F y)+b = (x+b) \vee_F (y+b)$. Now,

$$\begin{aligned}a + (x \vee_F y) + b &= [a + (x \vee_F y)] + b\\ &= [(a + x) \vee_F (a + y)] + b\\ &= (a + x + b) \vee_F (a + y + b)\end{aligned}$$

(ii) Similar to (i).

(iii) Since $\mu(x \wedge_F y, x) > 0$ and $\mu(x \wedge_F y, y) > 0$, we have $\mu(-x, -(x \wedge_F y) > 0$ and $\mu(-y, -(x \wedge_F y) > 0$, which implies that $-(x \wedge_F y)$ is the upper bound of the set $\{-x, -y\}$. Now, let $z \in G$ be the upper bound of the set $\{-x, -y\}$. Thus, $\mu(-x, z) > 0$ and $\mu(-y, z) > 0$ and then $\mu(-z, x) > 0$ and $\mu(-z, y) > 0$. Which implies that $-z \in G$ is the lower bound of the set $\{x, y\}$. Thus, $\mu(-z, x \wedge_F y) > 0$ and therefore $\mu(-(x \wedge_F y), z) > 0$. This shows that $-(x \wedge y)$ is the least upper bound of the set $\{-x, -y\}$. Therefore, $-(x \wedge_F y) = -x \vee_F -y$.

(iv) Similar to (iii).

□

THEOREM 3.17. *Let G be $fpo-group$. Then we have the following:*

(i) G is totally ordered if and only if $G = G^+ \cup (-G^+)$.
(ii) G is directed if and only if G is directed above (or G is directed below).
(iii) G is directed if and only if G^+ generates G (i.e., $G = G^+ + (-G^+)$)).
(iv) G is an fl-group if and only if $x \vee_F e$ exists $\forall x \in G$ (or $x \wedge_F e$ exists $\forall x \in G$).
(v) G is an fl-group if and only if G^+ is a fuzzy lattice and G^+ generates G.

Proof. (i) Suppose G is totally ordered. Since $G^+ \subseteq G$, and $-G^+ \subseteq G$, then $G^+ \cup (-G^+) \subseteq G$. Again, let $x \in G$. Since G is a totally ordered, either $\mu(e, x) > 0$ or $\mu(x, e) > 0$. Thus, $x \in G^+$ or $x \in -G^+$ and hence $x \in G^+ \cup (-G^+)$. Then $G \subseteq G^+ \cup (-G^+)$. Therefore, $G = G^+ \cup (-G^+)$. Conversely, suppose $G = G^+ \cup (-G^+)$. Let $x, y \in G$. Then $x - y \in G$, and hence $x - y \in G^+$ or $x - y \in -G^+$. Then, $\mu(e, x - y) > 0$ or $\mu(x - y, e) > 0$ and thus $\mu(y, x) > 0$ or $\mu(x, y) > 0$. This shows that G is totally ordered.

(ii) Suppose G is directed; then G is directed above. Conversely, suppose G is directed above. Let $x, y \in G$, then $-x, -y \in G$. Since G is directed above, there exists $z \in G$ such that $\mu(-x, z) > 0$ and $\mu(-y, z) > 0$. Then we have $\mu(-z, x) > 0$ and $\mu(-z, y) > 0$. Therefore, G is directed below, and hence G is directed.

(iii) Suppose G is directed. Let $x \in G$, then there exist $z \in G$ such that $\mu(z, x) > 0$ and $\mu(z, e) > 0$. Then $x - z \in G^+$ and $z \in -G^+$. Since $x = (x - z) + z \in G^+ + -(G^+)$. Then $G \subseteq G^+ + -(G^+)$, and hence $G = G^+ + -(G^+)$. Conversely, Suppose $G = G^+ + (-G^+)$, and let $x, y \in G$. Then $x - y \in G^+ + (-G^+)$. Therefore, $x - y = a - b$ for $a, b \in G^+$. Since, $\mu(e, b) > 0$, then $\mu(a - b, a) > 0$ and hence $\mu(x - y, a) > 0$. Thus, $\mu(x, a + y) > 0$. Again, since $\mu(e, a) > 0$, then $\mu(y, a + y) > 0$. Therefore, G is directed above, and by the above result, G is directed.

(iv) Suppose G is an fl-group. Then, $x \vee_F e \in G$ for all $x \in G$. Conversely, suppose $x \vee_F e \in G$ for all $x \in G$. Now let $x, y \in G$. Then, $x - y \in G$, and $(x-y) \vee_F e = e \vee_F (x-y)$ exists in G. Then $(x-x) \vee_F (x-y) = x + (-x \vee_F -y) \in G$. Thus, $z = x + (-(x \wedge_F y)) \in G$, and then $-x + z = -(x \wedge_F y) \in G$, and also $x \wedge_F y \in G$. Again, let $x, y \in G$, then $-x, -y \in G$. Thus, $-x \wedge_F -y \in G$, and then $-(x \vee_F y) \in G$ and hence $x \vee_F y \in G$. Therefore, (G, μ) is a fuzzy lattice, and hence $(G, +, \mu)$ is fl-group.

(v) Suppose G is fl-group. Let $x, y \in G^+$. Then $\mu(e, x) > 0$ and $\mu(e, y) > 0$. Thus, $\mu(e, x \wedge_F y) > 0$ and $\mu(e, x \vee_F y) > 0$. This implies $x \wedge_F y \in G^+$ and $x \vee_F y \in G^+$. Therefore, G^+ is fuzzy lattice. In addition, since G is fl-group, then G is a fuzzy lattice, and since every fuzzy lattice is directed then by above (iii), G^+ generates G. Conversely, suppose G^+ is a fuzzy lattice and G^+ generates G. Now, let $x \in G$. Then $x \in G^+ - G^+$. Thus, $x = a - b$ for $a, b \in G^+$. Let $g = \sup_{G^+}\{a, b\}$. Therefore, $g \in G^+$ is the least upper bound of $\{a, b\}$ in G^+. Again, since $\mu(a, g) > 0$ and $\mu(b, g) > 0$, then $g \in G$ is the upper bound of $\{a, b\}$ in G. Again, let $h \in G$ be any upper bound of $\{a, b\}$ in G. Thus, $\mu(a, h) > 0$ and $\mu(b, h) > 0$. Since $\mu(e, a) > 0$ then $\mu(e, h) > 0$ and hence $h \in G^+$ and hence $h \in G^+$ is the upper bound of $\{a, b\}$ in G^+. Thus, $\mu(g, h) > 0$ and hence g is least upper bound of $\{a, b\}$ in G. Therefore, $a \vee_F b$ exists and is equal to $g \in G$. Now consider,

$$x \vee_F e = (a - b) \vee_F (b - b) = (a \vee_F b) + (-b) = g + (-b) = g - b \in G$$

Thus, $x \vee_F e$ exists in G for all $x \in G$. Thus, by (iv) above G is the fl-group.

□

In the following theorems, we will see the characterization of fl- groups in terms of their level subsets and the support.

THEOREM 3.18. *Let μ be a fuzzy relation on G. If $(G, +, \mu_\alpha)$ is an l-group for all $\alpha \in (0, 1]$. Then $(G, +, \mu)$ is an fl-group.*

Proof. Suppose $(G, +, \mu_\alpha)$ is an l-group for all $\alpha \in (0, 1]$. Then (G, μ_α) is a lattice for all $\alpha \in (0, 1]$ and $(G, +)$ is a group. Then, by Theorem (2.29), we see that (G, μ) is a fuzzy lattice. Now let $x, y, a, b \in G$ be such that $\mu(x, y) > 0$. Suppose $\mu(x, y) = \alpha > 0$, then $(x, y) \in \mu_\alpha$. Since $(G, +, \mu_\alpha)$ is an l-group, then $(a+x+b, a+y+b) \in \mu_\alpha$. Thus $\mu(a + x + b, a + y + b) \geq \alpha$ and hence $\mu(a + x + b, a + y + b) \geq \mu(x, y)$. Therefore, $(G, +, \mu)$ is a fl-group. □

THEOREM 3.19. *Let μ be a fuzzy relation on G, and $S(\mu)$ be the support of the fuzzy relation on G. If $(G, +, \mu)$ is fl-group, then $(G, +, S(\mu))$ is an l-group.*

Proof. Suppose that $(G, +, \mu)$ is the fl-group. Then $(G, +)$ is a group, and (G, μ) is a fuzzy lattice. Then, by Theorem (2.32), $(G, S(\mu))$ is a lattice on G. Now let $x, y, a, b \in G$ be such that $(x, y) \in S(\mu)$. Then $\mu(x, y) > 0$, and since $(G, +, \mu)$ is a fl-group we have $\mu(a+x+b, a+y+b) > 0$, which implies $(a+x+b, a+y+b) \in S(\mu)$. Therefore, $(G, +, S(\mu))$ is an l group. □

THEOREM 3.20. *Let G be an fl-group. Then G is an infinitely distributive fuzzy lattice, and hence G is a distributive fuzzy lattice.*

Proof. Let $\{x_\alpha\}_{\alpha \in \Delta}$ and $a \in G$. Suppose $\vee_\alpha x_\alpha \in G$ and say $x = \vee_\alpha x_\alpha$. Then for all $\alpha \in \Delta$, $\mu(x_\alpha, x) > 0$ and hence $\mu(e, x - x_\alpha) > 0$. This implies $\mu(a, x - x_\alpha + a) > 0$ for all $\alpha \in \Delta$ Then

$$\mu(a \wedge_F x, (x - x_\alpha + a) \wedge_F x) > 0 \tag{11}$$

So,

$$\mu(a \wedge_F x, (x - x_\alpha) + (a \wedge_F x_\alpha)) > 0 \tag{12}$$

Then

$$\mu((a \wedge_F x) - (a \wedge_F x_\alpha), (x - x_\alpha)) > 0 \quad \text{for } \alpha \in \Delta \tag{13}$$

And so

$$\mu(\wedge_{\alpha\in\Delta}[(a \wedge_F x) - (a \wedge_F x_\alpha)], \wedge_{\alpha\in\Delta}(x - x_\alpha)) > 0 \tag{14}$$

Since, $\mu(x_\alpha, x) > 0$ for all $\alpha \in \Delta$. Then $\mu(a \wedge_F x_\alpha, a \wedge_F x) > 0$, and so $\mu(e, (a \wedge_F x) - (a \wedge_F x_\alpha)) > 0$. Therefore,

$$\mu(e, \wedge_{\alpha\in\Delta}[(a \wedge_F x) - (a \wedge_F x_\alpha)]) > 0 \tag{15}$$

Again since,

$$e = x - x = x - \vee_{F_\alpha} x_\alpha = \wedge_\alpha (x - x_\alpha)$$

Then from Equations 14 we have,

$$\mu(\wedge_{\alpha\in\Delta}[(a \wedge_F x) - (a \wedge_F x_\alpha), e) > 0 \tag{16}$$

Hence, by equations 15, 16, and by antisymmetric of μ, we have,

$$\wedge_{\alpha\in\Delta}[(a \wedge_F x) - (a \wedge_F x_\alpha)] = e$$

Thus,

(17) $$(a \wedge_F x) - \vee_{\alpha\in\Delta}(a \wedge_F x_\alpha) = e$$

And also,

(18) $$a \wedge_F x = \vee_{\alpha\in\Delta}(a \wedge_F x_\alpha)$$

Then $a \wedge_F (\vee_{F_{\alpha\in\Delta}} x_\alpha) = \vee_{F_{\alpha\in\Delta}}(a \wedge_F x_\alpha)$. Thus, G is infinitely meet distributive fuzzy lattice. Hence, it is an infinitely distributive fuzzy lattice and, therefore, it is a distributive fuzzy lattice. □

Definition 3.21. Let G be an fl-group and $x \in G$. Then

(i) The positive part of x, denoted by x^+, is defined as $x^+ = x \vee_F e$.
(ii) The negative part of x, denoted by x^-, defined as $x^- = -x \vee_F e$.
(iii) The absolute value of x, denoted by $|x|$, is defined as $|x| = x \vee_F (-x)$.

Now, we state the following theorems without proof, as they are easy to prove.

Theorem 3.22. *Let G be an fl-group. Then*

(a) $x - (x \wedge_F y) + y = x \vee_F y$.
(b) $|x - y| = (x \vee_F y) - (x \wedge_F y)$.
(c) $x = x^+ - x^-$.
(d) $|x| = x^+ + x^-$ *and* $x + |x| = |x| + x$.
(e) $x^+ \wedge_F x^- = e$ *and hence* $x^+ + x^- = x^- + x^+$.
(f) If x, y *are disjoint, then* $\{x^+, x^-, y^+, y^-\}$ *are mutually disjoint and mutually commute.*

Theorem 3.23. *Let G be a fl-group. Then the following are equivalent:*

(i) $x \wedge_F y = e$.
(ii) $x + y = x \vee_F y$.
(iii) $(x - y)^+ = x$ *and* $(x - y)^- = y$.

Theorem 3.24. *Let G be a fl-group. Then we have the following:*

(a) $nx^+ \wedge_F nx^- = e$ *for* $n \in \mathbb{N}$.
(b) $(nx)^+ = nx^+$.
(c) If $(x - y)^+ + y = x \vee_F y$.
(d) In addition; if x and y commute, then $n(x \vee_F y) = nx \vee_F ny$ *and* $n(x \wedge_F y) = nx \wedge_F ny$.

4. fl-subgroups of Fuzzy Lattice Ordered Groups

In this section, we introduce the notion of fl-subgroups of an fl-groups and study their related properties. We also provide several characterizations of the quotient of fl-subgroups of fl-groups and obtain fuzzy analogs results of the classical theory of lattice-ordered groups.

Definition 4.1. Let $(G, +, \mu)$ be an fl-group and $S \subseteq G$. Then $(S, +, \mu)$ is said to be a fl-subgroup of G if

(a) S is sub group of G.
(b) S is a fuzzy sublattice of G.

It is easy to prove the following result:

Lemma 4.2. *Let $(G, +, \mu)$ be an fl-group. Suppose $\{G_i\}_{i\in I}$ is a family of fl-subgroups of G. Then $\cap_{i\in I} G_i$ is the fl-subgroup of G. Thus, the set of fl subgroups of G forms a complete lattice under set inclusion.*

Remark 4.3. The union of fl-subgroups may not be a fl-subgroup.

Definition 4.4. Let (G, μ) be a fuzzy poset. Let $C \subseteq G$. Then C is said to be a a fuzzy convex subset of G with respect to a fuzzy relation μ. If $x, y \in C$ and $z \in G$ such that $\mu(x, z) > 0$ and $\mu(z, y) > 0$. Then $z \in C$.

Definition 4.5. Let $(G, +, \mu)$ be an fl- group. Let $(C, +, \mu)$ be an fl subgroup of $(G, +, \mu)$. Then C is said to be a fuzzy convex fl-subgroup of G if C is a fuzzy convex set in G with respect to μ.

Theorem 4.6. *Let G be an fl-group and S be a subgroup of G. Then S is fl-subgroup of G if and only if $x \in S \Rightarrow x^+ \in S$.*

Proof. Suppose S is fl-subgroup of G. Let $x \in S$. Since $e \in S$, then $x^+ = x \vee e \in S$. Conversely, Suppose $x^+ \in S$ for all $x \in S$.

(a) Let $x, y \in S$. Then $x - y \in S$ and hence $(x - y)^+ \in S$.
$\Longrightarrow (x - y)^+ + y \in S$
$\Longrightarrow ((x - y) \vee_F e) + y \in S$
$\Longrightarrow ((x - y) \vee_F (y - y)) + y \in S$
$\Longrightarrow ((x \vee_F y) + -y) + y \in S$
$\Longrightarrow x \vee_F y \in S$

(b) Let $x, y \in S$. Then
$\Longrightarrow -x, -y \in S$
$\Longrightarrow -x \vee_F -y \in S$
$\Longrightarrow -(x \wedge_F y) \in S$
$\Longrightarrow x \wedge_F y \in S$

From (a) and (b), S is a fuzzy sublattice of G. Therefore, S is a fl-subgroup of G. □

Theorem 4.7. *Let $(G, +, \mu)$ be an fpo- group and C be subgroup of G. Define the set G/C called the quotient of G by C by ;*

$$G/C = \{x + C : x \in G\}$$

And also define a fuzzy relation $\mu_C : G/C \times G/C \longrightarrow [0, 1]$ by,

$$\mu_C(x + C, y + C) = \sup\{\mu(x, y + c) : c \in C\} \tag{19}$$

Then $(G/C, \mu_C)$ is a fuzzy poset if and only if C is a fuzzy convex set on G with respect to μ.

Proof. Suppose $(G/C, \mu_C)$ is a fuzzy poset. Let $x, y \in C$ and $a \in G$ be such that $\mu(x, a) > 0$ and $\mu(a, y) > 0$. Then, we have $\mu_C(x + C, a + C) > 0$ and $\mu_C(a + C, y + C) > 0$. Thus, $\mu_C(C, a + C) > 0$ and $\mu_C(a + C, C) > 0$. Then by antisymmetric of μ_C, we have $C = a + C$, and then $a \in C$. Therefore, C is a fuzzy convex set.

Conversely, suppose C is a fuzzy convex set with respect to μ

(a) Let $x + C \in G/C$ for $x \in G$. Since, $1 = \mu(x, x) \le \mu_C(x + C, x + C)$. Then μ_C is reflexive.

(b) Let $x+C, y+C \in G/C$ such that $\mu_C(x+C, y+C) > 0$ and $\mu_C(y+C, x+C) > 0$. Then $\mu(x, y+c) > 0$ and $\mu(y, x+d) > 0$ for some $c, d \in C$. And also $\mu(-y+x, c) > 0$ and $\mu(-d, -y+x) > 0$. Since C is fuzzy convex and $c, -d \in C$, we have $-y+x \in C$. So, $x+C = y+C$. This shows that μ_C is antisymmetric.

(c) Let $x+C, y+C, z+C \in G/C$ for $x, y, z \in G$. Then

$$\begin{aligned}
\mu_C(x+C, z+C) &= \sup\{\mu(x, z+c) : c \in C\} \\
&\geq \sup\{\min(\mu(x, y+d), \mu(y+d, z+c)) : c, d \in C\} \\
&= \min(\sup\{\mu(x, y+d) : d \in C\}, \sup\{\mu(y+d, z+c) : c, d \in C\}) \\
&= \min(\sup\{\mu(x, y+d) : d \in C\}, \sup\{\mu(y, z+c-d) : c, d \in C\}) \\
&= \min(\sup\{\mu(x, y+d) : d \in C\}, \sup\{\mu(y, z+k) : k \in C\}) \\
&= \min(\mu_C(x+C, y+C), \mu_C(y+C, z+C)) \\
\mu_C(x+C, z+C) &\geq \min(\mu_C(x+C, y+C), \mu_C(y+C, z+C))
\end{aligned}$$

Thus, $\mu_C(x+C, z+C) \geq \sup_{y+C \in G/C}\{\min(\mu_C(x+C, y+C), \mu_C(y+C, z+C))\}$, and then, μ_C is transitive.

Therefore, $(G/C, \mu_C)$ is a fuzzy poset.

□

Theorem 4.8. *Let G be an fl-group. Let C be a fuzzy convex subgroup of G. Then the following are equivalent.*

(a) C is fl-subgroup of G.

(b) G/C is fuzzy lattice. Furthermore, G/C is a distributive fuzzy lattice and the map $\psi : G \longrightarrow G/C$ defined by $\psi(x) = x + C$ is a fuzzy lattice homomorphism.

Proof. $(a) \Rightarrow (b)$: Suppose that C is an fl-subgroup of G. Since C is a fuzzy convex subgroup of G, by the above theorem we see that $(G/C, \mu_C)$ is a fuzzy poset. Let $x+C, y+C \in G/C$ for $x, y \in G$. Since $\mu(x, x \vee_F y) > 0$ and $\mu(y, x \vee_F y) > 0$, then by the definition of μ_C, we have $\mu_C(x+C, (x \vee_F y)+C) > 0$ and $\mu_C(y+C, (x \vee_F y)+C) > 0$. Thus, $(x \vee_F y)+C$ is the upper bound of the set $\{x+C, y+C\}$ in G/C. Again, let $z+C$ be an upper bound of the set $\{x+C, y+C\}$ in G/C. Then $\mu_C(x+C, z+C) > 0$ and $\mu_C(y+C, z+C) > 0$, hence $\mu(x, z+c) > 0$ and $\mu(y, z+d) > 0$ for some $c, d \in C$. Then $\mu(x \vee_F y, (z+c) \vee (z+d)) > 0$ and hence $\mu(x \vee_F y, z+(c \vee_F d)) > 0$. Since $c \vee_F d \in C$. Then we have $\mu_C((x \vee_F y)+C, z+C) > 0$. This shows that $(x \vee_F y)+C$ is the least upper bound of $\{x+C, y+C\}$. Therefore, $(x+C) \vee (y+C) = (x \vee_F y)+C$. Similarly, $(x+C) \wedge (y+C) = (x \wedge_F y)+C$. Hence, G/C is a fuzzy lattice. Again, since G is fl-group. G is a distributive fuzzy lattice, and hence G/C is also a distributive fuzzy lattice.

Now, define a map: $\psi : G \longrightarrow G/C$ by $\psi(x) = x+C$. Let $x, y \in G$. Then

$$\psi(x \vee_F y) = (x \vee_F y) + C = (x+C) \vee (y+C) = \psi(x) \vee \psi(y)$$

Similarly, $\psi(x \wedge_F y) = \psi(x) \wedge \psi(y)$. Therefore, ψ is a fuzzy lattice homomorphism.

$(b) \Rightarrow (a)$: Suppose G/C is fuzzy lattice. Now, let $x \in C$. Then

$$x^+ + C = (x \vee_F e) + C = (x+C) \vee (e+C) = C \vee C = C$$

Therefore, $x^+ \in C$ and therefore C is the fl subgroup of G.

□

THEOREM 4.9. *(a) Let $(G, +, \mu)$ be a fpo-group and C be a fuzzy convex normal subgroup of G. Then $(G/C, +, \mu_C)$ is fpo-group.*

(b) Let $(G, +, \mu)$ be a fl-group and C be a fuzzy convex normal fl-subgroup of G. Then $(G/C, +, \mu_C)$ is fl-group.

Proof. (a) Let $(G, +, \mu)$ be a fpo-group and C be a fuzzy convex normal subgroup of G. Then $(G/C, +)$ is a group and also $(G/C, \mu_C)$ is a fuzzy poset. Now, let $x + C, y + C, a + C, b + C \in G/C$. Then

$$\begin{aligned}\mu_C(x + C, y + C) &= \sup\{\mu(x, y + d) : d \in C\}\\ &\leq \sup\{\mu(a + x + b, a + y + d + b) : d \in C\}\\ &= \sup\{\mu(a + x + b, a + y + b - b + d + b) : d \in C\}\\ &= \mu_C(a + x + b + C, a + y + b + C) \ \text{(Since } C \text{ is normal, } -b + d + b \in C)\\ &= \mu_C((a + C) + (x + C) + (b + C), (a + C) + (y + C) + (b + C))\end{aligned}$$

Thus, μ_C is translation invariant. Therefore, $(G/C, +, \mu_C)$ is the fpo-group.

(b) Since C is fuzzy convex normal subgroup of G, by (a) above we have $(G/C, +, \mu_C)$ is fpo-group. Again, since C is a fl subgroup of G according to Theorem (4.8), G/C is a fuzzy lattice. Therefore, $(G/C, +, \mu_C)$ is fl-group.

□

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