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FACTORIZATION PROPERTIES ON THE COMPOSITE HURWITZ RINGS

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Abstract. Let $A \subseteq B$ be an extension of integral domains with characteristic zero. Let $H(A, B)$ and $h(A, B)$ be rings of composite Hurwitz series and composite Hurwitz polynomials, respectively. We simply call $H(A, B)$ and $h(A, B)$ composite Hurwitz rings of A and B. In this paper, we study when $H(A, B)$ and $h(A, B)$ are unique factorization domains (resp., GCD-domains, finite factorization domains, bounded factorization domains).

1. Introduction

Let R be an integral domain with quotient field K. The study of factorization in R has been significant attention in commutative algebra and semigroup theory. The classical situation is when R is a unique factorization domain (UFD), that is, when every nonzero nonunit of R is a finite product of irreducible elements of R, uniquely up to order and associates. In [1], Anderson et al. introduced several classes of integral domains satisfying conditions weaker than unique factorization. The factorizations have been studied extensively and there are many excellent results (see [9, 11, 22, 23] for UFD and [1–4, 8, 10, 13] for weaker than unique factorization).

We first introduce the various factorizations in [1] that we will study here. Following Cohn [8], we say that R is *atomic* if every nonzero nonunit of R is a product of a finite number of irreducible elements of R. We say that R satisfies the *ascending chain condition on principal ideals* (ACCP) if there does not exist an infinite strictly ascending chain of principal ideals of R. It is well known that any domain which satisfies ACCP is atomic. However, the converse is not true; atomic domain that does not satisfy ACCP was first constructed in [13]. According to Anderson et al. [1], we say that R is a *bounded factorization domain* (BFD) if R is atomic and for each nonzero nonunit of R there is a bound on the length of factorizations into products of irreducible elements, and a *finite factorization domain* (FFD) if R is atomic and each nonzero element of R has at most a finite number of nonassociate irreducible divisors.

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It is clear that UFDs are FFDs and that FFDs are BFDs. In general, we have the following:

$$\text{UFD} \Longrightarrow \text{FFD} \Longrightarrow \text{BFD} \Longrightarrow \text{ACCP} \Longrightarrow \text{atomic domain.}$$

Let R be a commutative ring with identity and $H(R)$ the set of formal expressions of the form $\sum_{n=0}^{\infty} a_n X^n$, where $a_n \in R$. Define addition and $*$-product on $H(R)$ as follows: For $f = \sum_{n=0}^{\infty} a_n X^n, g = \sum_{n=0}^{\infty} b_n X^n \in H(R)$,

$$f + g = \sum_{n=0}^{\infty} (a_n + b_n) X^n \text{ and } f * g = \sum_{n=0}^{\infty} c_n X^n,$$

where $c_n = \sum_{k=0}^{n} \binom{n}{k} a_k b_{n-k}$ and $\binom{n}{k} = \frac{n!}{(n-k)!k!}$ for nonnegative integers $n \geq k$. Under these two operations, $H(R)$ becomes a commutative ring with identity containing R [17]. In [18], the ring $H(R)$ is called a *ring of Hurwitz series* over R. The *ring of Hurwitz polynomials* $h(R)$ over R is the subring of $H(R)$ consisting of formal expressions of the form $\sum_{k=0}^{n} a_k X^k$. We simply call $H(R)$ and $h(R)$ the Hurwitz rings over R.

For an extension $A \subseteq B$ of commutative rings with identity, consider the sets $H(A, B) := \{f \in H(B) \mid \text{the constant term of } f \text{ belongs to } A\}$ and $h(A, B) := \{f \in h(B) \mid \text{the constant term of } f \text{ belongs to } A\}$. Then it is easy to see that $H(A, B)$ and $h(A, B)$ are subrings of $H(B)$ and $h(B)$, respectively. We call $H(A, B)$ (resp., $h(A, B)$) a *ring of composite Hurwitz series* (resp., *ring of composite Hurwitz polynomial*). We simply call $H(A, B)$ and $h(A, B)$ composite Hurwitz rings of A and B. For more information on (composite) Hurwitz rings, the readers can refer to [6, 7, 19–21].

It is known in [1, Proposition 2.2 and Theorem 5.1] that Noetherian domains and Krull domains are BFDs and FFDs, respectively. So the rings of polynomials and formal power series over a Noetherian domain (resp., Krull domain) are also BFDs (resp., FFDs). On the other hand, Hurwitz rings over a Noetherian domain (resp., Krull domain) need not be Noetherian domains (resp. Krull domains); it is known that for an integral domain R, $h(R)$ (resp., $H(R)$) is a Noetherian domain if and only if R is a Noetherian domain containing $\mathbb{Q}$ [7, Corollary 7.7], and $h(R)$ is a Krull domain if and only if R is a Krull domain containing $\mathbb{Q}$ [24, Theorem 4.5]. For example, $h(\mathbb{Z})$ and $H(\mathbb{Z})$ are neither Noetherian domains nor Krull domains. It is also known in [19, Theorem 2.4] that for an integral domain R with characteristic zero, R satisfies ACCP if and only if $h(R)$ (resp., $H(R)$) satisfies ACCP. Hence, the Hurwitz rings over a Noetherian domian (resp., Krull domain) with characteristic zero satisfy ACCP, so are atomic domains.

In this paper, we study the investigation of various factorization properties in the (composite) Hurwitz rings. In Section 2, we investigate conditions for (composite) Hurwitz rings to be (completely) integrally closed, and then study necessary and sufficient conditions for such rings to be UFDs (resp., GCD-domains, Krull domains). In Section 3, we give necessary and sufficient conditions for (composite) Hurwitz rings to be BFDs or FFDs.

For an integral domain R, let R^* denote its set of nonzero elements, $U(R)$ its group of units, and $R[\![X]\!]$ (resp.,$R[X]$) the ring of formal power series (resp., polynomials) over R. Throughout, $\mathbb{N}_0$, $\mathbb{Z}$, and $\mathbb{Q}$ denote the nonnegative integers, integers, and rational numbers, respectively. General references for any undefined terminology or notation are [12, 16].

2. Unique factorization domains

Let $A \subseteq B$ be an extension of commutative rings with identity. In this section, we determine the conditions for composite Hurwitz rings $H(A,B)$ and $h(A,B)$ to be (completely) integrally closed domains, and then characterize when $H(A,B)$ and $h(A,B)$ are UFDs (resp., GCD-domains, Krull domains).

We start with recalling the known results on composite Hurwitz rings which will be needed in the sequel. It is known in [6, Proposition 1.1] that for a commutative ring R with identity, $H(R)$ (resp., $h(R)$) is an integral domain if and only if R is an integral domain with characteristic zero. The following is a simple observation when composite Hurwitz rings are integral domains.

Lemma 2.1. *Let $A \subseteq B$ be an extension of commutative rings with identity. Then $H(A,B)$ (resp., $h(A,B)$) is an integral domain if and only if A and B are integral domains with characteristic zero.*

For a commutative ring R with identity, the mapping $\psi : R[\![X]\!] \to H(R)$ (resp., $\phi : R[X] \to h(R)$) defined by

$$\psi\left(\sum_{n=0}^{\infty} a_n X^n\right) = \sum_{n=0}^{\infty} n! a_n X^n \text{ (resp., } \phi\left(\sum_{k=0}^{n} a_k X^k\right) = \sum_{k=0}^{n} k! a_k X^k)$$

is a ring homomorphism [18, Proposition 2.3]; and ψ is an isomorphism if and only if ϕ is an isomorphism, if and only if R contains $\mathbb{Q}$ ([18, Proposition 2.4] and [7, Theorem 1.4 and Corollary 1.5]). These are extended to composite Hurwitz rings as follows.

Lemma 2.2. [20, Lemma 2.1] *Let $A \subseteq B$ be an extension of commutative rings with identity. Then the following conditions are equivalent.*

(i) B contains $\mathbb{Q}$.

(ii) The mapping $\psi : A + XB[\![X]\!] \to H(A,B)$ defined by $\psi\left(\sum_{i=0}^{\infty} a_i X^i\right) = \sum_{i=0}^{\infty} i! a_i X^i$ is a ring isomorphism.

(iii) The mapping $\phi : A + XB[X] \to h(A,B)$ defined by $\phi\left(\sum_{i=0}^{n} a_i X^k\right) = \sum_{i=0}^{n} i! a_i X^k$ is a ring isomorphism.

We are now ready to study when composite Hurwitz rings $H(A,B)$ and $h(A,B)$ are (completely) integrally closed.

Theorem 2.3. *Let $A \subseteq B$ be an extension of integral domains with characteristic zero. Then the following statements hold.*

(i) If $h(A,B)$ (resp., $H(A,B)$) is integrally closed, then A is integrally closed and $\mathbb{Q} \subseteq B$.

(ii) If $h(A,B)$ (resp., $H(A,B)$) is completely integrally closed, then $A = B$ is completely integrally closed and $\mathbb{Q} \subseteq B$.

Proof. Let R denote either $h(A,B)$ or $H(A,B)$. Since A is an integral domain with characteristic zero, we may assume that $\mathbb{Z} \subseteq A$.

(i) Suppose that R is integrally closed. Then, obviously, A is integrally closed. Let p be a prime number.

We claim that $(\frac{1}{p}X^k) * (\frac{1}{p}X^k) = \frac{1}{p^2}\frac{(2k)!}{k!k!}X^{2k} \in R$ for some $k > 1$.
For each $n \geq 1$, let $w(n)$ be the largest power of p dividing $n!$. Then $w(n) = \sum_{1\leq l}[\frac{n}{p^l}]$. Let $k = (p-1)p^2 + (p-1)p + (p-1)(= p^3 - 1)$. Then $w(k) = p^2 + p - 2$. Since $2k = p^3 + (p-1)p^2 + (p-1)p + (p-2)$, we have $w(2k) = 2p^2 + 2p - 1$. Hence, $w(2k) - 2w(k) = 3 \geq 2$, and thus p^2 divides $\frac{(2k)!}{k!k!}$. Therefore, $(\frac{1}{p}X^k) * (\frac{1}{p}X^k) = \frac{1}{p^2}\frac{(2k)!}{k!k!}X^{2k} \in R$ for some $k > 1$.
By the claim, $\frac{1}{p}X^k$ for some $k > 1$ is integral over R, and hence it should be in R, i.e., $\frac{1}{p} \in B$. Therefore, p is a unit element of B. Since p is an arbitrary prime number, any nonzero integer n is a unit element of B. Thus B contains $\mathbb{Q}$.

(ii) Suppose that R is completely integrally closed. Clearly, it is integrally closed, and hence $\mathbb{Q} \subseteq B$. By Lemma 2.2, either $R \cong A + XB[X]$ or $R \cong A + XB[\![X]\!]$ is completely integrally closed. Note that if $A + XB[X]$ or $A + XB[\![X]\!]$ is completely integrally closed, then $A = B$ ($\because$ Suppose that $A + XB[X]$ (resp., $A + XB[\![X]\!]$) is completely integrally closed. Let K be the quotient field of $A + XB[X]$ (resp., $A + XB[\![X]\!]$). For $0 \neq b \in B$, $b = \frac{bX}{X} \in K$. Then $b^nX \in A + XB[X]$ (resp., $b^nX \in A + XB[\![X]\!]$) for all $n \geq 1$. Hence b is almost integral over $A + XB[X]$ (resp., $A + XB[\![X]\!]$). So, $b \in A$.) Therefore, $A = B$ is a completely integrally closed domain containing $\mathbb{Q}$.

$\square$

Note that UFDs and Krull domains are completely integrally closed. When $A \neq B$, composite Hurwitz rings $h(A, B)$ and $H(A, B)$ are neither UFDs nor Krull domains. It is well known that an integral domains R is a UFD (resp., Krull domain) if and only if $R[X]$ is a UFD (resp., Krull domain). By applying Theorem 2.3 to UFDs and Krull domains, we recover the following which are same as [24, Theorems 4.2 and 4.5].

Corollary 2.4. [24, Theorems 4.2 and 4.5] *Let A be an integral domain with characteristic zero. Then the following statements are equivalent.*

(i) $h(A)$ is a UFD (resp., Krull domain).
(ii) A is a UFD (resp., Krull domain) and $\mathbb{Q} \subseteq A$.
(iii) A is a UFD (resp., Krull domain) and $h(A) \cong A[X]$.

The next result concerns the ring of Hurwitz series analog of Corollary 2.4. We note that if $R[\![X]\!]$ is a UFD, then R is a UFD, but the converse is not true [11, Example 19.6]. We also note that $R[\![X]\!]$ is a Krull domain if and only if R is a Krull domain [11, Proposition 1.7].

Corollary 2.5. *Let A be an integral domain with characteristic zero. Then the following statements holds.*

(i) If $H(A)$ is a UFD, then A is a UFD containing $\mathbb{Q}$.
(ii) $H(A)$ is a Krull domain if and only if A is a Krull domain containing $\mathbb{Q}$ if and only if A is a Krull domain and $H(A) \cong A[\![X]\!]$.

Recall that a GCD-domain is an integral domain with the property that any two elements have a greatest common divisor, equivalently, the intersection of any two principal ideals is a principal ideal. We now determine the conditions when $h(A, B)$

(resp., $H(A,B)$) is a GCD-domain. Let R be an integral domain. A saturated multiplicative closed subset S of R is a splitting multiplicative set of R if for each $r \in R$, $r = as$ for some $a \in R$ and $s \in S$ such that $aR \cap tR = atR$ for all $t \in S$. It is known [5, Theorems 2.10 and 2.11] that for an extension $A \subseteq B$ of integral domains, (1) $A + XB[X]$ is a GCD-domain if and only if A is a GCD-domain and $B = A_S$ for a splitting multiplicative set of A, and (2) $A + XB[\![X]\!]$ is a GCD-domain if and only if A is a GCD-domain, $B = A_S$ for a splitting multiplicative set of A, and $B[\![X]\!](= A_S[\![X]\!])$ is a GCD-domain.

Proposition 2.6. *Let $A \subseteq B$ be an extension of integral domains with characteristic zero. Then the following statements hold.*

(i) $h(A,B)$ is a GCD-domain if and only if A is a GCD-domain and $B = A_S$, where S is a splitting multiplicative set of A containing all prime numbers.

(ii) $H(A,B)$ is a GCD-domain if and only if A is a GCD-domain, $B = A_S$, where S is a splitting multiplicative set of A containing all prime numbers, and $H(B)(= H(A_S))$ is a GCD-domain.

Proof. The proofs of (i) and (ii) are almost same. So we give a proof of (i). (i) ($\Rightarrow$) Since a GCD-domain is integrally closed, it follows from Lemma 2.2 and Theorem 2.3 that $h(A,B) \cong A + XB[X]$ is a GCD-domain and $\mathbb{Q} \subseteq B$. By [5, Theorem 2.10], A is a GCD-domain and $B = A_S$, where S is a splitting multiplicative set of A. Since $\mathbb{Q} \subseteq B = A_S$ and S is a saturated multiplicative set, S contains all prime numbers. ($\Leftarrow$) By [5, Theorem 2.10], $A + XB[X]$ is a GCD-domain. Since S is a multiplicative set of A containing all prime numbers, every prime number is a unit in $B = A_S$. Thus $\mathbb{Q} \subseteq B$. By Lemma 2.2, $h(A,B) \cong A + XB[X]$ is a GCD-doamin. $\square$

It is well known that an integral domain R is a GCD-domain if and only if $R[X]$ is a GCD-domain. A UFD is a GCD-domain with ACCP. We note that $R[\![X]\!]$ need not be a GCD-domain if R is a GCD-domain (for example, let R be a UFD such that $R[\![X]\!]$ is not a UFD [11, Example 19.6]). When $A = B$ in Proposition 2.6, we obtain

Corollary 2.7. *Let A be an integral domain with characteristic zero. Then the following assertions holds.*

(i) $h(A)$ is a GCD-domain if and only if A is a GCD-domain containing $\mathbb{Q}$ if and only if A is a GCD-domain and $h(A) \cong A[X]$.

(ii) If $H(A)$ is a GCD-domain, then A is a GCD-domain containing $\mathbb{Q}$.

It is known [2, Corollary 1.7] that every saturated multiplicative set of a UFD is a splitting set. We now give some examples.

Example 2.8. *1. The Hurwitz rings $H(\mathbb{Z})$ and $h(\mathbb{Z})$ are not UFDs. Since $\mathbb{Z} + X\mathbb{Q}[\![X]\!] \cong H(\mathbb{Z},\mathbb{Q})$ (resp., $\mathbb{Z} + X\mathbb{Q}[X] \cong h(\mathbb{Z},\mathbb{Q})$) under the mapping ψ (resp., ϕ) in Lemma 2.2, the Hurwitz rings $H(\mathbb{Z})$ and $h(\mathbb{Z})$ contain subrings (which are UFDs) of the form $\psi(\mathbb{Z}[\![X]\!]) = \{\sum_{n=0}^{\infty} a_n X^n \in H(\mathbb{Z}) \mid a_n \in n!\mathbb{Z}\}$ and $\phi(\mathbb{Z}[X]) = \{\sum_{k=0}^{n} a_k X^k \in h(\mathbb{Z}) \mid a_k \in k!\mathbb{Z}\}$, respectively.*

2. We note that each overring of a PID R is a quotient ring of R. Let $A \subseteq B$ be overrings of $\mathbb{Z}$. Then $A + XB[X]$ and $A + XB[\![X]\!]$ are GCD-domains [5, Theorems 2.10 and 2.11]. It follows from Proposition 2.6 that $h(A,B)$ and $H(A,B)$ are GCD-domains if and only if $B = \mathbb{Q}$.

3. *Let R be a UFD containing $\mathbb{Z}$ and S be a saturated multiplicative subset of R. Then $R + XR_S[X]$ is a GCD-domain [5, Theorem 2.10]. It follows from Proposition 2.6 that $h(R, R_S)$ is a GCD-domain if and only if $p \in S$ for every prime number p.*
4. *Let $R = \mathbb{Z} + Y\mathbb{Q}[Y]$. Then R is a GCD-domain. So $R[X]$ is a GCD-domain, but $h(R)$ is not a GCD-domain.*

3. Bounded and finite factorization domains

We recall that an integral domain R is a bounded factorization domain (BFD) if it is atomic and for each nonzero nonunit $x \in R$, there is a positive integer N such that whenever $x = x_1 \cdots x_n$ for irreducible elements $x_1, \ldots, x_n$ of R, then $n \leq N$. In [1], Anderson et al. introduced length functions and characterized BFDs in terms of the existence of length functions. We start with recalling characterization of BFDs with length functions. For an integral domain R and nonnegative integer $\mathbb{N}_0$, a function $l : R^* \to \mathbb{N}_0$ is called a *length* function of R if it satisfies the following two properties : $l(x) = 0$ if and only if $x \in U(R)$, and $l(xy) \geq l(x) + l(y)$ for any $x, y \in R^*$.

Lemma 3.1. [1, Theorem 2.4] *Let R be an integral domain. Then the following statements are equivalent.*

(i) R is a BFD.
(ii) For each nonzero nonunit $x \in R$, there is a positive integer N such that whenever $x = x_1 \cdots x_n$ with each x_i a nonunit of R, then $n \leq N$.
(iii) There is a length function $l : R^* \to \mathbb{N}_0$.

We now consider the units of (composite) Hurwitz rings. Let R be a commutative ring with identity. It is shown that (1) a Hurwitz series $f = \sum_{i=0}^{\infty} a_i X^i$ is a unit in $H(R)$ if and only if a_0 is a unit in R [18, Proposition 2.5], and (2) a Hurwitz polynomial $f = \sum_{i=0}^{n} a_i X^i$ is a unit in $h(R)$ if and only if a_0 is a unit in R and for each $i \geq 1$, a_i is nilpotent or some power of a_i is with torsion [7, Theorem 3.1]. In [19, Lemma 2.2], these are extended to composite Hurwitz rings as follows.

Lemma 3.2. [19, Lemma 2.2] *Let $A \subseteq B$ be an extension of commutative rings with identity. Then the following assertions hold.*

(i) A composite Hurwitz series $f = \sum_{i=0}^{\infty} a_i X^i$ is a unit in $H(A, B)$ if and only if a_0 is a unit in A.
(ii) A composite Hurwitz polynomial $f = \sum_{i=0}^{n} a_i X^i$ is a unit in $h(A, B)$ if and only if a_0 is a unit in A and for each $i \geq 1$, a_i is nilpotent or some power of a_i is with torsion.

We now study when composite Hurwitz rings $H(A, B)$ and $h(A, B)$ are BFDs. We need the following definition in [4]. Let $A \subseteq B$ be an extension of integral domains. We say that B is a *bounded factorization domain with respect to A* (A-BFD) if for each nonzero nonunit $b \in B$, there is a positive integer N such that whenever $b = b_1 \cdots b_n$ with each $b_i \in B$ a nonunit, then at most N of the b_i's are in A. It is known [4, Proposition 2.1] that $A + XB[X]$ is a BFD if and only if $A + XB[\![X]\!]$ is a BFD if and only if $U(A) = U(B) \cap A$ and B is an A-BFD. The following, composite

Hurwitz rings analog for BFDs of $A+XB[X]$ and $A+XB[\![X]\!]$, can be obtained by the similar arguments as in the proof of [4, Proposition 2.1]. We include a proof for readers.

THEOREM 3.3. *Let $A \subseteq B$ be an extension of integral domains with characteristic zero. Then the following statements are equivalent.*

(i) $h(A,B)$ is a BFD.

(ii) $H(A,B)$ is a BFD.

(iii) $U(A)=U(B)\cap A$ and B is an A-BFD.

Proof. Put $R=h(A,B)$ and $T=H(A,B)$.

(i) $\Rightarrow$ (ii) Let R be a BFD. Then there is a length function $l_R : R^* \to \mathbb{N}_0$ by Lemma 3.1. Define a function $l_T : T^* \to \mathbb{N}_0$ by $l_T(\sum_{i=n}^{\infty} a_iX^i) = l_R(a_nX^n)+n$ for every $\sum_{i=n}^{\infty} a_iX^i \in T$ with $0 \neq a_n$. We claim that the function l_T is a length function. Clearly, $l_T(\sum_{i=n}^{\infty} a_iX^i)=0$ if and only if $n=0$ and a_0 is a unit in R, if and only if $n=0$ and a_0 is a unit in T by Lemma 3.2. Let $f,g \in T^*$. Write f,g as follows: $f=\sum_{i=n}^{\infty} a_iX^i$ and $g=\sum_{j=m}^{\infty} b_jX^j$ with $a_n \neq 0$ and $b_m \neq 0$. Then $f*g=\sum_{k=m+n}^{\infty} c_kX^k$, where $c_{m+n}=\binom{n+m}{n}a_nb_m$. Hence we have the following:

$$\begin{aligned} l_T(f*g) &= l_R(\binom{n+m}{n}a_nb_mX^{n+m})+n+m \\ &= l_R(a_nX^n * b_mX^m)+n+m \\ &\geq l_R(a_nX^n)+l_R(b_mX^m)+n+m = l_T(f)+l_T(g). \end{aligned}$$

Therefore, T is a BFD.

(ii) $\Rightarrow$ (iii) Suppose that T is a BFD. It is clear that $U(A) \subseteq U(B)\cap A$. Let $a \in U(B)\cap A$. Then $a^{-1} \in B$. Consider ascending chain $(\frac{1}{a^n}X)_{n\geq 1}$ of principal ideals of T. Since T is a BFD, T satisfies ACCP. So there exists a positive integer n such that $(\frac{1}{a^n}X)=(\frac{1}{a^m}X)$ for every $m \geq n$. Hence $\frac{1}{a^{n+1}}X=\frac{1}{a^n}X * f$ for some $f \in T$. So $a \in U(A)$. Therefore, $U(A)=U(B)\cap A$. We now show that B is an A-BFD. Let $b \in B^*$ be a nonunit. Since T is a BFD, there exists a positive integer n_0 such that bX can be the product of at most n_0 nonunits in T by Lemma 3.1. Consider the factorization of b as nonunits of B. Since $U(A)=U(B)\cap A$, we can write $b=a_1\cdots a_mb_1\cdots b_n$, where $a_1,\dots,a_m$ are nonunits of A, and $b_1,\dots,b_n$ are nonunits in $B\setminus A$. Note that $a_1,\dots,a_m$ are nonunits in T and $b_1\cdots b_nX$ is a nonunit in T by Lemma 3.2. Hence, $bX=(a_1\cdots a_m)*(b_1\cdots b_nX)$. Thus $m \leq n_0-1$. Therefore B is an A-BFD.

(iii) $\Rightarrow$ (i) Since B is an A-BFD and $U(A)=U(B)\cap A$, it is easy to show that A is a BFD. Let $f=\sum_{i=0}^{n} b_iX^i$ with $b_n \neq 0$ be a nonunit of R. If $\deg(f)=0$, then $f=b_0 \in A$. Since A is a BFD, there exists a positive integer N such that whenever $b_0=b_1\cdots b_n$ with each b_i a nonunit of R, then $n \leq N$. If $\deg(f)=n\geq 1$, then since B is an A-BFD, there exists a positive integer N such that the number of nonunit factors in A of a factorization of b_n in B is at most N. Since $\deg(f)=n$, a factorization of f in R has at most $N+n$ nonunit factors. Therefore, R is a BFD by Lemma 3.1.

□

When $A=B$ in Theorem 3.3, we obtain

Corollary 3.4. *Let A be an integral domain with characteristic zero. Then the following statements are equivalent.*

(i) A *is a BFD.*
(ii) $h(A)$ *is a BFD.*
(iii) $H(A)$ *is a BFD.*
(iv) $A[X]$ *is a BFD.*
(v) $A[\![X]\!]$ *is a BFD.*

We now give examples of (composite) Hurwitz rings with bounded factorization property which are not isomorphic to (composite) polynomial or power series rings.

Example 3.5. *1. The rings $H(\mathbb{Z})$ and $h(\mathbb{Z})$ are non-Noetherian BFDs.*

2. Let K be an algebraic number field and $\mathcal{O}_K$ be the ring of integers of K. Then $\mathcal{O}_K$ is a Dedekind domain. Since $\mathcal{O}_K$ is a finitely generated $\mathbb{Z}$-module, it follows from [14, Proposition 2.1] or [15, Theorem 4] that $\mathbb{Z}+X\mathcal{O}_K[\![X]\!]$ and $\mathbb{Z}+X\mathcal{O}_K[X]$ are Noetherian domains, hence BFDs. By [4, Proposition 2.1] and Theorem 3.3, $H(\mathbb{Z}, \mathcal{O}_K)$ and $h(\mathbb{Z}, \mathcal{O}_K)$ are BFDs. However, it follows from [20, Theorem 2.1] that $H(\mathbb{Z}, \mathcal{O}_K)$ and $h(\mathbb{Z}, \mathcal{O}_K)$ are non-Noetherian domains.

We recall that an integral domain R is a finite factorization domain (FFD) if each nonzero nonunit of R has only a finite number of nonassociate divisors. It is shown in [1, Proposition 5.3] that $R[X]$ is an FFD if and only if R is an FFD. The following, Hurwitz polynomial analog of polynomial ring, can be obtained by the similar arguments as in the proof of [1, Proposition 5.3]. We include a proof for readers.

Proposition 3.6. *Let R be an integral domain with characteristic zero. Then $h(R)$ is an FFD if and only if R is an FFD.*

Proof. If $h(R)$ is an FFD, then clearly R is an FFD. Suppose that R is an FFD with quotient field K. Let $0 \neq f \in h(R)$ be an nonunit. If f is constant, then f has only finitely many nonassociate factors since R is an FFD. We may assume that f is nonconstant. Suppose that f has an infinitely many nonassociate factors in $h(R)$. Note that by Lemma 2.2, $h(K) \cong K[X]$ is a UFD, and hence an FFD. So there is an infinite set of nonassociate factors, say $\{f_n\}_{n\geq 1}$, of f in $h(R)$ such that $f_1h(K) = f_nh(K)$ for each $n \geq 1$. Since the unit group of $h(K)$ is K^* by Lemma 3.2, every f_n has the same degree. Let a and a_n be the leading coefficients of f and f_n, respectively. Since R is an FFD, an infinite number of a_n's are associate in R. Hence, we may assume that $\{f_n\}_{n\geq 1}$ is an infinite set of nonassociate factors of f in $h(R)$ such that all the f_n's have the same leading coefficients and $f_1h(K) = f_nh(K)$. Since f_1 and f_n have the same leading coefficients and $f_1h(K) = f_nh(K)$, we have $f_1 = f_n$ for $n \geq 1$, which is a contradiction. $\square$

It is shown in [4, Proposition 3.1] that for an extension $A \subseteq B$ of integral domains, $A + XB[X]$ is an FFD if and only if B is an FFD and $U(B)/U(A)$ is finite. The following, composite Hurwitz polynomial analog of composite polynomial ring, can be obtained by the similar arguments as in the proof of [4, Proposition 3.1]. We include a proof for readers. For an extension $A \subseteq B$ of integral domains, let $[A : B] := \{x \in A | xB \subseteq A\}$.

Proposition 3.7. *Let $A \subseteq B$ be an extension of integral domains with characteristic zero. Then $h(A,B)$ is an FFD if and only if B is an FFD and $U(B)/U(A)$ is finite.*

Proof. ($\Rightarrow$) Suppose that $h(A,B)$ is an FFD. If $f \in h(B)$, then $X * f \in h(A,B)$. So $X \in [h(A,B) : h(B)]$. It follows from [3, Theorem 4] that $U(h(B))/U(h(A,B))$ is finite and $h(B)$ is an FFD. By Lemma 3.2, $U(h(B)) = U(B)$ and $U(h(A,B)) = U(A)$. Hence $U(B)/U(A)$ is finite. By Proposition 3.6, B is an FFD.
($\Leftarrow$) Suppose that B is an FFD and $U(B)/U(A)$ is finite. By Proposition 3.6, $h(B)$ is an FFD. Let K be a quotient field of $h(A,B)$. By Lemma 3.2, $U(A) = U(h(A,B)) \subseteq U(h(B)) \cap K^* \subseteq U(B)$. Since $U(B)/U(A)$ is finite, $(U(h(B)) \cap K^*)/U(h(A,B))$ is finite. It follows from [3, Theorem 3] that $h(A,B)$ is an FFD. □

Unlike the polynomial ring, the power series ring $R[\![X]\!]$ over an FFD R need not be an FFD [3, Example 10]. It is also shown in [3, Corollary 2] that if $R[\![X]\!]$ is an FFD, then R is completely integrally closed. The following, Hurwitz series analog of power series, can be obtained by the similar argument as in the proof of [3, Corollary 2]. We include a proof for readers.

Proposition 3.8. *Let R be an integral domain with characteristic zero. If $H(R)$ is an FFD, then R is completely integrally closed.*

Proof. Suppose that $H(R)$ is an FFD. Let K be the quotient field of R and $\alpha \in K^*$ be almost integral over R. There exists $0 \neq d \in R$ such that $d\alpha^n \in R$ for every $n \geq 1$. So $0 \neq d \in [R : R[\alpha]]$. Hence $d \in [H(R) : H(R[\alpha])]$. Since $H(R)$ is an FFD, it follows from [3, Theorem 4] that $U(H(R[\alpha]))/U(H(R))$ is finite. Suppose that $\alpha \notin R$. Note that $1 + \alpha x^n \in U(H(R[\alpha]))$ for every $n \neq 1$ by Lemma 3.2. Since $U(H(R[\alpha]))/U(H(R))$ is finite, we have $(1 + \alpha X^n)U(H(R)) = (1 + \alpha X^m)U(H(R))$ for some $m < n$. So we have

$$(1 + \alpha X^n)(1 + \alpha X^m)^{-1} = (1 + \alpha X^n)(1 - \alpha X^m + \cdots) = 1 - \alpha X^m + \cdots \in U(H(R)).$$

Hence $\alpha \in R$, a contradiction. □

Example 3.9. *Put $R := h(\mathbb{Z})$. Then R is an FFD by Proposition 3.6. Since R is not completely integrally closed by Theorem 2.3, $H(R)$ and $R[\![X]\!]$ are not FFDs.*

It is known in [4, Proposition 3.3] that for an extension $A \subseteq B$ of integral domains, $A + XB[\![X]\!]$ is an FFD if and only if $B[\![X]\!]$ is an FFD and $U(B)/U(A)$ is finite. The following can be obtained by the similar arguments as in the proof of [4, Proposition 3.3]. We include a proof for readers.

Proposition 3.10. *Let $A \subseteq B$ be an extension of integral domains with characteristic zero. Then $H(A,B)$ is an FFD if and only if $H(B)$ is an FFD and $U(B)/U(A)$ is finite.*

Proof. ($\Rightarrow$) Suppose that $H(A,B)$ is an FFD. Since $X \in [H(A,B), H(B)]$, it follows from [3, Theorem 4] that $U(H(B))/U(H(A,B))$ is finite and $H(B)$ is an FFD. By Lemma 3.2, $U(H(B)) \cong U(B)$ and $U(h(A,B)) \cong U(A)$. Hence $U(B)/U(A)$ is finite.
($\Leftarrow$) Suppose that $H(B)$ is an FFD and $U(B)/U(A)$ is finite. Let K be a quotient field of $h(A,B)$. By Lemma 3.2, $U(A) \cong U(H(A,B)) \subseteq U(H(B)) \cap K^* \cong U(B) \cap K^* \subseteq$

$U(B)$. So $(U(H(B)) \cap K^*)/U(H(A,B))$ is finite. It follows from [3, Theorem 3] that $H(A,B)$ is an FFD. $\square$

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