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GENERALIZED FIRST VARIATION AND GENERALIZED SEQUENTIAL FOURIER-FEYNMAN TRANSFORM

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Abstract. This paper is a further development of the recent results by the author and coworker on the generalized sequential Fourier-Feynman transform for functionals in a Banach algebra $\hat{\mathcal{S}}$ and some related functionals. We establish existence of the generalized first variation of these functionals. Also we investigate various relationships between the generalized sequential Fourier-Feynman transform, the generalized sequential convolution product and the generalized first variation of the functionals.

1. Introduction

Let $C_0[0,T]$ be the space of continuous functions $x(t)$ on $[0,T]$ such that $x(0)=0$. Let a subdivision σ of $[0,T]$ be given:

$$\sigma : 0=\tau_0<\tau_1<\cdots<\tau_m=T,$$

and let $X(t,\sigma,\vec{\xi})$ be a polygonal curve in $C_0[0,T]$ based on a subdivision σ and the real numbers $\vec{\xi}=\{\xi_k\}$, that is,

$$X(t,\sigma,\vec{\xi})=\frac{\xi_{k-1}(\tau_k-t)+\xi_k(t-\tau_{k-1})}{\tau_k-\tau_{k-1}}$$

when $\tau_{k-1}\le t\le\tau_k$, $k=1,2,\ldots,m$ and $\xi_0=0$. If there is a sequence of subdivisions $\{\sigma_n\}$, then σ, m and τ_k will be replaced by σ_n, m_n and $\tau_{n,k}$.

Let Z_h be the Gaussian process

$$Z_h(x,t)=\int_0^t h(s)\,dx(s),$$

where $h(\neq 0)$ is in $L_2[0,T]$ and the integral $\int_0^t h(s)\,dx(s)$ denotes the Paley-Wiener-Zygmund (PWZ) integral [7,11].

Note that Z_h is a Gaussian process with mean zero and covariance function

$$\int_{C_0[0,T]} Z_h(x,s)Z_h(x,t)\,dm(x)=\int_0^{\min\{s,t\}} h^2(u)\,du,$$

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where the integral on the left-hand side of the last expression denotes the Wiener integral. Of course if $h \equiv 1$ on $[0,T]$, then $Z_h(x,t) = x(t)$ is the standard Wiener process. The standard Wiener process is stationary in time, while the Gaussian process Z_h is non-stationary in time, unless h is equal to the constant function 1.

Let $q \neq 0$ be a given real number and let $F(x)$ be a functional defined on a subset of $C_0[0,T]$ containing all the polygonal curves in $C_0[0,T]$. Let $\{\sigma_n\}$ be a sequence of subdivisions such that the norm $\|\sigma_n\| \to 0$ and let $\{\lambda_n\}$ be a sequence of complex numbers with $\operatorname{Re}\lambda_n > 0$ such that $\lambda_n \to -iq$. Then if the integral in the right hand side of (1.1) exists for all n and if the following limit exists and is independent of the choice of the sequences $\{\sigma_n\}$ and $\{\lambda_n\}$, we say that the generalized sequential Feynman integral with parameter q exists and it is denoted by

$$\int^{\mathrm{sf}_q} F(Z_h(x,\cdot))\,dx = \lim_{n\to\infty}\int_{\mathbb{R}^{m_n}} W_{\lambda_n}(\sigma_n,\vec{\xi})F(Z_h(X(\cdot,\sigma_n,\vec{\xi}),\cdot))\,d\vec{\xi}, \tag{1.1}$$

where

$$\begin{aligned} W_\lambda(\sigma,\vec{\xi}) &= \gamma_{\sigma,\lambda}\exp\Big\{-\frac{\lambda}{2}\int_0^T\Big|\frac{dX}{dt}(t,\sigma_n,\vec{\xi})\Big|^2\,dt\Big\} \\ &= \gamma_{\sigma,\lambda}\exp\Big\{-\frac{\lambda}{2}\sum_{k=1}^m\frac{(\xi_k-\xi_{k-1})^2}{\tau_k-\tau_{k-1}}\Big\} \end{aligned}$$

and

$$\gamma_{\sigma,\lambda} = \Big(\frac{\lambda}{2\pi}\Big)^{m/2}\prod_{k=1}^m(\tau_k-\tau_{k-1})^{-1/2}.$$

When $h \equiv 1$ on $[0,T]$, the generalized sequential Feynman integral is reduced to the sequential Feynman integral $\int^{\mathrm{sf}_q} F(x)\,dx$ defined and studied in [3–5, 8].

Let $D[0,T]$ be the class of elements $x \in C_0[0,T]$ such that x is absolutely continuous on $[0,T]$ and its derivative $x' \in L_2[0,T]$.

Now we introduce the definitions of a generalized sequential Fourier-Feynman transform, a generalized sequential convolution product and a generalized first variation for functionals defined on $C_0[0,T]$. In defining all the three concepts and throughout this paper, we will assume that h, h_1 and h_2 are non-zero in $L_2[0,T]$.

Definition 1.1. Let q be a nonzero real number. For $y \in D[0,T]$, we define the generalized sequential Fourier-Feynman transform $\Gamma_{q,h}(F)$ of F by the formula

$$\Gamma_{q,h}(F)(y) = \int^{\mathrm{sf}_q} F(Z_h(x,\cdot)+y)\,dx \tag{1.2}$$

if it exists [14, 17].

Definition 1.2. Let q be a nonzero real number. For $y \in D[0,T]$, we define the generalized sequential convolution product $(F*G)_{q,h}$ of F and G by the formula

$$(F*G)_{q,h}(y) = \int^{\mathrm{sf}_q} F\Big(\frac{y+Z_h(x,\cdot)}{\sqrt{2}}\Big)G\Big(\frac{y-Z_h(x,\cdot)}{\sqrt{2}}\Big)\,dx \tag{1.3}$$

if it exists [14].

Definition 1.3. Let $x, y \in C_0[0,T]$. The generalized first variation of F in the direction y is defined by the formula

$$\delta_{h_1,h_2} F(x|y) = \frac{\partial}{\partial r} F(Z_{h_1}(x,\cdot) + r Z_{h_2}(y,\cdot))|_{r=0} \tag{1.4}$$

if it exists [7].

Remark 1.4. 1. When $h_1 = h_2 \equiv 1$ on $[0,T]$, the generalized first variation is reduced to the first variation $\delta F(x|y)$ which was defined and studied on [8,9,15].
2. Hence some of the results in [8] can be obtained as corollaries of the results in this paper. For example, Theorems 4.1, 4.2, 4,3 and 4.4 in [8] follow from Theorems 3.1, 3.5, 3.6 and 3.4 below, respectively.

For $u, v \in L_2[0,T]$, we let

$$\langle u, v \rangle = \int_0^T u(t)v(t)\,dt,$$

and for a subdivision σ of $[0,T]$, we let

$$\langle u, v \rangle_k = \int_{\tau_{k-1}}^{\tau_k} u(t)v(t)\,dt$$

for $k = 1, \dots, m$. If there is a sequence of subdivision $\{\sigma_n\}$, then $\langle u, v\rangle_k$ will be replaced by $\langle u, v\rangle_{n,k}$.

Let $\mathcal{M} = \mathcal{M}(L_2[0,T])$ be the class of complex measures of finite variation defined on $\mathcal{B}(L_2[0,T])$, the Borel measurable subsets of $L_2[0,T]$.

In this paper, we work with three classes of functionals. Now we describe these classes of functionals, that is, expressions (1.5), (1.9) and (1.10), after which we will describe more the results of this paper.

A functional F defined on a subset of $C_0[0,T]$ that contains $D[0,T]$ is said to be an element of $\hat{\mathcal{S}} = \hat{\mathcal{S}}(L_2[0,T])$ if there exists a measure $f \in \mathcal{M}$ such that for $x \in D[0,T]$,

$$F(x) = \int_{L_2[0,T]} \exp\{i\langle u, x'\rangle\}\,df(u). \tag{1.5}$$

Note that $\hat{\mathcal{S}}$ with the norm $\|F\| = \|f\|$ is a Banach algebra [3]. For some Banach algebras which are useful to study Feynman integral and related topics, see [2, 3].

The second and third classes of functionals are different from but are closely related with the expression (1.5).

Let $\mathcal{T}$ be the set of functions Ψ defined on $\mathbb{R}$ by

$$\Psi(r) = \int_{\mathbb{R}} \exp\{irs\}\,d\rho(s), \tag{1.6}$$

where ρ is a complex Borel measure of bounded variation on $\mathbb{R}$. For $s \in \mathbb{R}$, let $\gamma(s)$ be the function $u \in L_2[0,T]$ such that $u(t) = s$ for $0 \le t \le T$; thus $\gamma : \mathbb{R} \to L_2[0,T]$ is continuous. For $E \in \mathcal{B}(L_2[0,T])$, let

$$\psi(E) = \rho(\gamma^{-1}(E)). \tag{1.7}$$

Thus $\psi \in \mathcal{M}$. Transforming the right hand member of (1.6), we have for $x \in D[0,T]$,

$$\Psi(x(T)) = \int_{L_2[0,T]} \exp\{i\langle u, x'\rangle\}\,d\psi(u), \tag{1.8}$$

and $\Psi(x(T))$, considered as a functional of x, is an element of $\hat{\mathcal{S}}$.

For $x \in D[0,T]$, let

$$F(x) = G(x)\Psi(x(T)), \tag{1.9}$$

where $G \in \hat{\mathcal{S}}$ and $\Psi \in \mathcal{T}$ are given by (1.5) with corresponding measure g in $\mathcal{M}$ and (1.6), respectively. Since $\hat{\mathcal{S}}$ is a Banach algebra, we know that the functional F in (1.9) is an element of $\hat{\mathcal{S}}$.

Let $f \in \mathcal{M}$ and Φ be a bounded measurable functional defined on $L_2[0,T]$, and let

$$F(x) = \int_{L_2[0,T]} \exp\{i\langle u, x'\rangle\}\Phi(u)\, df(u), \tag{1.10}$$

for $x \in D[0,T]$.

These functionals were studied in [4–6, 8, 14, 17] and are often employed in the application of the Feynman integral to quantum theory. Especially the function Ψ in (1.6) corresponds to the initial condition associated with Schrödinger equation.

We are now ready to discuss the results of this paper. In Section 2, we summarize the existences and expressions for the generalized sequential Fourier-Feynman transform from [17], and for the generalized sequential convolution product [14].

In Section 3, we establish existences and expressions for the generalized first variation of the functionals that we work with in this paper. Moreover we obtain some relationships involving the generalized sequential Fourier-Feynman transform and the generalized first variation. In the last section, using the results in Sections 2 and 3, we obtain some relationships involving the generalized sequential convolution product and the generalized first variation.

2. Generalized sequential Fourier-Feynman transform and generalized sequential convolution product

For the convenience of the readers, we introduce some results from [14, 17] on the existences and explicit expressions for the generalized sequential Fourier-Feynman transform and the generalized sequential convolution product of functionals that we work with in this paper.

In Theorems 2.1, 2.2 and 2.3 below, we summarize some results on the generalized sequential Fourier-Feynman transform [17], while in Theorems 2.4, 2.5 and 2.6, we summarize some results on the generalized sequential convolution product [14] with modified forms which are applicable in this paper.

Theorem 2.1 (Theorem 3.4 in [17]). *Let $F \in \hat{\mathcal{S}}$ be given by* (1.5) *and q be a nonzero real number. Then the generalized sequential Fourier-Feynman transform $\Gamma_{q,h}(F)(y)$ exists and is given by the formula*

$$\Gamma_{q,h}(F)(y) = \int_{L_2[0,T]} \exp\Big\{i\langle u, y'\rangle - \frac{i}{2q}\|uh\|_2^2\Big\}\, df(u) \tag{2.1}$$

for $y \in D[0,T]$. Furthermore, as a function of y, $\Gamma_{q,h}(F)(y)$ is an element of $\hat{\mathcal{S}}$. In fact,

$$\Gamma_{q,h}(F)(y) = \int_{L_2[0,T]} \exp\{i\langle u, y'\rangle\}\, df^t_{q,h}(u) \tag{2.2}$$

for $y \in D[0,T]$, *where* $f_{q,h}^t$ *is the measure in* $\mathcal{M}$ *defined by*

$$f_{q,h}^t(E) = \int_E \exp\Big\{-\frac{i}{2q}\|uh\|_2^2\Big\}\, df(u) \tag{2.3}$$

for $E \in \mathcal{B}(L_2[0,T])$.

THEOREM 2.2 (Theorem 3.7 in [17]). *For* $x \in D[0,T]$, *let* $F(x) = G(x)\Psi(x(T))$ *be given by* (1.9) *and* q *be a nonzero real number. Then the generalized sequential Fourier-Feynman transform* $\Gamma_{q,h}(F)(y)$ *exists and is given by the formula*

$$\Gamma_{q,h}(F)(y) = \int_{L_2[0,T]}\int_{\mathbb{R}} \exp\Big\{i\langle u+s, y'\rangle - \frac{i}{2q}\|(u+s)h\|_2^2\Big\}\, d\rho(s)\, dg(u) \tag{2.4}$$

for $y \in D[0,T]$. *Furthermore, as a function of* y, $\Gamma_{q,h}(F)(y)$ *is an element of* $\hat{\mathcal{S}}$. *In fact,*

$$\Gamma_{q,h}(F)(y) = \int_{L_2[0,T]} \exp\{i\langle u, y'\rangle\}\, dg_{\psi;q,h}^t(u) \tag{2.5}$$

for $y \in D[0,T]$, *where* $g_{\psi;q,h}^t$ *is the measure in* $\mathcal{M}$ *defined by* (2.3) *replacing* f *with* g_ψ, *and* g_ψ *is the measure defined by* $g_\psi(E) = \int_{L_2[0,T]} g(E-u)\, d\psi(u)$ *for* $E \in \mathcal{B}(L_2[0,T])$, *and* ψ *is given by* (1.7).

THEOREM 2.3 (Theorem 3.8 in [17]). *Let* F *be given by* (1.10) *and* q *be a nonzero real number. Then the generalized sequential Fourier-Feynman transform* $\Gamma_{q,h}(F)(y)$ *exists and is given by the formula*

$$\Gamma_{q,h}(F)(y) = \int_{L_2[0,T]} \exp\Big\{i\langle u, y'\rangle - \frac{i}{2q}\|uh\|_2^2\Big\}\Phi(u)\, df(u) \tag{2.6}$$

for $y \in D[0,T]$. *Furthermore, as a function of* y, $\Gamma_{q,h}(F)(y)$ *is an element of* $\hat{\mathcal{S}}$. *In fact,*

$$\Gamma_{q,h}(F)(y) = \int_{L_2[0,T]} \exp\{i\langle u, y'\rangle\}\, df_{\phi;q,h}^t(u) \tag{2.7}$$

for $y \in D[0,T]$, *where* $f_{\phi;q,h}^t$ *is the measure in* $\mathcal{M}$ *defined by* (2.3) *replacing* f *with* f_ϕ, *and* f_ϕ *is the measure defined by* $f_\phi(E) = \int_E \Phi(u)\, df(u)$ *for* $E \in \mathcal{B}(L_2[0,T])$.

In [14], the author and coworker investigated the existence of the generalized sequential convolution product for functionals that we work with in this paper. Also they showed that the generalized sequential Fourier-Feynman transform of the generalized sequential convolution product is a product of the generalized sequential Fourier-Feynman transforms of these functionals.

THEOREM 2.4 (Theorem 3.3 in [14]). *Let* $F_j \in \hat{\mathcal{S}}$ *be given by* (1.5) *with corresponding measures* f_j *in* $\mathcal{M}$ *for* $j = 1,2$. *Then for each nonzero real number* q, *the generalized sequential convolution product* $(F_1 * F_2)_{q,h}$ *exists and is given by*

$$(F_1 * F_2)_{q,h}(y) = \int_{L_2^2[0,T]} \exp\Big\{\frac{i}{\sqrt{2}}\langle u_1+u_2, y'\rangle - \frac{i}{4q}\|(u_1-u_2)h\|_2^2\Big\}\, df_1(u_1)\, df_2(u_2) \tag{2.8}$$

for $y \in D[0,T]$. *Furthermore, as a function of* $y \in D[0,T]$, $(F_1 * F_2)_{q,h}(y)$ *is an element of* $\hat{\mathcal{S}}$. *In fact,*

$$(F_1 * F_2)_{q,h}(y) = \int_{L_2[0,T]} \exp\{i\langle u, y'\rangle\}\, d(f_1 * f_2)^c_{q,h}(u), \tag{2.9}$$

for $y \in D[0,T]$, *where*

$$(f_1 * f_2)^c_{q,h} = (f_1 * f_2)_{q,h} \circ \eta^{-1} \tag{2.10}$$

is the measure in $\mathcal{M}$, *and*

$$(f_1 * f_2)_{q,h}(E) = \int_E \exp\Big\{-\frac{i}{4q}\|(u_1 - u_2)h\|_2^2\Big\}\, df_1(u_1)\, df_2(u_2) \tag{2.11}$$

for $E \in \mathcal{B}(L_2^2[0,T])$ *and* $\eta : L_2^2[0,T] \to L_2[0,T]$ *is a function defined by* $\eta(u_1, u_2) = \frac{u_1+u_2}{\sqrt{2}}$.

Theorem 2.5 (Theorem 3.4 in [14]). *For* $x \in D[0,T]$, *let* $F_j(x) = G_j(x)\Psi_j(x(T))$ *where* $G_j \in \hat{\mathcal{S}}$ *and* $\Psi_j \in \mathcal{T}$ *are given by* (1.5) *with corresponding measures* g_j *in* $\mathcal{M}$ *and* (1.6), *respectively for* $j = 1, 2$. *Then for each nonzero real number* q, *the generalized sequential convolution product* $(F_1 * F_2)_{q,h}$ *exists and is given by*

$$\begin{aligned}(F_1 * F_2)_{q,h}(y) = \int_{L_2^2[0,T]} \int_{\mathbb{R}^2} \exp\Big\{\frac{i}{\sqrt{2}}\langle u_1 + u_2 + s_1 + s_2, y'\rangle \\ - \frac{i}{4q}\|(u_1 - u_2 + s_1 - s_2)h\|_2^2\Big\}\, d\rho_1(s_1)\, d\rho_2(s_2)\, dg_1(u_1)\, dg_2(u_2)\end{aligned} \tag{2.12}$$

for $y \in D[0,T]$. *Furthermore, as a function of* $y \in D[0,T]$, $(F_1 * F_2)_{q,h}(y)$ *is an element of* $\hat{\mathcal{S}}$. *In fact,*

$$(F_1 * F_2)_{q,h}(y) = \int_{L_2[0,T]} \exp\{i\langle u, y'\rangle\}\, d(g_{1,\psi_1} * g_{2,\psi_2})^c_{q,h}(u), \tag{2.13}$$

for $y \in D[0,T]$, *where* $(g_{1,\psi_1} * g_{2,\psi_2})^c_{q,h}$ *is the measure in* $\mathcal{M}$ *defined by* (2.10) *and* (2.11) *replacing* f_j *with* g_{j,ψ_j}, *and* $g_{j,\psi_j} \in \mathcal{M}$ *is given by* $g_{j,\psi_j}(E) = \int_{L_2[0,T]} g_j(E - u)\, d\psi_j(u)$, $E \in \mathcal{B}(L_2[0,T])$ *for* $j = 1, 2$.

Theorem 2.6 (Theorem 3.5 in [14]). *Let* F_j *be given by* (1.10) *with corresponding bounded measurable functional* Φ_j *defined on* $L_2[0,T]$ *for* $j = 1, 2$. *Then for each nonzero real number* q, *the generalized sequential convolution product* $(F_1 * F_2)_{q,h}$ *exists and is given by*

$$\begin{aligned}(F_1 * F_2)_{q,h}(y) = \int_{L_2^2[0,T]} \exp\Big\{\frac{i}{\sqrt{2}}\langle u_1 + u_2, y'\rangle - \frac{i}{4q}\|(u_1 - u_2)h\|_2^2\Big\} \\ \times \Phi_1(u_1)\Phi_2(u_2)\, df_1(u_1)\, df_2(u_2)\end{aligned} \tag{2.14}$$

for $y \in D[0,T]$. *Furthermore, as a function of* $y \in D[0,T]$, $(F_1 * F_2)_{q,h}(y)$ *is an element of* $\hat{\mathcal{S}}$. *In fact,*

$$(F_1 * F_2)_{q,h}(y) = \int_{L_2[0,T]} \exp\{i\langle w, y'\rangle\}\, d(f_{1,\phi_1} * f_{2,\phi_2})^c_{q,h}(w), \tag{2.15}$$

for $y \in D[0,T]$, *where* $(f_{1,\phi_1} * f_{2,\phi_2})^c_{q,h}$ *is the measure in* $\mathcal{M}$ *with* $(f_{1,\phi_1} * f_{2,\phi_2})_{q,h}$ *defined by* (2.10) *and* (2.11) *replacing* f_j *with* f_{j,ϕ_j}, *and* $f_{j,\phi_j} \in \mathcal{M}$ *is given by* $f_{j,\phi_j}(E) = \int_E \Phi_j(v)\, df_j(v)$, $E \in \mathcal{B}(L_2[0,T])$ *for* $j = 1, 2$.

Remark 2.7. In Theorems 2.4, 2.5 and 2.6, we considered the generalized sequential convolution product $(F_1 * F_2)_{q,h}$ of the same type of functionals F_1 and F_2. But F_1 and F_2 are not necessarily of the same type of functionals. That is, even if F_1 and F_2 are different type of functionals, $(F_1 * F_2)_{q,h}(y)$ exists and belongs to $\hat{\mathcal{S}}$ as a function of $y \in D[0,T]$. For the explicit expressions for $(F_1 * F_2)_{q,h}$ when F_1 and F_2 are different type of functionals, see Theorem 3.6 in [14].

3. Generalized first variation and generalized sequential Fourier-Feynman transform

In this section we establish existences and explicit expressions of the generalized first variation for functionals studied in Section 2. Also we investigate relationships between the generalized sequential Fourier-Feynman transform and the generalized first variation of the functionals. To guarantee the existences of the generalized first variation $\delta_{h_1,h_2}F(x|y)$, we need further assumptions on F or h_j for $j = 1, 2$ as we see in the following theorems.

Theorem 3.1. *Let* $F \in \hat{\mathcal{S}}$ *be given by* (1.5) *with* $\int_{L_2[0,T]} \|uh_2\|_2\, d|f|(u) < \infty$ *and let* $y \in D[0,T]$. *Then the generalized first variation* $\delta_{h_1,h_2}F(x|y)$ *exists and is given by*

$$\delta_{h_1,h_2}F(x|y) = \int_{L_2[0,T]} i\langle uh_2, y'\rangle \exp\{i\langle uh_1, x'\rangle\}\, df(u) \tag{3.1}$$

for $x \in D[0,T]$. *Furthermore, as a function of* $x \in D[0,T]$, $\delta_{h_1,h_2}F(x|y)$ *is an element of* $\hat{\mathcal{S}}$. *In fact,*

$$\delta_{h_1,h_2}F(x|y) = \int_{L_2[0,T]} \exp\{i\langle u, x'\rangle\}\, df^v_{y,h_1,h_2}(u) \tag{3.2}$$

for $x \in D[0,T]$, *where*

$$f^v_{y,h_1,h_2} = f_{y,h_2} \circ \mu^{-1}_{h_1} \tag{3.3}$$

with $f_{y,h_2}(E) = i\int_E \langle uh_2, y'\rangle\, df(u)$ *for* $E \in \mathcal{B}(L_2[0,T])$ *and* $\mu_{h_1} : L_2[0,T] \to L_2[0,T]$ *is a function defined by* $\mu_{h_1}(u) = uh_1$.

Proof. For $x, y \in D[0,T]$, we have

$$\begin{aligned}\delta_{h_1,h_2}F(x|y) &= \frac{\partial}{\partial r}\Big(\int_{L_2[0,T]} \exp\Big\{i\Big\langle u, \frac{d}{dt}(Z_{h_1}(x,\cdot) + rZ_{h_2}(y,\cdot))\Big\rangle\Big\}\, df(u)\Big)\Big|_{r=0}\\ &= \frac{\partial}{\partial r}\Big(\int_{L_2[0,T]} \exp\{i\langle uh_1, x'\rangle + ir\langle uh_2, y'\rangle\}\, df(u)\Big)\Big|_{r=0}.\end{aligned}$$

Since

$$\int_{L_2[0,T]} |\langle uh_2, y'\rangle|\, d|f|(u) \le \|y'\|_2 \int_{L_2[0,T]} \|uh_2\|_2\, d|f|(u) < \infty,$$

we can pass the partial derivative under the integral sign to obtain (3.1). It is obvious that f^v_{y,h_1,h_2} is a measure in $\mathcal{M}$ and so $\delta_{h_1,h_2}F(x|y)$ can be rewritten as (3.2) which completes the proof. □

Next, we establish the existence of the generalized first variation of the functionals we considered in Theorems 2.2 and 2.3.

THEOREM 3.2. *For $x \in D[0,T]$, let $F(x) = G(x)\Psi(x(T))$ be given as in Theorem 2.2. Further assume that $\int_{L_2[0,T]}\int_{\mathbb{R}} \|(u+s)h_2\|_2\, d|\rho|(s)\, d|g|(u) < \infty$ and let $y \in D[0,T]$. Then the generalized first variation $\delta_{h_1,h_2}F(x|y)$ exists and is given by*

$$\delta_{h_1,h_2}F(x|y) = \int_{L_2[0,T]}\int_{\mathbb{R}} i\langle (u+s)h_2, y'\rangle \exp\{i\langle (u+s)h_1, x'\rangle\}\, d\rho(s)\, dg(u) \tag{3.4}$$

for $x \in D[0,T]$. Furthermore, as a function of $x \in D[0,T]$, $\delta_{h_1,h_2}F(x|y)$ is an element of $\hat{\mathcal{S}}$. In fact,

$$\delta_{h_1,h_2}F(x|y) = \int_{L_2[0,T]} \exp\{i\langle u, x'\rangle\}\, dg^v_{\psi;y,h_1,h_2}(u) \tag{3.5}$$

for $x \in D[0,T]$, where $g^v_{\psi;y,h_1,h_2}$ is the measure in $\mathcal{M}$ defined by (3.3) replacing f with g_ψ, and g_ψ is the measure in Theorem 2.2.

Proof. Since $\hat{\mathcal{S}}$ is a Banach algebra, F belongs to $\hat{\mathcal{S}}$, and using Theorem 6.1 in [2] and Theorem 2.3 in [8] we know that it can be expressed as

$$F(x) = \int_{L_2[0,T]} \exp\{i\langle u, x'\rangle\}\, dg_\psi(u),$$

where g_ψ is defined in Theorem 2.2. Since

$$\begin{aligned}\int_{L_2[0,T]} \|uh_2\|_2\, d|g_\psi|(u) &= \int_{L_2^2[0,T]} \|(u+w)h_2\|_2\, d|g|(u)\, d|\psi|(w)\\ &= \int_{L_2[0,T]}\int_{\mathbb{R}} \|(u+s)h_2\|_2\, d|\rho|(s)\, d|g|(u) < \infty,\end{aligned}$$

we can apply Theorem 3.1 to obtain

$$\delta_{h_1,h_2}F(x|y) = \int_{L_2[0,T]} i\langle wh_2, y'\rangle \exp\{i\langle wh_1, x'\rangle\}\, dg_\psi(w)$$

for $x \in D[0,T]$. By the unsymmetric Fubini theorem [2] and the transformation $u = w - v$, we have

$$\delta_{h_1,h_2}F(x|y) = \int_{L_2^2[0,T]} i\langle (u+v)h_2, y'\rangle \exp\{i\langle (u+v)h_1, x'\rangle\}\, dg(u)\, d\psi(v)$$

for $x \in D[0,T]$. Finally by the definitions (1.6) and (1.7) for Ψ and ψ, and the Fubini theorem, we obtain (3.4). Moreover by the same method as in Theorems 2.2 and 3.1, we see that $\delta_{h_1,h_2}F(x|y)$ is given by (3.5), and belongs to $\hat{\mathcal{S}}$ as a function of $x \in D[0,T]$. □

THEOREM 3.3. *Let F be given as in Theorem 2.3 with $\int_{L_2[0,T]} \|uh_2\|_2 |\Phi(u)|\, d|f|(u) < \infty$ and let $y \in D[0,T]$. Then the generalized first variation $\delta_{h_1,h_2} F(x|y)$ exists and is given by*

$$\delta_{h_1,h_2} F(x|y) = \int_{L_2[0,T]} i\langle uh_2, y'\rangle \exp\{i\langle uh_1, x'\rangle\}\Phi(u)\, df(u) \tag{3.6}$$

for each $x \in D[0,T]$. Furthermore, as a function of $x \in D[0,T]$, $\delta_{h_1,h_2} F(x|y)$ is an element of $\hat{\mathcal{S}}$. In fact,

$$\delta_{h_1,h_2} F(x|y) = \int_{L_2[0,T]} \exp\{i\langle u, x'\rangle\}\, df^v_{\phi;y,h_1,h_2}(u) \tag{3.7}$$

for $x \in D[0,T]$, where $f^v_{\phi;y,h_1,h_2}$ is the measure in $\mathcal{M}$ defined by (3.3) replacing f with f_ϕ, and f_ϕ is the measure in Theorem 2.3.

Proof. By Theorem 2.4 in [17], we know that F belongs to $\hat{\mathcal{S}}$ and is expressed as $F(x) = \int_{L_2[0,T]} \exp\{i\langle u, x'\rangle\}\, df_\phi(u)$, where f_ϕ is defined as in Theorem 2.3. Since

$$\int_{L_2[0,T]} \|uh_2\|_2\, d|f_\phi|(u) = \int_{L_2[0,T]} \|uh_2\|_2 |\Phi(u)|\, d|f|(u) < \infty,$$

we can apply Theorem 3.1 to obtain

$$\delta_{h_1,h_2} F(x|y) = \int_{L_2[0,T]} i\langle uh_2, y'\rangle \exp\{i\langle uh_1, x'\rangle\}\, df_\phi(u)$$

for $x \in D[0,T]$. Replacing $df_\phi(u)$ by $\Phi(u)\, df(u)$, we obtain (3.6). Moreover by the same method as in the proof of Theorem 3.2, we see that $\delta_{h_1,h_2} F(x|y)$ is given by (3.7), and belongs to $\hat{\mathcal{S}}$ as a function of $x \in D[0,T]$. □

As commented in Remark 2.5 of [17], at present we do not know whether the functional

$$F(x) = G(x)\Psi(x(T)),$$

where $G \in \hat{\mathcal{S}}$ and $\Psi \in L_1(\mathbb{R})$, has the generalized sequential Feynman integrable or the generalized sequential Fourier-Feynman transform. But we can show that F has the generalized first variation as in the following theorem.

THEOREM 3.4. *For $x \in D[0,T]$, let $F(x) = G(x)\Psi(x(T))$, where $G \in \hat{\mathcal{S}}$ is given by (1.5) and $\Psi \in L_1(\mathbb{R})$. Further assume that $\int_{L_2[0,T]} \int_{\mathbb{R}} \|uh_2\|_2\, d|g|(u) < \infty$, Ψ' exists and let $y \in D[0,T]$. Then the generalized first variation $\delta_{h_1,h_2} F(x|y)$ exists and is given by*

$$\delta_{h_1,h_2} F(x|y) = \delta_{h_1,h_2} G(x|y)\Psi(Z_{h_1}(x,T)) + G(Z_{h_1}(x,\cdot))\Psi'(Z_{h_1}(x,T))Z_{h_2}(y,T) \tag{3.8}$$

for $x \in D[0,T]$.

Proof. For $x, y \in D[0,T]$, we have

$$\begin{aligned}
\delta_{h_1,h_2} F(x|y) &= \frac{\partial}{\partial r}\{G(Z_{h_1}(x,\cdot) + rZ_{h_2}(y,\cdot))\Psi(Z_{h_1}(x,T) + rZ_{h_2}(y,T))\}|_{r=0} \\
&= \frac{\partial}{\partial r}\{G(Z_{h_1}(x,\cdot) + rZ_{h_2}(y,\cdot))\}|_{r=0}\Psi(Z_{h_1}(x,T)) \\
&\quad + G(Z_{h_1}(x,\cdot)\frac{\partial}{\partial r}\{\Psi(Z_{h_1}(x,T) + rZ_{h_2}(y,T))\}|_{r=0}
\end{aligned}$$

and this is equal to the right hand side of (3.8) as we wished. □

Next we discuss relationships between the generalized sequential Fourier-Feynman transform and the generalized first variation of functionals we worked with in Theorems 3.1, 3.2 and 3.3. In Theorem 3.5 below, we consider $\delta_{h_1,h_2}F(x|y)$ as a function of x, while in Theorems 3.6 and 3.7, we consider $\delta_{h_1,h_2}F(x|y)$ as a function of y.

THEOREM 3.5. *Let F be given as in Theorems 3.1, 3.2 and 3.3 with corresponding assumptions in the theorems. Let $y \in D[0,T]$ and let q be a nonzero real number. Then we have*

$$\Gamma_{q,h}(\delta_{h_1,h_2}F(\cdot|y))(x) = \delta_{h_1,h_2}\Gamma_{q,hh_1}(F)(x|y) \tag{3.9}$$

for $x \in D[0,T]$.

Proof. Since the generalized first variation $\delta_{h_1,h_2}F(x|y)$ belongs to $\hat{\mathcal{S}}$ as a function of $x \in D[0,T]$ and has the expressions (3.2), (3.5) or (3.7), we apply Theorem 2.1 to obtain the left hand side of (3.9). On the other hand, since the generalized sequential Fourier-Feynman transform $\Gamma_{q,hh_1}F(x)$ belongs to $\hat{\mathcal{S}}$ and has the expressions (2.2), (2.5) or (2.7), we apply Theorem 3.1 to obtain the right hand side of (3.9). For example, if $F \in \hat{\mathcal{S}}$ and $\int_{L_2[0,T]} \|uh_2\|_2\, d|f|(u) < \infty$, then

$$\begin{aligned}\Gamma_{q,h}(\delta_{h_1,h_2}F(\cdot|y))(x) &= \int_{L_2[0,T]} \exp\Big\{i\langle u,x'\rangle - \frac{i}{2q}\|uh\|_2^2\Big\}\, df^v_{y,h_1,h_2}(u)\\ &= \int_{L_2[0,T]} i\langle uh_2,y'\rangle \exp\Big\{i\langle uh_1,x'\rangle - \frac{i}{2q}\|uhh_1\|_2^2\Big\}\, df(u),\end{aligned}$$

and

$$\begin{aligned}\delta_{h_1,h_2}\Gamma_{q,hh_1}F(x|y) &= \int_{L_2[0,T]} i\langle uh_2,y'\rangle \exp\{i\langle uh_1,x'\rangle\}\, df^t_{q,hh_1}(u)\\ &= \int_{L_2[0,T]} i\langle uh_2,y'\rangle \exp\Big\{i\langle uh_1,x'\rangle - \frac{i}{2q}\|uhh_1\|_2^2\Big\}\, df(u),\end{aligned}$$

where the second equality follows from the definition of f^t_{q,hh_1} in Theorem 2.1. Hence we complete the proof of (3.9) for the functionals in Theorem 3.1. By the same method it is easy to see that the relationship (3.9) holds for the functionals in Theorems 3.2 and 3.3. □

Since the first variation $\delta_{h_1,h_2}F(x|y)$ does not belong to $\hat{\mathcal{S}}$ as a function of $y \in D[0,T]$, we can not apply Theorem 2.1 for the expressions $\delta_{h_1,h_2}F(x|y)$ obtained in Theorems 3.1, 3.2 and 3.3. Instead, we use Definition 1.1 to get $\Gamma_{q,h}(\delta_{h_1,h_2}F(x|\cdot))(y)$.

THEOREM 3.6. *Let F be given as in Theorems 3.1, 3.2 and 3.3 with corresponding assumptions in the theorems. Let $x \in D[0,T]$ and let q be a nonzero real number. Then we have*

$$\Gamma_{q,h}(\delta_{h_1,h_2}F(x|\cdot))(y) = \delta_{h_1,h_2}F(x|y) \tag{3.10}$$

for $y \in D[0,T]$.

Proof. We only prove the case when F is given as in Theorem 3.1, and leave the proofs for the rest cases to the reader because they are similar. Let $\sigma : 0 = \tau_0 <$

$\tau_1 < \cdots < \tau_m = T$ be a subdivision of $[0,T]$. Then using the expression (3.1) for the generalized first variation of F we have

$$\begin{aligned}&\delta_{h_1,h_2}F(x|Z_h(X(\cdot,\sigma,\vec{\xi}),\cdot)+y)\\&=\int_{L_2[0,T]}\Big\{i\sum_{k=1}^{m}\frac{\xi_k-\xi_{k-1}}{\tau_k-\tau_{k-1}}\langle uh_2,h\rangle_k+i\langle uh_2,y'\rangle\Big\}\exp\{i\langle uh_1,x'\rangle\}\,df(u).\end{aligned}$$

Let λ be a complex number with $\operatorname{Re}\lambda>0$, and let

$$I_{\sigma,\lambda}(\delta_{h_1,h_2}F(x|\cdot))(y)=\int_{\mathbb{R}^m}W_\lambda(\sigma,\vec{\xi})\delta_{h_1,h_2}F(x|Z_h(X(\cdot,\sigma,\vec{\xi}),\cdot)+y)\,d\vec{\xi}.$$

By the Fubini theorem, we have

$$\begin{aligned}I_{\sigma,\lambda}(\delta_{h_1,h_2}F(x|\cdot))(y)&=\gamma_{\sigma,\lambda}\int_{L_2[0,T]}\int_{\mathbb{R}^m}\Big\{i\sum_{k=1}^{m}\frac{\xi_k-\xi_{k-1}}{\tau_k-\tau_{k-1}}\langle uh_2,h\rangle_k+i\langle uh_2,y'\rangle\Big\}\\&\qquad\times\exp\Big\{-\frac{\lambda}{2}\sum_{k=1}^{m}\frac{(\xi_k-\xi_{k-1})^2}{\tau_k-\tau_{k-1}}+i\langle uh_1,x'\rangle\Big\}\,d\vec{\xi}\,df(u).\end{aligned}$$

Evaluating the m-dimensional Riemann integral on the right hand side, we have

$$I_{\sigma,\lambda}(\delta_{h_1,h_2}F(x|\cdot))(y)=\int_{L_2[0,T]}i\langle uh_2,y'\rangle\exp\{i\langle uh_1,x'\rangle\}\,df(u).$$

Now let $\{\sigma_n\}$ be a sequence of subdivisions of $[0,T]$ such that $\|\sigma_n\|\to 0$, and let $\{\lambda_n\}$ be a sequence of complex numbers such that $\operatorname{Re}\lambda_n>0$ and $\lambda_n\to -iq$ as $n\to\infty$. Since the expression on the right hand side of the last expression is independent of σ and λ, we have that

$$\begin{aligned}\Gamma_{q,h}(\delta_{h_1,h_2}F(x|\cdot))(y)&=\int^{\mathrm{sf}_q}\delta_{h_1,h_2}F(x|Z_h(x,\cdot)+y)\,dx\\&=\lim_{n\to\infty}I_{\sigma_n,\lambda_n}(\delta_{h_1,h_2}F(x|\cdot))(y)\\&=\int_{L_2[0,T]}i\langle uh_2,y'\rangle\exp\{i\langle uh_1,x'\rangle\}\,df(u),\end{aligned}$$

which is equal to $\delta_{h_1,h_2}F(x|y)$ in (3.1), and this completes the proof. □

In this paper, we use the generalized sequential Feynman integral to define the generalized sequential Fourier-Feynman transform. Similarly (generalized) analytic Fourier-Feynman transform can be defined using the concept of (generalized) analytic Feynman integral. Many works on the (generalized) analytic Fourier-Feynman can be seen in, for example, [1, 7, 10, 12, 13, 15]. The relationships (3.9) and (3.10) are the same as the relationships (24) and (26) in [7], respectively, for the generalized analytic Fourier-Feynman transform and the generalized first variation of functionals in the Banach algebra $\mathcal{S}$ which was introduced in [2].

4. Generalized first variation and generalized sequential convolution product

In this section we establish relationships involving the generalized first variation and generalized sequential convolution product for functionals that we worked with in the previous sections.

In Theorems 4.1, 4.2 and 4.3, we take the generalized first variation of the generalized sequential convolution product, while in Theorems 4.4, 4.5 and 4.6, we take the generalized sequential convolution product of the generalized first variation with respect to the first argument of the variation.

THEOREM 4.1. *Let* $F_j \in \hat{\mathcal{S}}$ *be given by* (1.5) *with* $\int_{L_2[0,T]} \|uh_2\|_2 \, d|f_j|(u) < \infty$ *for* $j = 1, 2$ *and let* $y \in D[0,T]$. *Then the generalized first variation* $\delta_{h_1,h_2}(F_1 * F_2)_{q,h}(x|y)$ *exists, belongs to* $\hat{\mathcal{S}}$ *as a function of* $x \in D[0,T]$, *and is given by*

$$\delta_{h_1,h_2}(F_1 * F_2)_{q,h}(x|y) = \int_{L_2[0,T]} \exp\{i\langle u, x'\rangle\} \, d(f_1 * f_2)^{c;v}_{q,h;y,h_1,h_2}(u) \tag{4.1}$$

for $x \in D[0,T]$, *where* $(f_1 * f_2)^{c;v}_{q,h;y,h_1,h_2}$ *is the measure in* $\mathcal{M}$ *defined by* (3.3) *replacing* f *with* $(f_1 * f_2)^c_{q,h}$ *in Theorem* 2.4. *In addition, the generalized first variation in* (4.1) *can be expressed explicitly as*

$$\frac{i}{\sqrt{2}} \int_{L_2^2[0,T]} \langle (u_1 + u_2)h_2, y'\rangle \exp\Big\{ \frac{i}{\sqrt{2}} \langle (u_1 + u_2)h_1, x'\rangle - \frac{i}{4q} \|(u_1 - u_2)h\|_2^2 \Big\} \, df_1(u_1) \, df_2(u_2) \tag{4.2}$$

for $x \in D[0,T]$.

Proof. Since $(F_1 * F_2)_{q,h}(y)$ belongs to $\hat{\mathcal{S}}$ and is expressed as (2.9), in order to apply Theorem 3.1 it is enough to show that the measure $(f_1 * f_2)^c_{q,h}$ satisfies the assumption in Theorem 3.1. In fact,

$$\begin{aligned}\int_{L_2[0,T]} \|uh_2\|_2 \, d|(f_1 * f_2)^c_{q,h}|(u) &= \frac{1}{\sqrt{2}} \int_{L_2^2[0,T]} \|(u_1 + u_2)h_2\|_2 \, d|(f_1 * f_2)_{q,h}|(u_1, u_2) \\ &\le \frac{1}{\sqrt{2}} \int_{L_2^2[0,T]} (\|u_1 h_2\|_2 + \|u_2 h_2\|_2) \, d|f_1|(u_1) \, d|f_2|(u_2)\end{aligned}$$

which is finite, since f_j belongs to $\mathcal{M}$ with $\int_{L_2[0,T]} \|uh_2\|_2 \, d|f_j|(u) < \infty$ for $j = 1, 2$. Now we apply Theorem 3.1 to the expression (2.9) to obtain (4.1). To find an explicit expression for (4.1), we start with the expression (3.1). Then we have

$$\begin{aligned}\delta_{h_1,h_2}(F_1 * F_2)_{q,h}(x|y) &= \int_{L_2[0,T]} i\langle uh_2, y'\rangle \exp\{i\langle uh_1, x'\rangle\} \, d(f_1 * f_2)^c_{q,h}(u) \\ &= \frac{i}{\sqrt{2}} \int_{L_2^2[0,T]} \langle (u_1 + u_2)h_2, y'\rangle \exp\Big\{ \frac{i}{\sqrt{2}} \langle (u_1 + u_2)h_1, x'\rangle \Big\} \\ &\quad \times d(f_1 * f_2)_{q,h}(u_1, u_2)\end{aligned}$$

where the second equality follows from the definition of the measure $(f_1 * f_2)^c_{q,h}$ in Theorem 2.4. Finally by the definition (2.11) of $(f_1 * f_2)_{q,h}$ in Theorem 2.4 we know that the last expression is equal to the expression (4.2), and this completes the proof. □

THEOREM 4.2. *For $x \in D[0,T]$, let $F_j(x) = G_j(x)\Psi_h(x(T))$ be given as in Theorem 2.5 with $\int_{L_2[0,T]}\int_{\mathbb{R}} \|(u+s)h_2\|_2 \, d|\rho|(s)\, d|g_j|(u) < \infty$ for $j = 1,2$ and let $y \in D[0,T]$. Then the generalized first variation $\delta_{h_1,h_2}(F_1 * F_2)_{q,h}(x|y)$ exists, belongs to $\hat{\mathcal{S}}$ as a function of $x \in D[0,T]$, and is given by*

$$\delta_{h_1,h_2}(F_1 * F_2)_{q,h}(x|y) = \int_{L_2[0,T]} \exp\{i\langle u, x'\rangle\}\, d(g_{1,\psi_1} * g_{2,\psi_2})^{c;v}_{q,h;y,h_1,h_2}(u) \tag{4.3}$$

for $x \in D[0,T]$, where $(g_{1,\psi_1} * g_{2,\psi_2})^{c;v}_{q,h;y,h_1,h_2}$ is the measure in $\mathcal{M}$ defined by (3.3) replacing f with $(g_{1,\psi_1} * g_{2,\psi_2})^{c}_{q,h}$ in Theorem 2.5. In addition, the generalized first variation in (4.3) can be expressed explicitly as

$$\begin{aligned} &\frac{i}{\sqrt{2}}\int_{L_2^2[0,T]}\int_{\mathbb{R}^2} \langle (u_1+u_2+s_1+s_2)h_2, y'\rangle \exp\Big\{\frac{i}{\sqrt{2}}\langle (u_1+u_2+s_1+s_2)h_1, x'\rangle \\ &- \frac{i}{4q}\|(u_1-u_2+s_1-s_2)h\|_2^2\Big\}\, d\rho_1(s_1)\, d\rho_2(s_2)\, dg_1(u_1)\, dg_2(u_2) \end{aligned} \tag{4.4}$$

for $x \in D[0,T]$.

Proof. Note that

$$\begin{aligned} &\int_{L_2[0,T]} \|uh_2\|_2 \, d|(g_{1,\psi_1} * g_{2,\psi_2})^{c}_{q,h}|(u) \\ &= \frac{1}{\sqrt{2}}\int_{L_2^2[0,T]} \|(u_1+u_2)h_2\|_2 \, d|(g_{1,\psi_1} * g_{2,\psi_2})_{q,h}|(u_1,u_2) \\ &\le \frac{1}{\sqrt{2}}\int_{L_2^2[0,T]}\int_{\mathbb{R}^2} (\|(u_1+s_1)h_2\|_2 + \|(u_2+s_2)h_2\|_2)\, d|\rho_1|(s_1)\, d|\rho_2|(s_2)\, d|g_1|(u_1)\, d|g_2|(u_2) \end{aligned}$$

which is finite, since g_j belongs to $\mathcal{M}$ with $\int_{L_2[0,T]} \|(u+s)h_2\|_2 \, d|\rho|(s)\, d|g_j|(u) < \infty$ for $j = 1,2$. Now we apply Theorem 3.1 to the expression (2.13) to obtain (4.3). Similar method as in the proof of Theorem 4.1 and the definitions of the corresponding measures in Theorems 2.5 and 3.1 give the expression (4.4). □

THEOREM 4.3. *Let F_j be given as in Theorem 2.6 with $\int_{L_2[0,T]} \|uh_2\|_2 |\Phi_j(u)|\, d|f_j|(u) < \infty$ for $j = 1,2$ and let $y \in D[0,T]$. Then the generalized first variation $\delta_{h_1,h_2}(F_1 * F_2)_{q,h}(x|y)$ exists, belongs to $\hat{\mathcal{S}}$ as a function of $x \in D[0,T]$, and is given by*

$$\delta_{h_1,h_2}(F_1 * F_2)_{q,h}(x|y) = \int_{L_2[0,T]} \exp\{i\langle u, x'\rangle\}\, d(f_{1,\phi_1} * f_{2,\phi_2})^{c;v}_{q,h;y,h_1,h_2}(u) \tag{4.5}$$

for $x \in D[0,T]$, where $(f_{1,\phi_1} * f_{2,\phi_2})^{c;v}_{q,h;y,h_1,h_2}$ is the measure in $\mathcal{M}$ defined by (3.3) replacing f with $(f_{1,\phi_1} * f_{2,\phi_2})^{c}_{q,h}$ in Theorem 2.6. In addition, the generalized first variation in (4.5) can be expressed explicitly as

$$\begin{aligned} &\frac{i}{\sqrt{2}}\int_{L_2^2[0,T]} \langle (u_1+u_2)h_2, y'\rangle \exp\Big\{\frac{i}{\sqrt{2}}\langle (u_1+u_2)h_1, x'\rangle - \frac{i}{4q}\|(u_1-u_2)h\|_2^2\Big\} \\ &\times \Phi_1(u_1)\Phi_2(u_2)\, df_1(u_1)\, df_2(u_2) \end{aligned} \tag{4.6}$$

for $x \in D[0,T]$.

Proof. Note that

$$\begin{aligned}
&\int_{L_2[0,T]} \|uh_2\|_2 \, d|(f_{1,\phi_1} * f_{2,\phi_2})^c_{q,h}|(u) \\
&= \frac{1}{\sqrt{2}} \int_{L_2^2[0,T]} \|(u_1+u_2)h_2\|_2 \, d|(f_{1,\phi_1} * f_{2,\phi_2})_{q,h}|(u_1,u_2) \\
&\leq \frac{1}{\sqrt{2}} \int_{L_2^2[0,T]} (\|u_1h_2\|_2 + \|u_2h_2\|_2)|\Phi_1(u_1)||\Phi_2(u_2)| \, d|f_1|(u_1) \, d|f_2|(u_2)
\end{aligned}$$

which is finite, since f_j belongs to $\mathcal{M}$ with $\int_{L_2[0,T]} \|uh_2\|_2 |\Phi_j(u)| \, d|f_j|(u) < \infty$ for $j = 1, 2$. Now we apply Theorem 3.1 to the expression (2.15) to obtain (4.5). Similar method as in the proof of Theorem 4.1 and the definitions of the corresponding measures in Theorems 2.6 and 3.1 give the expression (4.6). □

Since we know from Theorems 3.1, 3.2 and 3.3 that the generalized first variation of the functionals we work with in this paper exists and is an element of $\hat{\mathcal{S}}$ as a function of $x \in D[0,T]$, we can obtain the generalized sequential convolution product of the generalized first variation as in the following theorems.

THEOREM 4.4. *Let F_j be given as in Theorem 4.1 for $j = 1, 2$ and let $y \in D[0,T]$. Then the generalized sequential convolution product $(\delta_{h_1,h_2}F_1(\cdot|y) * \delta_{h_1,h_2}F_2(\cdot|y))_{q,h}(x)$ exists, belongs to $\hat{\mathcal{S}}$ as a function of $x \in D[0,T]$, and is given by*

(4.7)
$$(\delta_{h_1,h_2}F_1(\cdot|y) * \delta_{h_1,h_2}F_2(\cdot|y))_{q,h}(x) = \int_{L_2[0,T]} \exp\{i\langle u, x'\rangle\} \, d(f^v_{1;y,h_1,h_2} * f^v_{2;y,h_1,h_2})^c_{q,h}(u)$$

for $x \in D[0,T]$, where $(f^v_{1;y,h_1,h_2} * f^v_{2;y,h_1,h_2})^c_{q,h}$ is the measure in $\mathcal{M}$ defined as in Theorem 2.4 replacing f_j with $f^v_{j;y,h_1,h_2}$ in (3.3). In addition, the generalized sequential convolution product in (4.7) can be expressed explicitly as

$$-\int_{L_2^2[0,T]} \langle u_1h_2, y'\rangle\langle u_2h_2, y'\rangle \exp\Big\{\frac{i}{\sqrt{2}}\langle (u_1+u_2)h_1, x'\rangle - \frac{i}{4q}\|(u_1-u_2)hh_1\|_2^2\Big\} \, df_1(u_1) \, df_2(u_2) \tag{4.8}$$

for $x \in D[0,T]$.

Proof. Since $\delta_{h_1,h_2}F_j(x|y)$ belongs to $\hat{\mathcal{S}}$ as a function of $x \in D[0,T]$ for $j = 1, 2$ and expressed as (3.2), we apply Theorem 2.4 to obtain (4.7). Moreover, by the definitions of the corresponding measures in Theorems 2.4 and 3.1, we have the expression (4.8). □

THEOREM 4.5. *Let F_j be given as in Theorem 4.2 for $j = 1, 2$ and let $y \in D[0,T]$. Then the generalized sequential convolution product $(\delta_{h_1,h_2}F_1(\cdot|y) * \delta_{h_1,h_2}F_2(\cdot|y))_{q,h}(x)$ exists, belongs to $\hat{\mathcal{S}}$ as a function of $x \in D[0,T]$, and is given by*

(4.9)
$$(\delta_{h_1,h_2}F_1(\cdot|y) * \delta_{h_1,h_2}F_2(\cdot|y))_{q,h}(x) = \int_{L_2[0,T]} \exp\{i\langle u, x'\rangle\} \, d(g^v_{1,\psi_1;y,h_1,h_2} * g^v_{2,\psi_2;y,h_1,h_2})^c_{q,h}(u)$$

for $x \in D[0,T]$, where $(g^v_{1,\psi_1;y,h_1,h_2} * g^v_{2,\psi_2;y,h_1,h_2})^c_{q,h}$ is the measure in $\mathcal{M}$ defined as in Theorem 2.4 replacing f_j with $g^v_{j,\psi_j;y,h_1,h_2}$ in Theorem 3.2. In addition, the generalized *sequential convolution product in* (4.9) *can be expressed explicitly as*

(4.10)

$$- \int_{L_2^2[0,T]} \int_{\mathbb{R}^2} \langle (u_1+s_1)h_2, y'\rangle \langle (u_2+s_2)h_2, y'\rangle \exp\Big\{ \frac{i}{\sqrt{2}} \langle (u_1+u_2+s_1+s_2)h_1, x'\rangle - \frac{i}{4q}\|(u_1-u_2+s_1-s_2)hh_1\|_2^2 \Big\}\, d\rho_1(s_1)\, d\rho_2(s_2)\, df_1(u_1)\, df_2(u_2)$$

for $x \in D[0,T]$.

Proof. Since $\delta_{h_1,h_2}F_j(x|y)$ belongs to $\hat{\mathcal{S}}$ as a function of $x \in D[0,T]$ for $j=1,2$ and expressed as (3.5), we apply Theorem 2.4 to obtain (4.9). Moreover, by the definitions of the corresponding measures in Theorems 2.4 and 3.2, we have the expression (4.10). □

Theorem 4.6. *Let F_j be given as in Theorem* 4.3 *for* $j=1,2$ *and let* $y \in D[0,T]$. *Then the generalized sequential convolution product* $(\delta_{h_1,h_2}F_1(\cdot|y) * \delta_{h_1,h_2}F_2(\cdot|y))_{q,h}(x)$ *exists, belongs to $\hat{\mathcal{S}}$ as a function of $x \in D[0,T]$, and is given by*

(4.11)

$$(\delta_{h_1,h_2}F_1(\cdot|y) * \delta_{h_1,h_2}F_2(\cdot|y))_{q,h}(x) = \int_{L_2[0,T]} \exp\{i\langle u, x'\rangle\}\, d(f^v_{1,\phi_1;y,h_1,h_2} * f^v_{2,\phi_2;y,h_1,h_2})^c_{q,h}(u)$$

for $x \in D[0,T]$, *where* $(f^v_{1,\phi_1;y,h_1,h_2} * f^v_{2,\phi_2;y,h_1,h_2})^c_{q,h}$ *is the measure in $\mathcal{M}$ defined as in Theorem* 2.4 *replacing f_j with $f^v_{j,\phi_j;y,h_1,h_2}$ in Theorem* 3.3. *In addition, the generalized sequential convolution product in* (4.11) *can be expressed explicitly as*

$$- \int_{L_2^2[0,T]} \langle u_1h_2, y'\rangle \langle u_2h_2, y'\rangle \exp\Big\{ \frac{i}{\sqrt{2}} \langle (u_1+u_2)h_1, x'\rangle - \frac{i}{4q}\|(u_1-u_2)hh_1\|_2^2 \Big\} \times \Phi_1(u_1)\Phi_2(u_2)\, df_1(u_1)\, df_2(u_2) \tag{4.12}$$

for $x \in D[0,T]$.

Proof. Since $\delta_{h_1,h_2}F_j(x|y)$ belongs to $\hat{\mathcal{S}}$ as a function of $x \in D[0,T]$ for $j=1,2$ and expressed as (3.7), we apply Theorem 2.4 to obtain (4.11). Moreover, by the definitions of the corresponding measures in Theorems 2.4 and 3.3, we have the expression (4.12). □

The expressions (4.2) and (4.8) are the same as the expressions (27) and (28) in [7], respectively, for the generalized first variation and the generalized analytic convolution product of functionals in the Banach algebra $\mathcal{S}$.

In Theorems 4.1 through 4.6, we considered relationships between the generalized first variation and the generalized sequential convolution product of the same type of functionals. But as we commented in Remark 2.6, the generalized sequential convolution product $(F_1 * F_2)_{q,h}$ exists and belongs to $\hat{\mathcal{S}}$ even if F_1 and F_2 are different type of functionals. Hence all the results in this section can naturally be extended to different type of functionals F_1 and F_2.

References

[1] J.M. Ahn, K.S. Chang, B.S. Kim and I. Yoo, *Fourier-Feynman transform, convolution and first variation*, Acta Math. Hungar. **100**, (2003), 215-235.

[2] R.H. Cameron and D.A. Storvick, *Some Banach algebras of analytic Feynman integrable functionals*, pp. 18-67 in Analytic functions (Kozubnik, 1979), Lecture Notes in Math. **798**, Springer, Berlin, 1980.

[3] R.H. Cameron and D.A. Storvick, *A simple definition of the Feynman integral, with applications*, Mem. Amer. Math. Soc. No. **288**, Amer. Math. Soc., 1983.

[4] R.H. Cameron and D.A. Storvick, *Sequential Fourier-Feynman transforms*, Annales Acad. Scient. Fenn. **10** (1985), 107–111.

[5] R.H. Cameron and D.A. Storvick, *New existence theorems and evaluation formulas for sequential Feynman integrals*, Proc. London Math. Soc. **52** (1986), 557–581.

[6] R.H. Cameron and D.A. Storvick, *New existence theorems and evaluation formulas for analytic Feynman integrals*, Deformations Math. Struct., Complex Analy. Phys. Appl., Kluwer Acad. Publ., Dordrecht (1989), 297–308.

[7] K.S. Chang, D.H. Cho, B.S. Kim, T.S. Song and I. Yoo, *Relationships involving generalized Fourier-Feynman transform, convolution and first variation*, Integral Transform. Spec. Funct. **16** (2005), 391–405.

[8] K.S. Chang, D.H. Cho, B.S. Kim, T.S. Song and I. Yoo, *Sequential Fourier-Feynman transform, convolution and first variation*, Trans. Amer. Math. Soc. **360** (2008), 1819–1838.

[9] K.S. Chang, B.S. Kim and I. Yoo, *Fourier-Feynman transform, convolution and first variation of functionals on abstract Wiener space*, Integral Transform. Spec. Funct. **10** (2000), 179–200.

[10] S.J. Chang and J.G. Choi, *Analytic Fourier-Feynman transforms and convolution products associated with Gaussian processes on Wiener space*, Banach J. Math. Anal. **11** (2017), 785–807.

[11] D.M. Chung, C. Park and D. Skoug, *Generalized Feynman integrals via conditional Feynman integrals*, Michigan Math. J. **40** (1993), 377–391.

[12] T. Huffman, C. Park and D. Skoug, *Analytic Fourier-Feynman transforms and convolution*, Trans. Amer. Math. Soc. **347** (1995), 661–673.

[13] T. Huffman, C. Park and D. Skoug, *Convolutions and Fourier-Feynman transforms of functionals involving multiple integrals*, Michigan Math. J. **43** (1996), 247-261.

[14] B.S. Kim and I. Yoo, *Generalized sequential convolution product for the generalized sequential Fourier-Feynman transform*, Korean J. Math. **29** (2021), 321–332.

[15] C. Park, D. Skoug and D. Storvick, *Relationships among the first variation, the convolution product, and the Fourier-Feynman transform*, Rocky Mountain J. Math. **28** (1998), 1447–1468.

[16] D. Skoug and D. Storvick, *A survey of results involving transforms and convolutions in function space*, Rocky Mountain J. Math. **34** (2004), 1147–1176.

[17] I. Yoo and B.S. Kim, *Generalized sequential Fourier-Feynman transform*, Rocky Mountain J. Math. **51** (2021), 2251–2268.

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