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# APPROXIMATION OF SOLUTIONS THROUGH THE FIBONACCI WAVELETS AND MEASURE OF NONCOMPACTNESS TO NONLINEAR VOLTERRA-FREDHOLM FRACTIONAL INTEGRAL EQUATIONS

Supriya Kumar Paul and Lakshmi Narayan Mishra*

Abstract. This paper consists of two significant aims. The first aim of this paper is to establish the criteria for the existence of solutions to nonlinear Volterra-Fredholm (V-F) fractional integral equations on $[0, L]$, where $0 < L < \infty$. The fractional integral is described here in the sense of the Katugampola fractional integral of order $\lambda > 0$ and with the parameter $\beta > 0$. The concepts of the fixed point theorem and the measure of noncompactness are used as the main tools to prove the existence of solutions. The second aim of this paper is to introduce a computational method to obtain approximate numerical solutions to the considered problem. This method is based on the Fibonacci wavelets with collocation technique. Besides, the results of the error analysis and discussions of the accuracy of the solutions are also presented. To the best knowledge of the authors, this is the first computational method for this generalized problem to obtain approximate solutions. Finally, two examples are discussed with the computational tables and convergence graphs to interpret the efficiency and applicability of the presented method.

## 1. Introduction

Integral equations are among the most significant tools in the fields of scientific inquiry and applied mathematics. Recent years have seen a major increase in interest in the theory of fractional integral equations, which is now a significant field of nonlinear analysis. In several references, the authors have discussed the existence, stability, or other qualitative characteristics of solutions to different kinds of problems via the application of fixed point theorems and measure of noncompactness [8, 17, 20, 24, 25, 28, 34, 35, 38, 39]. The papers [6, 7, 12, 16, 22, 27, 29–33] describe the advancement of fractional calculus and provide explanations of some of its wide applications in engineering and science. Basically, there are three types of integral equations in the literature. These are Fredholm, Volterra, and Volterra-Fredholm integral equations. A variety of physical phenomena in the fields of airfoil theory, elasticity, molecular conduction, elastic constant problems, and contact problems can

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* Corresponding author.

be described by the Volterra-Fredholm (V-F) integral equations [1,2,9,26]. These integral equations are a combination of Volterra and Fredholm integral equations. To solve such type of integral equation, several analytical and numerical techniques have been established by various researchers. Here's a few observations of numerical techniques: Micula [21] presented a method for the Fredholm-Volterra integral equations of second kind. Maleknejad et al. [18] introduced a method to solve the nonlinear V-F integral equations by using Legendre polynomials. Mirzaee et al. [23] presented a technique for numerical solution of nonlinear and linear V-F integral equations. Maleknejad et al. [19] presented the Adomian decomposition method for the system of V-F integral equations. Yusufoğlu et al. [42] suggested a technique based on interpolation to solve linear V-F integral equations. Didgar et al. [13] studied on Taylor expansion for the solution of V-F integral equations and systems of V-F integral equations.

Recently, in 2021, Geçmen et al. [14] introduced a method by using Hosoya polynomials for the following V-F integral equation,

$$\psi(\mu) = Q(\mu) + \int_0^{\mu} \mathcal{K}_1(\mu,\vartheta)\psi(\vartheta)d\vartheta + \int_0^1 \mathcal{K}_2(\mu,\vartheta)\psi(\vartheta)d\vartheta. \tag{1}$$

In 2022, Amin et al. [3, 4] studied on the solvability and presented a computational technique for the solution of V-F fractional integral equations as follows:

$$\psi(\mu) = Q(\mu) + \frac{\delta(\mu)}{\Gamma(\lambda)} \int_0^{\mu} (\mu-\vartheta)^{\lambda-1} \psi(\vartheta)d\vartheta + \frac{\xi(\mu)}{\Gamma(\lambda)} \int_0^{L} (L-\vartheta)^{\lambda-1} \psi(\vartheta)d\vartheta, \tag{2}$$

and

$$\psi(\mu) = Q(\mu) + \frac{\delta(\mu)}{\Gamma(\lambda)} \int_0^{\mu} (\mu-\vartheta)^{\lambda-1} \mathcal{F}_1(\vartheta,\psi(\vartheta))d\vartheta + \frac{\xi(\mu)}{\Gamma(\lambda)} \int_0^{L} (L-\vartheta)^{\lambda-1} \mathcal{F}_2(\vartheta,\psi(\vartheta))d\vartheta. \tag{3}$$

In this paper, we consider a generalized nonlinear V-F fractional integral equation in the sense of Katugampola fractional integral, i.e.,

$$\begin{aligned}\psi(\mu) = Q(\mu,\psi(\mu)) &+ \frac{\delta(\mu,\psi(\mu))}{\Gamma(\lambda)} \int_0^{\mu} \vartheta^{\beta-1} \left(\frac{\mu^\beta - \vartheta^\beta}{\beta}\right)^{\lambda-1} \mathcal{K}_1(\mu,\vartheta)\mathcal{F}_1(\vartheta,\psi(\vartheta))d\vartheta \\ &+ \frac{\xi(\mu,\psi(\mu))}{\Gamma(\lambda)} \int_0^{L} \vartheta^{\beta-1} \left(\frac{L^\beta - \vartheta^\beta}{\beta}\right)^{\lambda-1} \mathcal{K}_2(\mu,\vartheta)\mathcal{F}_2(\vartheta,\psi(\vartheta))d\vartheta, \quad \mu\in[0,L],\end{aligned} \tag{4}$$

where $\beta > 0$, $\lambda > 0$, $0 < L < \infty$, and $\mathcal{K}_1, \mathcal{K}_2 : [0, L] \times [0, L] \to \mathbb{R}$, $Q, \delta, \xi : [0, L] \times \mathbb{R} \to \mathbb{R}$, $\mathcal{F}_1, \mathcal{F}_2 : [0, L] \times \mathbb{R} \to \mathbb{R}$ are all continuous functions.

Remark 1.1. In particular, when $\beta = 1$, $\xi(\mu,\psi(\mu)) = \xi(\mu)$, $\delta(\mu,\psi(\mu)) = \delta(\mu)$, $Q(\mu,\psi(\mu)) = Q(\mu)$, $\mathcal{K}_1(\mu,\vartheta) = \mathcal{K}_2(\mu,\vartheta) = 1$, then Eq. (4) reduces to the form of Eq. (3). Together with, when $\mathcal{F}_1(\vartheta,\psi(\vartheta)) = \mathcal{F}_2(\vartheta,\psi(\vartheta)) = \psi(\vartheta)$, then Eq. (4) reduces to Eq. (2). Also, when $\beta = 1$, $\lambda = 1$, $\xi(\mu,\psi(\mu)) = \delta(\mu,\psi(\mu)) = 1$, $Q(\mu,\psi(\mu)) = Q(\mu)$, and $\mathcal{F}_1(\vartheta,\psi(\vartheta)) = \mathcal{F}_2(\vartheta,\psi(\vartheta)) = \psi(\vartheta)$, then Eq. (4) reduces to the form of Eq. (1).

Analysis of the existence criteria for the solutions of various types of integral equations is an essential part of the study. One can use these requirements to identify the situation under which the problem's solution exists. Thus, the first aim of this paper is to establish the requirements for the existence of solutions to Eq. (4). The concepts of measure of noncompactness and fixed-point approaches are significant in this sense.

Even though it is known that Eq. (4) has a solution, due to its complicated form, it is not always possible to discover the analytical solution. Therefore, the second aim of this paper is to present a computational technique to obtain the approximate numerical solutions of Eq. (4). Fibonacci wavelets, introduced by Sabermahani et al. [36] in 2019, are used in this study to formulate the computational technique for Eq. (4). These wavelets are a novel class because they are not based on orthogonal functions. As far as we know, this study presents the first reference based on the numerical approach for Eq. (4).

This paper is arranged as follows: Notations and supporting information are included in Section 2. The requirements for the existence of solutions are discussed in 3. In Section 4, a method for finding approximate solutions is discussed. Error analysis is included in Section 5. Section 6 provides two examples to interpret the efficiency and applicability of the presented method and Section 7 provides conclusions and suggestions for further research.

## 2. Notations and auxiliary facts

Let $\mathbb{R}$ be the set of real numbers and $\mathbb{R}_+ = [0, \infty)$. Assume that $(D, \|.\|)$ is a real Banach space with zero element $\theta$. Denote by $\mathcal{B}(\psi, \varkappa)$ the closed ball in $D$ with radius $\varkappa$ and centered at $\psi$. We will write $\mathcal{B}_\varkappa$ to denote the ball $\mathcal{B}(\theta, \varkappa)$. Let $Conv\,\mathcal{V}$ and $\overline{\mathcal{V}}$ denote the convex hull and closure of $\mathcal{V}$, respectively. Denote by $\mathcal{M}_D$ the family of all nonempty and bounded subsets of $D$. and by $\mathcal{N}_D$ its subfamily consisting of all relatively compact subsets.

Definition 2.1. [15,41] The Katugampola fractional integral of $\psi : [a, b] \to \mathbb{R}$ is defined as:

$$ {}^{\beta}_{a}\mathcal{I}^{\lambda}\psi(\mu) = \frac{(\beta+1)^{1-\lambda}}{\Gamma(\lambda)} \int_a^\mu \vartheta^\beta (\mu^{\beta+1} - \vartheta^{\beta+1})^{\lambda-1} \psi(\vartheta) d\vartheta, $$

where $\lambda > 0$ and $\beta \neq -1$ are real numbers.

Definition 2.2. [5] A mapping $\mathcal{T} : \mathcal{M}_D \to \mathbb{R}_+$ is said to be a measure of noncompactness in $D$ if it satisfies the following conditions:

- $(C_1)$ The family $ker\,\mathcal{T} = \{\mathcal{V} \in \mathcal{M}_D : \mathcal{T}(\mathcal{V}) = 0\}$ is nonempty and $ker\,\mathcal{T} \subset \mathcal{N}_D$.
- $(C_2)$ $\mathcal{V} \subset \mathcal{V}_1 \Rightarrow \mathcal{T}(\mathcal{V}) \leq \mathcal{T}(\mathcal{V}_1)$.
- $(C_3)$ $\mathcal{T}(\overline{\mathcal{V}}) = \mathcal{T}(Conv\,\mathcal{V}) = \mathcal{T}(\mathcal{V})$.
- $(C_4)$ $\mathcal{T}(\gamma\mathcal{V} + (1-\gamma)\mathcal{V}_1) \leq \gamma\mathcal{T}(\mathcal{V}) + (1-\gamma)\mathcal{T}(\mathcal{V}_1)$, $\forall\, 0 \leq \gamma \leq 1$.
- $(C_5)$ If $(\mathcal{V}_n)$ is a sequence of closed sets from $\mathcal{M}_D$ such that $\mathcal{V}_{n+1} \subset \mathcal{V}_n$ for $n = 1, 2, \ldots$, and if $\lim_{n\to\infty} \mathcal{T}(\mathcal{V}_n) = 0$ then $\mathcal{V}_\infty = \bigcap_{n=1}^\infty \mathcal{V}_n$ is nonempty.

Theorem 2.3. [10] *Let $\Delta$ be a nonempty, bounded, closed and convex subset of $D$ and let $\mathcal{S} : \Delta \to \Delta$ be a continuous mapping such that $\exists$ a constant $\kappa \in [0, 1)$ and for any nonempty subset $W$ of $\Delta$ satisfying $\mathcal{T}(\mathcal{S}W) \leq \kappa\mathcal{T}(W)$. Then $\mathcal{S}$ has a fixed point in $\Delta$.*

In this paper, we will work in the Banach space $C([0, L], \mathbb{R})$, which consists of all continuous functions $\psi : [0, L] \to \mathbb{R}$ with the norm $\|\psi\| = \sup\{|\psi(\mu)| : \mu \in [0, L]\}$.

Definition 2.4. [5] Let $\mathcal{X}$ be a nonempty and bounded subset of $C([0,L],\mathbb{R})$. For $\epsilon>0$ and $\psi\in\mathcal{X}$, we denote by $\omega(\psi,\epsilon)$ the modulus of continuity of the function $\psi$, i.e.,
$\omega(\psi,\epsilon)=\sup\{|\psi(s)-\psi(t)|: s,t\in[0,L], |s-t|\le\epsilon\}$.

Furthermore, let $\omega(\mathcal{X},\epsilon)$ and $\omega_0(\mathcal{X})$ be defined by

$$\omega(\mathcal{X},\epsilon)=\sup\{\omega(\psi,\epsilon):\psi\in\mathcal{X}\},$$

and

$$\omega_0(\mathcal{X})=\lim_{\epsilon\to 0}\omega(\mathcal{X},\epsilon).$$

Then the function $\omega_0(\mathcal{X})$ is a measure of noncompactness in $C([0,L],\mathbb{R})$.

**2.1. Fibonacci polynomials.** In general, Fibonacci polynomials are defined as follows [36]:

$$\bar{P}_\sigma(\mu)=\begin{cases}1, & \sigma=0,\\ \mu, & \sigma=1,\\ \mu\bar{P}_{\sigma-1}(\mu)+\bar{P}_{\sigma-2}(\mu), & \sigma>1.\end{cases} \tag{5}$$

Moreover, these polynomials can also be expressed in the power form:

$$\bar{P}_\sigma(\mu)=\sum_{j=0}^{\lfloor\frac{\sigma}{2}\rfloor}\binom{\sigma-j}{j}\mu^{\sigma-2j},\qquad \sigma\ge 0. \tag{6}$$

Lemma 2.5. *( [36]). If $\bar{P}_\sigma(\mu)$ $(\sigma=0,1,\dots,M)$ are Fibonacci polynomials, then*

$$\int_0^1\bar{P}_\eta(\mu)\bar{P}_\sigma(\mu)\,d\mu=\sum_{j=0}^{\lfloor\frac{\eta}{2}\rfloor}\sum_{l=0}^{\lfloor\frac{\sigma}{2}\rfloor}\binom{\eta-j}{j}\binom{\sigma-l}{l}\frac{1}{\eta+\sigma-2j-2l+1}. \tag{7}$$

**2.2. Fibonacci wavelets.** Fibonacci wavelets are defined as follows [36]:

$$\Phi_{\sigma,\eta}(\mu)=\begin{cases}2^{\frac{k-1}{2}}\widehat{P}_\eta(2^{k-1}\mu-\sigma+1), & \frac{\sigma-1}{2^{k-1}}\le\mu<\frac{\sigma}{2^{k-1}},\\ 0, & \text{otherwise},\end{cases} \tag{8}$$

with

$$\widehat{P}_\eta(\mu)=\frac{1}{\sqrt{w_\eta}}\bar{P}_\eta(\mu),$$

and

$$w_\eta=\int_0^1\bar{P}_\eta^2(\mu)\,d\mu,$$

where $w_\eta$, $\eta=0,1,\dots,M-1$, can be computed by Eq. (7), $\eta$ is the order of the Fibonacci polynomials and $\sigma=1,2,\dots,2^{k-1}$, where $k$ is a positive integer.

Now for $M=3$, $k=2$, we get

$$\Phi_{1,0}(\mu)=\begin{cases}\sqrt{2}, & 0\le\mu<\frac{1}{2}\\ 0, & \text{otherwise}\end{cases},\qquad \Phi_{1,1}(\mu)=\begin{cases}2\sqrt{6}\mu, & 0\le\mu<\frac{1}{2}\\ 0, & \text{otherwise}\end{cases},$$

$$\Phi_{1,2}(\mu)=\begin{cases}\sqrt{\frac{15}{14}}(1+4\mu^2), & 0\le\mu<\frac{1}{2}\\ 0, & \text{otherwise}\end{cases},$$

$$\Phi_{2,0}(\mu)=\begin{cases}\sqrt{2}, & \frac{1}{2}\le\mu<1\\ 0, & \text{otherwise}\end{cases},\qquad \Phi_{2,1}(\mu)=\begin{cases}\sqrt{6}(2\mu-1), & \frac{1}{2}\le\mu<1\\ 0, & \text{otherwise}\end{cases},$$

$$\Phi_{2,2}(\mu)=\begin{cases}\sqrt{\frac{30}{7}}(2\mu^2-2\mu+1), & \frac{1}{2}\le\mu<1\\ 0, & \text{otherwise}\end{cases}.$$

Additionally, Fibonacci wavelets graphs for $M=3$, $k=2$ are shown in Figure 1.

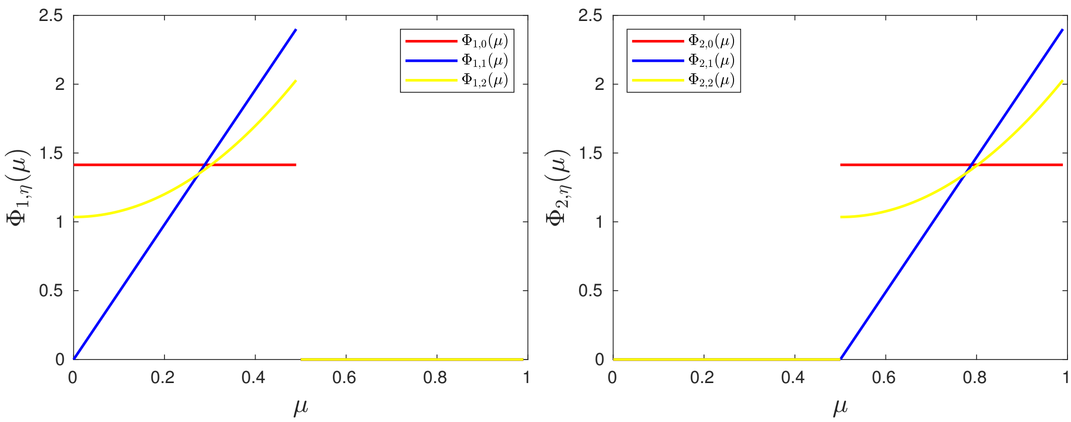


FIGURE 1. Graphical representation of Fibonacci wavelets for $M=3$, $k=2$.

LEMMA 2.6. [11] *Let $\psi:\mathbb{R}_+\to\mathbb{R}_+$ be a concave function with $\psi(0)=0$. Then $\psi(s+t)\le\psi(s)+\psi(t)$, for any $s,t\in\mathbb{R}_+$.*

LEMMA 2.7. *Let $\psi:\mathbb{R}_+\to\mathbb{R}_+$ be the function defined by $\psi(\mu)=\mu^\beta$.*

(*i*) *If $\beta\ge1$ and $\mu_2,\mu_1\in[0,L]$ with $\mu_1<\mu_2$, then $\mu_2^\beta-\mu_1^\beta\le\beta L^{\beta-1}(\mu_2-\mu_1)$.*
(*ii*) *If $0<\beta<1$ and $\mu_2,\mu_1\in[0,L]$ with $\mu_1<\mu_2$, then $\mu_2^\beta-\mu_1^\beta\le(\mu_2-\mu_1)^\beta$.*

*Proof.* (*i*) For $\beta=1$ the result is clear. Let $\beta>1$. By applying the Mean Value Theorem to the function $\psi$ on the interval $[\mu_1,\mu_2]$, we get

$$\mu_2^\beta-\mu_1^\beta=\beta\zeta^{\beta-1}(\mu_2-\mu_1),\qquad 0\le\mu_1<\zeta<\mu_2\le L.$$

This gives us

$$\mu_2^\beta-\mu_1^\beta\le\beta L^{\beta-1}(\mu_2-\mu_1),\qquad \mu_1,\mu_2\in[0,L],\quad \mu_1<\mu_2.$$

(*ii*) For $0<\beta<1$, since $\psi''(\mu)=\beta(\beta-1)\mu^{\beta-2}\le0$ for $\mu\in\mathbb{R}_+$, $\psi$ is concave, and as $\psi(0)=0$, by Lemma 2.6, $\psi(s+t)\le\psi(s)+\psi(t)$, for any $s,t\in\mathbb{R}_+$.
Therefore, for $\mu_2,\mu_1\in[0,L]$ with $\mu_2>\mu_1$, we get
$\psi(\mu_2)=\psi(\mu_2-\mu_1+\mu_1)\le\psi(\mu_2-\mu_1)+\psi(\mu_1)$,
this implies that
$\mu_2^\beta-\mu_1^\beta\le(\mu_2-\mu_1)^\beta$. □

## 3. Qualitative analysis

In this section, we will discuss the existence of solutions for Eq. (4) with the concept of measure of noncompactness.

THEOREM 3.1. *We assume the following conditions for Eq. (4):*

($A_1$) $Q : [0, L]\times\mathbb{R} \to \mathbb{R}$ *is a continuous function. Also, there exist constants* $M_1, Q^* \geq 0$ *such that*
$|Q(\mu, \psi_1) - Q(\mu, \psi_2)| \leq M_1|\psi_1 - \psi_2|$, *and* $|Q(\mu, 0)| \leq Q^*$, *for all* $\mu \in [0, L]$, $\psi_1, \psi_2 \in \mathbb{R}$.

($A_2$) $\delta, \xi : [0, L]\times\mathbb{R} \to \mathbb{R}$ *are continuous functions and there exist constants* $d_1, d_2 \geq 0$ *such that*
$\big|\delta(\mu, \psi_1) - \delta(\mu, \psi_2)\big| \leq d_1\big|\psi_1 - \psi_2\big|$, *and*
$\big|\xi(\mu, \psi_1) - \xi(\mu, \psi_2)\big| \leq d_2\big|\psi_1 - \psi_2\big|$, *for all* $\mu \in [0, L]$, *and* $\psi_1, \psi_2 \in \mathbb{R}$.
*Moreover, there exist constants* $M_2, M_3 \geq 0$ *such that* $|\delta(\mu, 0)| \leq M_2$, *and* $|\xi(\mu, 0)| \leq M_3$, *for all* $\mu \in [0, L]$.

($A_3$) $\mathcal{K}_1, \mathcal{K}_2 : [0, L] \times [0, L] \to \mathbb{R}$ *are continuous functions and there exist constants* $l_1, l_2 > 0$ *such that* $|\mathcal{K}_1(\mu, \vartheta)| \leq l_1$, *and* $|\mathcal{K}_2(\mu, \vartheta)| \leq l_2$, *for all* $\mu, \vartheta \in [0, L]$.

($A_4$) $\mathcal{F}_1, \mathcal{F}_2 : [0, L] \times \mathbb{R} \to \mathbb{R}$ *are continuous functions and there exist non-decreasing functions* $\Omega_1, \Omega_2 : \mathbb{R}_+ \to \mathbb{R}_+$ *such that*
$\big|\mathcal{F}_1(\vartheta, \psi)\big| \leq \Omega_1(|\psi|)$, *and*
$\big|\mathcal{F}_2(\vartheta, \psi)\big| \leq \Omega_2(|\psi|)$, *for all* $\vartheta \in [0, L]$, *and* $\psi \in \mathbb{R}$.

($A_5$) *There exists a number* $\varkappa > 0$ *satisfies the inequality*
$(M_1\varkappa + Q^*) + \frac{l_1(d_1\varkappa+M_2)\Omega_1(\varkappa)\beta^{-\lambda}}{\Gamma(\lambda+1)}L^{\beta\lambda} + \frac{l_2(d_2\varkappa+M_3)\Omega_2(\varkappa)\beta^{-\lambda}}{\Gamma(\lambda+1)}L^{\beta\lambda} \leq \varkappa.$
*Moreover,* $\left(M_1 + \frac{d_1l_1\Omega_1(\varkappa)}{\Gamma(\lambda+1)}\beta^{-\lambda}L^{\beta\lambda} + \frac{d_2l_2\Omega_2(\varkappa)}{\Gamma(\lambda+1)}\beta^{-\lambda}L^{\beta\lambda}\right) < 1$ *also holds.*

*Then under the conditions* ($A_1$)$-$($A_5$), *Eq. (4) has at least one solution in* $C([0, L], \mathbb{R})$.

*Proof.* Let us define an operator $\mathcal{U}$ on the space $C([0, L], \mathbb{R})$ as

$$(\mathcal{U}\psi)(\mu) = Q(\mu, \psi(\mu)) + \frac{\delta(\mu, \psi(\mu))}{\Gamma(\lambda)}\int_0^\mu \vartheta^{\beta-1}\left(\frac{\mu^\beta - \vartheta^\beta}{\beta}\right)^{\lambda-1}\mathcal{K}_1(\mu, \vartheta)\mathcal{F}_1(\vartheta, \psi(\vartheta))d\vartheta$$
$$+ \frac{\xi(\mu, \psi(\mu))}{\Gamma(\lambda)}\int_0^L \vartheta^{\beta-1}\left(\frac{L^\beta - \vartheta^\beta}{\beta}\right)^{\lambda-1}\mathcal{K}_2(\mu, \vartheta)\mathcal{F}_2(\vartheta, \psi(\vartheta))d\vartheta. \tag{9}$$

Rewriting Eq. (9) as follows:

$$(\mathcal{U}\psi)(\mu) = Q(\mu, \psi(\mu)) + \delta(\mu, \psi(\mu)) \cdot (T\psi)(\mu) + \xi(\mu, \psi(\mu)) \cdot (G\psi)(\mu), \tag{10}$$

where

$$(T\psi)(\mu) = \frac{1}{\Gamma(\lambda)}\int_0^\mu \vartheta^{\beta-1}\left(\frac{\mu^\beta - \vartheta^\beta}{\beta}\right)^{\lambda-1}\mathcal{K}_1(\mu, \vartheta)\mathcal{F}_1(\vartheta, \psi(\vartheta))d\vartheta,$$
$$(G\psi)(\mu) = \frac{1}{\Gamma(\lambda)}\int_0^L \vartheta^{\beta-1}\left(\frac{L^\beta - \vartheta^\beta}{\beta}\right)^{\lambda-1}\mathcal{K}_2(\mu, \vartheta)\mathcal{F}_2(\vartheta, \psi(\vartheta))d\vartheta.$$

**Step 1.** We have to show that the operator $\mathcal{U}$ maps $C([0,L],\mathbb{R})$ into itself. To establish this, it is enough to show that if $\psi \in C([0,L],\mathbb{R})$, then $T\psi, G\psi \in C([0,L],\mathbb{R})$. To do this, let $\epsilon > 0$ be fixed and $\mu_2, \mu_1 \in [0,L]$ with $\mu_2 > \mu_1$ such that $|\mu_2 - \mu_1| \leq \epsilon$, then we get

$$\begin{aligned}
&\big|(T\psi)(\mu_2) - (T\psi)(\mu_1)\big| \\
&\leq \frac{1}{\Gamma(\lambda)}\bigg|\int_0^{\mu_2} \vartheta^{\beta-1}\left(\frac{\mu_2^\beta - \vartheta^\beta}{\beta}\right)^{\lambda-1} \mathcal{K}_1(\mu_2,\vartheta)\mathcal{F}_1(\vartheta,\psi(\vartheta))d\vartheta \\
&\quad - \int_0^{\mu_2} \vartheta^{\beta-1}\left(\frac{\mu_2^\beta - \vartheta^\beta}{\beta}\right)^{\lambda-1} \mathcal{K}_1(\mu_1,\vartheta)\mathcal{F}_1(\vartheta,\psi(\vartheta))d\vartheta\bigg| \\
&\quad + \frac{1}{\Gamma(\lambda)}\bigg|\int_0^{\mu_2} \vartheta^{\beta-1}\left(\frac{\mu_2^\beta - \vartheta^\beta}{\beta}\right)^{\lambda-1} \mathcal{K}_1(\mu_1,\vartheta)\mathcal{F}_1(\vartheta,\psi(\vartheta))d\vartheta \\
&\quad - \int_0^{\mu_1} \vartheta^{\beta-1}\left(\frac{\mu_2^\beta - \vartheta^\beta}{\beta}\right)^{\lambda-1} \mathcal{K}_1(\mu_1,\vartheta)\mathcal{F}_1(\vartheta,\psi(\vartheta))d\vartheta\bigg| \\
&\quad + \frac{1}{\Gamma(\lambda)}\bigg|\int_0^{\mu_1} \vartheta^{\beta-1}\left(\frac{\mu_2^\beta - \vartheta^\beta}{\beta}\right)^{\lambda-1} \mathcal{K}_1(\mu_1,\vartheta)\mathcal{F}_1(\vartheta,\psi(\vartheta))d\vartheta \\
&\quad - \int_0^{\mu_1} \vartheta^{\beta-1}\left(\frac{\mu_1^\beta - \vartheta^\beta}{\beta}\right)^{\lambda-1} \mathcal{K}_1(\mu_1,\vartheta)\mathcal{F}_1(\vartheta,\psi(\vartheta))d\vartheta\bigg|,
\end{aligned}$$

i.e.,

$$\begin{aligned}
&\big|(T\psi)(\mu_2) - (T\psi)(\mu_1)\big| \\
&\leq \frac{1}{\Gamma(\lambda)}\int_0^{\mu_2} \vartheta^{\beta-1}\left(\frac{\mu_2^\beta - \vartheta^\beta}{\beta}\right)^{\lambda-1} \big|\mathcal{K}_1(\mu_2,\vartheta) - \mathcal{K}_1(\mu_1,\vartheta)\big|\big|\mathcal{F}_1(\vartheta,\psi(\vartheta))\big|d\vartheta \\
&\quad + \frac{1}{\Gamma(\lambda)}\int_{\mu_1}^{\mu_2} \vartheta^{\beta-1}\left(\frac{\mu_2^\beta - \vartheta^\beta}{\beta}\right)^{\lambda-1} \big|\mathcal{K}_1(\mu_1,\vartheta)\mathcal{F}_1(\vartheta,\psi(\vartheta))\big|d\vartheta \\
&\quad + \frac{\beta^{1-\lambda}}{\Gamma(\lambda)}\int_0^{\mu_1} \vartheta^{\beta-1}\bigg|\left(\mu_2^\beta - \vartheta^\beta\right)^{\lambda-1} - \left(\mu_1^\beta - \vartheta^\beta\right)^{\lambda-1}\bigg|\big|\mathcal{K}_1(\mu_1,\vartheta)\mathcal{F}_1(\vartheta,\psi(\vartheta))\big|d\vartheta
\end{aligned}$$

$$
\begin{aligned}
&\leq \frac{\omega(\mathcal{K}_1,\epsilon)\beta^{1-\lambda}\Omega_1(\|\psi\|)}{\Gamma(\lambda)}\int_0^{\mu_2}\vartheta^{\beta-1}\left(\mu_2^\beta-\vartheta^\beta\right)^{\lambda-1}d\vartheta \\
&\quad+\frac{l_1\beta^{1-\lambda}\Omega_1(\|\psi\|)}{\Gamma(\lambda)}\int_{\mu_1}^{\mu_2}\vartheta^{\beta-1}\left(\mu_2^\beta-\vartheta^\beta\right)^{\lambda-1}d\vartheta \\
&\quad+\frac{l_1\beta^{1-\lambda}\Omega_1(\|\psi\|)}{\Gamma(\lambda)}\int_0^{\mu_1}\vartheta^{\beta-1}\left|\left(\mu_2^\beta-\vartheta^\beta\right)^{\lambda-1}-\left(\mu_1^\beta-\vartheta^\beta\right)^{\lambda-1}\right|d\vartheta \\
&\leq \frac{\omega(\mathcal{K}_1,\epsilon)\beta^{-\lambda}\Omega_1(\|\psi\|)}{\Gamma(\lambda+1)}L^{\beta\lambda}+\frac{l_1\beta^{-\lambda}\Omega_1(\|\psi\|)}{\Gamma(\lambda+1)}\left(\mu_2^\beta-\mu_1^\beta\right)^\lambda \\
&\quad+\frac{l_1\beta^{1-\lambda}\Omega_1(\|\psi\|)}{\Gamma(\lambda)}\int_0^{\mu_1}\vartheta^{\beta-1}\left|\left(\mu_2^\beta-\vartheta^\beta\right)^{\lambda-1}-\left(\mu_1^\beta-\vartheta^\beta\right)^{\lambda-1}\right|d\vartheta, \qquad (11)
\end{aligned}
$$

where $\omega(\mathcal{K}_1,\epsilon)=\sup\left\{\left|\mathcal{K}_1(\mu_2,\vartheta)-\mathcal{K}_1(\mu_1,\vartheta)\right|:\mu_2,\mu_1,\vartheta\in[0,L],|\mu_2-\mu_1|\leq\epsilon\right\}$.
It can be observed that

(12)

$$
\int_0^{\mu_1}\vartheta^{\beta-1}\left|\left(\mu_2^\beta-\vartheta^\beta\right)^{\lambda-1}-\left(\mu_1^\beta-\vartheta^\beta\right)^{\lambda-1}\right|d\vartheta=\begin{cases}\frac{1}{\beta\lambda}\left[\left(\mu_2^{\beta\lambda}-\mu_1^{\beta\lambda}\right)-\left(\mu_2^\beta-\mu_1^\beta\right)^\lambda\right], & \lambda>1,\\ 0, & \lambda=1,\\ \frac{1}{\beta\lambda}\left[\left(\mu_2^\beta-\mu_1^\beta\right)^\lambda-\left(\mu_2^{\beta\lambda}-\mu_1^{\beta\lambda}\right)\right], & \lambda<1.\end{cases}
$$

Following are the ten cases that we need to observe now:

Case 1: When $0<\beta<1$, $\lambda>1$, and $0<\beta\lambda<1$, then by using Eq. (12) and Lemma 2.7 we get

(13)

$$
\begin{aligned}
\left|(T\psi)(\mu_2)-(T\psi)(\mu_1)\right| &\leq \frac{\omega(\mathcal{K}_1,\epsilon)\beta^{-\lambda}\Omega_1(\|\psi\|)}{\Gamma(\lambda+1)}L^{\beta\lambda}+\frac{l_1\beta^{-\lambda}\Omega_1(\|\psi\|)}{\Gamma(\lambda+1)}\left(\mu_2-\mu_1\right)^{\beta\lambda} \\
&\quad+\frac{l_1\beta^{-\lambda}\Omega_1(\|\psi\|)}{\Gamma(\lambda+1)}\left[\left(\mu_2-\mu_1\right)^{\beta\lambda}-\left(\mu_2-\mu_1\right)^{\beta\lambda}\right].
\end{aligned}
$$

Case 2: When $0<\beta<1$, $\lambda>1$, and $\beta\lambda\geq 1$, then by using Eq. (12) and Lemma 2.7 we get

(14)

$$
\begin{aligned}
\left|(T\psi)(\mu_2)-(T\psi)(\mu_1)\right| &\leq \frac{\omega(\mathcal{K}_1,\epsilon)\beta^{-\lambda}\Omega_1(\|\psi\|)}{\Gamma(\lambda+1)}L^{\beta\lambda}+\frac{l_1\beta^{-\lambda}\Omega_1(\|\psi\|)}{\Gamma(\lambda+1)}\left(\mu_2-\mu_1\right)^{\beta\lambda} \\
&\quad+\frac{l_1\beta^{-\lambda}\Omega_1(\|\psi\|)}{\Gamma(\lambda+1)}\left[\beta\lambda L^{\beta\lambda-1}\left(\mu_2-\mu_1\right)-\left(\mu_2-\mu_1\right)^{\beta\lambda}\right].
\end{aligned}
$$

Case 3: When $0 < \beta < 1$, and $\lambda = 1$, then by using Eq. (12) and Lemma 2.7 we get

(15)
$$\left|(T\psi)(\mu_2) - (T\psi)(\mu_1)\right| \leq \frac{\omega(\mathcal{K}_1, \epsilon)\beta^{-1}\Omega_1(\|\psi\|)}{\Gamma(2)} L^{\beta} + \frac{l_1\beta^{-1}\Omega_1(\|\psi\|)}{\Gamma(2)} (\mu_2 - \mu_1)^{\beta}.$$

Case 4: When $0 < \beta < 1$, $\lambda < 1$, and $0 < \beta\lambda < 1$, then by using Eq. (12) and Lemma 2.7 we get

(16)
$$\begin{aligned}\left|(T\psi)(\mu_2) - (T\psi)(\mu_1)\right| \leq{}& \frac{\omega(\mathcal{K}_1, \epsilon)\beta^{-\lambda}\Omega_1(\|\psi\|)}{\Gamma(\lambda+1)} L^{\beta\lambda} + \frac{l_1\beta^{-\lambda}\Omega_1(\|\psi\|)}{\Gamma(\lambda+1)} (\mu_2 - \mu_1)^{\beta\lambda} \\ &+ \frac{l_1\beta^{-\lambda}\Omega_1(\|\psi\|)}{\Gamma(\lambda+1)} \left[(\mu_2 - \mu_1)^{\beta\lambda} - (\mu_2 - \mu_1)^{\beta\lambda}\right].\end{aligned}$$

Case 5: When $0 < \beta < 1$, $\lambda < 1$, and $\beta\lambda \geq 1$, then by using Eq. (12) and Lemma 2.7 we get

(17)
$$\begin{aligned}\left|(T\psi)(\mu_2) - (T\psi)(\mu_1)\right| \leq{}& \frac{\omega(\mathcal{K}_1, \epsilon)\beta^{-\lambda}\Omega_1(\|\psi\|)}{\Gamma(\lambda+1)} L^{\beta\lambda} + \frac{l_1\beta^{-\lambda}\Omega_1(\|\psi\|)}{\Gamma(\lambda+1)} (\mu_2 - \mu_1)^{\beta\lambda} \\ &+ \frac{l_1\beta^{-\lambda}\Omega_1(\|\psi\|)}{\Gamma(\lambda+1)} \left[(\mu_2 - \mu_1)^{\beta\lambda} - \beta\lambda L^{\beta\lambda-1}(\mu_2 - \mu_1)\right].\end{aligned}$$

Case 6: When $\beta \geq 1$, $\lambda > 1$, and $0 < \beta\lambda < 1$, then by using Eq. (12) and Lemma 2.7 we get

(18)
$$\begin{aligned}\left|(T\psi)(\mu_2) - (T\psi)(\mu_1)\right| \leq{}& \frac{\omega(\mathcal{K}_1, \epsilon)\beta^{-\lambda}\Omega_1(\|\psi\|)}{\Gamma(\lambda+1)} L^{\beta\lambda} + \frac{l_1\Omega_1(\|\psi\|)}{\Gamma(\lambda+1)} L^{\lambda(\beta-1)} (\mu_2 - \mu_1)^{\lambda} \\ &+ \frac{l_1\beta^{-\lambda}\Omega_1(\|\psi\|)}{\Gamma(\lambda+1)} \left[(\mu_2 - \mu_1)^{\beta\lambda} - \beta^{\lambda} L^{\lambda(\beta-1)}(\mu_2 - \mu_1)^{\lambda}\right].\end{aligned}$$

Case 7: When $\beta \geq 1$, $\lambda > 1$, and $\beta\lambda \geq 1$, then by using Eq. (12) and Lemma 2.7 we get

(19)
$$\begin{aligned}\left|(T\psi)(\mu_2) - (T\psi)(\mu_1)\right| \leq{}& \frac{\omega(\mathcal{K}_1, \epsilon)\beta^{-\lambda}\Omega_1(\|\psi\|)}{\Gamma(\lambda+1)} L^{\beta\lambda} + \frac{l_1\Omega_1(\|\psi\|)}{\Gamma(\lambda+1)} L^{\lambda(\beta-1)} (\mu_2 - \mu_1)^{\lambda} \\ &+ \frac{l_1\beta^{-\lambda}\Omega_1(\|\psi\|)}{\Gamma(\lambda+1)} \left[\beta\lambda L^{\beta\lambda-1}(\mu_2 - \mu_1) - \beta^{\lambda} L^{\lambda(\beta-1)}(\mu_2 - \mu_1)^{\lambda}\right].\end{aligned}$$

Case 8: When $\beta \geq 1$, and $\lambda = 1$, then by using Eq. (12) and Lemma 2.7 we get

(20)
$$\left|(T\psi)(\mu_2) - (T\psi)(\mu_1)\right| \leq \frac{\omega(\mathcal{K}_1, \epsilon)\beta^{-1}\Omega_1(\|\psi\|)}{\Gamma(2)} L^{\beta} + \frac{l_1\Omega_1(\|\psi\|)}{\Gamma(2)} L^{\beta-1} (\mu_2 - \mu_1).$$

Case 9: When $\beta \geq 1$, $\lambda < 1$, and $0 < \beta\lambda < 1$, then by using Eq. (12) and Lemma 2.7 we get

(21)
$$\begin{aligned}\left|(T\psi)(\mu_2) - (T\psi)(\mu_1)\right| \leq & \frac{\omega(\mathcal{K}_1, \epsilon)\beta^{-\lambda}\Omega_1(\|\psi\|)}{\Gamma(\lambda+1)} L^{\beta\lambda} + \frac{l_1\Omega_1(\|\psi\|)}{\Gamma(\lambda+1)} L^{\lambda(\beta-1)} \left(\mu_2 - \mu_1\right)^{\lambda} \\ & + \frac{l_1\beta^{-\lambda}\Omega_1(\|\psi\|)}{\Gamma(\lambda+1)} \left[\beta^{\lambda} L^{\lambda(\beta-1)} \left(\mu_2 - \mu_1\right)^{\lambda} - \left(\mu_2 - \mu_1\right)^{\beta\lambda}\right].\end{aligned}$$

Case 10: When $\beta \geq 1$, $\lambda < 1$, and $\beta\lambda \geq 1$, then by using Eq. (12) and Lemma 2.7 we get

(22)
$$\begin{aligned}\left|(T\psi)(\mu_2) - (T\psi)(\mu_1)\right| \leq & \frac{\omega(\mathcal{K}_1, \epsilon)\beta^{-\lambda}\Omega_1(\|\psi\|)}{\Gamma(\lambda+1)} L^{\beta\lambda} + \frac{l_1\Omega_1(\|\psi\|)}{\Gamma(\lambda+1)} L^{\lambda(\beta-1)} \left(\mu_2 - \mu_1\right)^{\lambda} \\ & + \frac{l_1\beta^{-\lambda}\Omega_1(\|\psi\|)}{\Gamma(\lambda+1)} \left[\beta^{\lambda} L^{\lambda(\beta-1)} \left(\mu_2 - \mu_1\right)^{\lambda} - \beta\lambda L^{\beta\lambda-1} \left(\mu_2 - \mu_1\right)\right].\end{aligned}$$

In all cases, by utilizing the uniform continuity of the function $\mathcal{K}_1$ on $[0, L] \times [0, L]$, we obtain $\omega(\mathcal{K}_1, \epsilon) \to 0$ as $\epsilon \to 0$. As $\mu_2 \to \mu_1$, it follows from all the above cases that $\left|(T\psi)(\mu_2) - (T\psi)(\mu_1)\right| \to 0$.

Hence, we can say $T\psi \in C([0, L], \mathbb{R})$.

Also,

$$\begin{aligned}&\left|(G\psi)(\mu_2) - (G\psi)(\mu_1)\right| \\ &= \left|\frac{1}{\Gamma(\lambda)} \int_0^L \vartheta^{\beta-1} \left(\frac{L^\beta - \vartheta^\beta}{\beta}\right)^{\lambda-1} \mathcal{K}_2(\mu_2, \vartheta)\mathcal{F}_2(\vartheta, \psi(\vartheta))d\vartheta \right. \\ &\quad \left. - \frac{1}{\Gamma(\lambda)} \int_0^L \vartheta^{\beta-1} \left(\frac{L^\beta - \vartheta^\beta}{\beta}\right)^{\lambda-1} \mathcal{K}_2(\mu_1, \vartheta)\mathcal{F}_2(\vartheta, \psi(\vartheta))d\vartheta\right| \\ &\leq \frac{1}{\Gamma(\lambda)} \int_0^L \vartheta^{\beta-1} \left(\frac{L^\beta - \vartheta^\beta}{\beta}\right)^{\lambda-1} \left|\mathcal{K}_2(\mu_2, \vartheta) - \mathcal{K}_2(\mu_1, \vartheta)\right|\left|\mathcal{F}_2(\vartheta, \psi(\vartheta))\right|d\vartheta \\ &\leq \frac{\omega(\mathcal{K}_2, \epsilon)}{\Gamma(\lambda)} \int_0^L \vartheta^{\beta-1} \left(\frac{L^\beta - \vartheta^\beta}{\beta}\right)^{\lambda-1} \Omega_2(|\psi(\vartheta)|)d\vartheta \\ &\leq \frac{\omega(\mathcal{K}_2, \epsilon)\beta^{-\lambda}\Omega_2(\|\psi\|)}{\Gamma(\lambda+1)} L^{\beta\lambda}, \qquad (23)\end{aligned}$$

where $\omega(\mathcal{K}_2, \epsilon) = \sup\left\{\left|\mathcal{K}_2(\mu_2, \vartheta) - \mathcal{K}_2(\mu_1, \vartheta)\right| : \mu_2, \mu_1, \vartheta \in [0, L], |\mu_2 - \mu_1| \leq \epsilon\right\}$.

By utilizing the uniform continuity of the function $\mathcal{K}_2$ on $[0, L] \times [0, L]$, we obtain $\omega(\mathcal{K}_2, \epsilon) \to 0$ as $\epsilon \to 0$. Thus, it follows that $G\psi \in C([0, L], \mathbb{R})$, and consequently, $\mathcal{U}\psi \in C([0, L], \mathbb{R})$.

**Step 2.** We will show that $\mathcal{U}$ maps $\mathcal{B}_{\varkappa}$ into itself.

Let $\psi \in C([0,L],\mathbb{R})$ be such that $\|\psi\| \leq \varkappa$. Then, for all $\mu \in [0,L]$, we get

$$\begin{aligned}
&\big|(\mathcal{U}\psi)(\mu)\big| \\
&= \Big|Q(\mu,\psi(\mu)) + \frac{\delta(\mu,\psi(\mu))}{\Gamma(\lambda)}\int_0^{\mu} \vartheta^{\beta-1}\left(\frac{\mu^\beta-\vartheta^\beta}{\beta}\right)^{\lambda-1}\mathcal{K}_1(\mu,\vartheta)\mathcal{F}_1(\vartheta,\psi(\vartheta))d\vartheta \\
&\quad + \frac{\xi(\mu,\psi(\mu))}{\Gamma(\lambda)}\int_0^{L} \vartheta^{\beta-1}\left(\frac{L^\beta-\vartheta^\beta}{\beta}\right)^{\lambda-1}\mathcal{K}_2(\mu,\vartheta)\mathcal{F}_2(\vartheta,\psi(\vartheta))d\vartheta\Big| \\
&\leq \big(\big|Q(\mu,\psi(\mu))-Q(\mu,0)\big| + \big|Q(\mu,0)\big|\big) \\
&\quad + \frac{\big(\big|\delta(\mu,\psi(\mu))-\delta(\mu,0)\big|+\big|\delta(\mu,0)\big|\big)}{\Gamma(\lambda)}\int_0^{\mu} \vartheta^{\beta-1}\left(\frac{\mu^\beta-\vartheta^\beta}{\beta}\right)^{\lambda-1}\big|\mathcal{K}_1(\mu,\vartheta)\mathcal{F}_1(\vartheta,\psi(\vartheta))\big|d\vartheta \\
&\quad + \frac{\big(\big|\xi(\mu,\psi(\mu))-\xi(\mu,0)\big|+\big|\xi(\mu,0)\big|\big)}{\Gamma(\lambda)}\int_0^{L} \vartheta^{\beta-1}\left(\frac{L^\beta-\vartheta^\beta}{\beta}\right)^{\lambda-1}\big|\mathcal{K}_2(\mu,\vartheta)\mathcal{F}_2(\vartheta,\psi(\vartheta))\big|d\vartheta \\
&\leq (M_1|\psi(\mu)|+Q^*) + \frac{l_1\,(d_1|\psi(\mu)|+M_2)}{\Gamma(\lambda)}\int_0^{\mu} \vartheta^{\beta-1}\left(\frac{\mu^\beta-\vartheta^\beta}{\beta}\right)^{\lambda-1}\Omega_1(|\psi(\vartheta)|)d\vartheta
\end{aligned}$$

$$\begin{aligned}
&\quad + \frac{l_2\,(d_2|\psi(\mu)|+M_3)}{\Gamma(\lambda)}\int_0^{L} \vartheta^{\beta-1}\left(\frac{L^\beta-\vartheta^\beta}{\beta}\right)^{\lambda-1}\Omega_2(|\psi(\vartheta)|)d\vartheta \\
&\leq (M_1\|\psi\|+Q^*) + \frac{l_1\,(d_1\|\psi\|+M_2)\,\Omega_1(\|\psi\|)}{\Gamma(\lambda)}\int_0^{\mu} \vartheta^{\beta-1}\left(\frac{\mu^\beta-\vartheta^\beta}{\beta}\right)^{\lambda-1}d\vartheta \\
&\quad + \frac{l_2\,(d_2\|\psi\|+M_3)\,\Omega_2(\|\psi\|)}{\Gamma(\lambda)}\int_0^{L} \vartheta^{\beta-1}\left(\frac{L^\beta-\vartheta^\beta}{\beta}\right)^{\lambda-1}d\vartheta \\
&\leq (M_1\varkappa+Q^*) + \frac{l_1\,(d_1\varkappa+M_2)\,\Omega_1(\varkappa)\beta^{1-\lambda}}{\Gamma(\lambda)}\int_0^{\mu} \vartheta^{\beta-1}\left(\mu^\beta-\vartheta^\beta\right)^{\lambda-1}d\vartheta \\
&\quad + \frac{l_2\,(d_2\varkappa+M_3)\,\Omega_2(\varkappa)\beta^{1-\lambda}}{\Gamma(\lambda)}\int_0^{L} \vartheta^{\beta-1}\left(L^\beta-\vartheta^\beta\right)^{\lambda-1}d\vartheta \\
&\leq (M_1\varkappa+Q^*) + \frac{l_1\,(d_1\varkappa+M_2)\,\Omega_1(\varkappa)\beta^{-\lambda}}{\Gamma(\lambda+1)}L^{\beta\lambda} + \frac{l_2\,(d_2\varkappa+M_3)\,\Omega_2(\varkappa)\beta^{-\lambda}}{\Gamma(\lambda+1)}L^{\beta\lambda}.
\end{aligned}$$

Thus, by the assumption $(A_5)$, we get $\|\mathcal{U}\psi\| \leq \varkappa$, which implies that $\mathcal{U}$ maps $\mathcal{B}_{\varkappa}$ into itself.

**Step 3.** We prove that $\mathcal{U}$ is continuous on $\mathcal{B}_{\varkappa}$.

To do this, let $\{\psi_n\}$ be a sequence in $\mathcal{B}_{\varkappa}$ and $\psi \in \mathcal{B}_{\varkappa}$ such that $\psi_n \to \psi$ as $n \to \infty$.

Then, for $\mu \in [0, L]$, we get

$$
\begin{aligned}
&\big|(\mathcal{U}\psi_n)(\mu) - (\mathcal{U}\psi)(\mu)\big| \\
&= \big|Q(\mu, \psi_n(\mu)) + \delta(\mu, \psi_n(\mu)) \cdot (T\psi_n)(\mu) + \xi(\mu, \psi_n(\mu)) \cdot (G\psi_n)(\mu) \\
&\quad - Q(\mu, \psi(\mu)) - \delta(\mu, \psi(\mu)) \cdot (T\psi)(\mu) - \xi(\mu, \psi(\mu)) \cdot (G\psi)(\mu)\big| \\
&\le \big|Q(\mu, \psi_n(\mu)) - Q(\mu, \psi(\mu))\big| + \big|\delta(\mu, \psi_n(\mu)) \cdot (T\psi_n)(\mu) - \delta(\mu, \psi_n(\mu)) \cdot (T\psi)(\mu)\big| \\
&\quad + \big|\delta(\mu, \psi_n(\mu)) \cdot (T\psi)(\mu) - \delta(\mu, \psi(\mu)) \cdot (T\psi)(\mu)\big| \\
&\quad + \big|\xi(\mu, \psi_n(\mu)) \cdot (G\psi_n)(\mu) - \xi(\mu, \psi_n(\mu)) \cdot (G\psi)(\mu)\big| \\
&\quad + \big|\xi(\mu, \psi_n(\mu)) \cdot (G\psi)(\mu) - \xi(\mu, \psi(\mu)) \cdot (G\psi)(\mu)\big| \\
&\le M_1|\psi_n(\mu) - \psi(\mu)| \\
&\quad + \frac{\big|\delta(\mu, \psi_n(\mu))\big|}{\Gamma(\lambda)} \int_0^\mu \vartheta^{\beta-1} \left(\frac{\mu^\beta - \vartheta^\beta}{\beta}\right)^{\lambda-1} \big|\mathcal{K}_1(\mu, \vartheta)\big|\big|\mathcal{F}_1(\vartheta, \psi_n(\vartheta)) - \mathcal{F}_1(\vartheta, \psi(\vartheta))\big| d\vartheta \\
&\quad + \frac{\big|\delta(\mu, \psi_n(\mu)) - \delta(\mu, \psi(\mu))\big|}{\Gamma(\lambda)} \int_0^\mu \vartheta^{\beta-1} \left(\frac{\mu^\beta - \vartheta^\beta}{\beta}\right)^{\lambda-1} \big|\mathcal{K}_1(\mu, \vartheta)\mathcal{F}_1(\vartheta, \psi(\vartheta))\big| d\vartheta \\
&\quad + \frac{\big|\xi(\mu, \psi_n(\mu))\big|}{\Gamma(\lambda)} \int_0^L \vartheta^{\beta-1} \left(\frac{L^\beta - \vartheta^\beta}{\beta}\right)^{\lambda-1} \big|\mathcal{K}_2(\mu, \vartheta)\big|\big|\mathcal{F}_2(\vartheta, \psi_n(\vartheta)) - \mathcal{F}_2(\vartheta, \psi(\vartheta))\big| d\vartheta \\
&\quad + \frac{\big|\xi(\mu, \psi_n(\mu)) - \xi(\mu, \psi(\mu))\big|}{\Gamma(\lambda)} \int_0^L \vartheta^{\beta-1} \left(\frac{L^\beta - \vartheta^\beta}{\beta}\right)^{\lambda-1} \big|\mathcal{K}_2(\mu, \vartheta)\mathcal{F}_2(\vartheta, \psi(\vartheta))\big| d\vartheta \\
&\le M_1|\psi_n(\mu) - \psi(\mu)| \\
&\quad + \frac{l_1 \left(\big|\delta(\mu, \psi_n(\mu)) - \delta(\mu, 0)\big| + \big|\delta(\mu, 0)\big|\right) \omega_{\mathcal{F}_1}(\epsilon)}{\Gamma(\lambda)} \int_0^\mu \vartheta^{\beta-1} \left(\frac{\mu^\beta - \vartheta^\beta}{\beta}\right)^{\lambda-1} d\vartheta \\
&\quad + \frac{l_1 d_1 |\psi_n(\mu) - \psi(\mu)|}{\Gamma(\lambda)} \int_0^\mu \vartheta^{\beta-1} \left(\frac{\mu^\beta - \vartheta^\beta}{\beta}\right)^{\lambda-1} \Omega_1(|\psi(\vartheta)|) d\vartheta \\
&\quad + \frac{l_2 \left(\big|\xi(\mu, \psi_n(\mu)) - \xi(\mu, 0)\big| + \big|\xi(\mu, 0)\big|\right) \omega_{\mathcal{F}_2}(\epsilon)}{\Gamma(\lambda)} \int_0^L \vartheta^{\beta-1} \left(\frac{L^\beta - \vartheta^\beta}{\beta}\right)^{\lambda-1} d\vartheta \\
&\quad + \frac{l_2 d_2 |\psi_n(\mu) - \psi(\mu)|}{\Gamma(\lambda)} \int_0^L \vartheta^{\beta-1} \left(\frac{L^\beta - \vartheta^\beta}{\beta}\right)^{\lambda-1} \Omega_2(|\psi(\vartheta)|) d\vartheta
\end{aligned}
$$

$$\begin{aligned}
&\leq M_1\|\psi_n-\psi\| + \frac{l_1(d_1\|\psi_n\|+M_2)\omega_{\mathcal{F}_1}(\epsilon)}{\Gamma(\lambda)}\int_0^{\mu}\vartheta^{\beta-1}\left(\frac{\mu^\beta-\vartheta^\beta}{\beta}\right)^{\lambda-1}d\vartheta\\
&\quad+\frac{l_1d_1\|\psi_n-\psi\|\Omega_1(\|\psi\|)}{\Gamma(\lambda)}\int_0^{\mu}\vartheta^{\beta-1}\left(\frac{\mu^\beta-\vartheta^\beta}{\beta}\right)^{\lambda-1}d\vartheta\\
&\quad+\frac{l_2(d_2\|\psi_n\|+M_3)\omega_{\mathcal{F}_2}(\epsilon)}{\Gamma(\lambda)}\int_0^{L}\vartheta^{\beta-1}\left(\frac{L^\beta-\vartheta^\beta}{\beta}\right)^{\lambda-1}d\vartheta\\
&\quad+\frac{l_2d_2\|\psi_n-\psi\|\Omega_2(\|\psi\|)}{\Gamma(\lambda)}\int_0^{L}\vartheta^{\beta-1}\left(\frac{L^\beta-\vartheta^\beta}{\beta}\right)^{\lambda-1}d\vartheta,
\end{aligned}$$

i.e.,

$$\begin{aligned}
&\left|(\mathcal{U}\psi_n)(\mu)-(\mathcal{U}\psi)(\mu)\right|\\
&\leq M_1\|\psi_n-\psi\|\\
&\quad+\frac{l_1(d_1\|\psi_n\|+M_2)\omega_{\mathcal{F}_1}(\epsilon)\beta^{-\lambda}L^{\beta\lambda}}{\Gamma(\lambda+1)}\\
&\quad+\frac{l_1d_1\|\psi_n-\psi\|\Omega_1(\|\psi\|)}{\Gamma(\lambda+1)}\beta^{-\lambda}L^{\beta\lambda}\\
&\quad+\frac{l_2(d_2\|\psi_n\|+M_3)\omega_{\mathcal{F}_2}(\epsilon)\beta^{-\lambda}L^{\beta\lambda}}{\Gamma(\lambda+1)}\\
&\quad+\frac{l_2d_2\|\psi_n-\psi\|\Omega_2(\|\psi\|)}{\Gamma(\lambda+1)}\beta^{-\lambda}L^{\beta\lambda},
\end{aligned}$$

where $\omega_{\mathcal{F}_1}(\epsilon)=\sup\big\{\left|\mathcal{F}_1(\vartheta,\psi_n)-\mathcal{F}_1(\vartheta,\psi)\right| : \vartheta\in[0,L] \text{ and } \psi_n,\psi\in[-\varkappa,\varkappa]; |\psi_n-\psi|\leq\epsilon\big\}$, and

$\omega_{\mathcal{F}_2}(\epsilon)=\sup\big\{\left|\mathcal{F}_2(\vartheta,\psi_n)-\mathcal{F}_2(\vartheta,\psi)\right| : \vartheta\in[0,L] \text{ and } \psi_n,\psi\in[-\varkappa,\varkappa]; |\psi_n-\psi|\leq\epsilon\big\}$.

Now, $\psi_n\to\psi$ as $n\to\infty$, this implies that $\|\mathcal{U}\psi_n-\mathcal{U}\psi\|\to 0$ as $n\to\infty$. Hence the operator $\mathcal{U}$ is continuous on $\mathcal{B}_\varkappa$.

**Step 4.** Let $\mathcal{X}$ be a non-empty subset of $\mathcal{B}_\varkappa$ and let $\epsilon>0$ be fixed. Further, we choose $\psi\in\mathcal{X}$ and $\mu_1,\mu_2\in[0,L]$ with $\mu_2>\mu_1$ such that $|\mu_2-\mu_1|\leq\epsilon$. Then, we obtain

$$\begin{aligned}
&\big|(\mathcal{U}\psi)(\mu_2) - (\mathcal{U}\psi)(\mu_1)\big| \\
&= \big|Q(\mu_2, \psi(\mu_2)) + \delta(\mu_2, \psi(\mu_2)) \cdot (T\psi)(\mu_2) + \xi(\mu_2, \psi(\mu_2)) \cdot (G\psi)(\mu_2) \\
&\quad - Q(\mu_1, \psi(\mu_1)) - \delta(\mu_1, \psi(\mu_1)) \cdot (T\psi)(\mu_1) - \xi(\mu_1, \psi(\mu_1)) \cdot (G\psi)(\mu_1) \\
&\quad + \delta(\mu_2, \psi(\mu_2)) \cdot (T\psi)(\mu_1) - \delta(\mu_2, \psi(\mu_2)) \cdot (T\psi)(\mu_1) \\
&\quad + \xi(\mu_2, \psi(\mu_2)) \cdot (G\psi)(\mu_1) - \xi(\mu_2, \psi(\mu_2)) \cdot (G\psi)(\mu_1)\big| \\
&\leq \big|Q(\mu_2, \psi(\mu_2)) - Q(\mu_1, \psi(\mu_1))\big| \\
&\quad + \big|\delta(\mu_2, \psi(\mu_2))\big| \cdot \big|(T\psi)(\mu_2) - (T\psi)(\mu_1)\big| \\
&\quad + \big|\delta(\mu_2, \psi(\mu_2)) - \delta(\mu_1, \psi(\mu_1))\big| \cdot \big|(T\psi)(\mu_1)\big| \\
&\quad + \big|\xi(\mu_2, \psi(\mu_2))\big| \cdot \big|(G\psi)(\mu_2) - (G\psi)(\mu_1)\big| \\
&\quad + \big|\xi(\mu_2, \psi(\mu_2)) - \xi(\mu_1, \psi(\mu_1))\big| \cdot \big|(G\psi)(\mu_1)\big|,
\end{aligned}$$

i.e.,

$$\begin{aligned}
&\big|(\mathcal{U}\psi)(\mu_2) - (\mathcal{U}\psi)(\mu_1)\big| \\
&\leq \big|Q(\mu_2, \psi(\mu_2)) - Q(\mu_2, \psi(\mu_1))\big| + \big|Q(\mu_2, \psi(\mu_1)) - Q(\mu_1, \psi(\mu_1))\big| \\
&\quad + \big(\big|\delta(\mu_2, \psi(\mu_2)) - \delta(\mu_2, 0)\big| + \big|\delta(\mu_2, 0)\big|\big) \cdot \big|(T\psi)(\mu_2) - (T\psi)(\mu_1)\big| \\
&\quad + \big(\big|\delta(\mu_2, \psi(\mu_2)) - \delta(\mu_2, \psi(\mu_1))\big| + \big|\delta(\mu_2, \psi(\mu_1)) - \delta(\mu_1, \psi(\mu_1))\big|\big) \cdot \big|(T\psi)(\mu_1)\big| \\
&\quad + \big(\big|\xi(\mu_2, \psi(\mu_2)) - \xi(\mu_2, 0)\big| + \big|\xi(\mu_2, 0)\big|\big) \cdot \big|(G\psi)(\mu_2) - (G\psi)(\mu_1)\big| \\
&\quad + \big(\big|\xi(\mu_2, \psi(\mu_2)) - \xi(\mu_2, \psi(\mu_1))\big| + \big|\xi(\mu_2, \psi(\mu_1)) - \xi(\mu_1, \psi(\mu_1))\big|\big) \cdot \big|(G\psi)(\mu_1)\big| \\
&\leq M_1|\psi(\mu_2) - \psi(\mu_1)| + \omega(Q, \epsilon) + (d_1|\psi(\mu_2)| + M_2) \cdot \big|(T\psi)(\mu_2) - (T\psi)(\mu_1)\big| \\
&\quad + \big(d_1\big|\psi(\mu_2) - \psi(\mu_1)\big| + \omega(\delta, \epsilon)\big) \cdot \big|(T\psi)(\mu_1)\big| \\
&\quad + (d_2|\psi(\mu_2)| + M_3) \cdot \big|(G\psi)(\mu_2) - (G\psi)(\mu_1)\big| \\
&\quad + \big(d_2\big|\psi(\mu_2) - \psi(\mu_1)\big| + \omega(\xi, \epsilon)\big) \cdot \big|(G\psi)(\mu_1)\big|,
\end{aligned} \tag{24}$$

where

$\omega(\xi, \epsilon) = \sup \{ |\xi(\mu_2, \psi) - \xi(\mu_1, \psi)| : \mu_1, \mu_2 \in [0, L], \psi \in [-\varkappa, \varkappa], |\mu_2 - \mu_1| \leq \epsilon \}$,

$\omega(Q, \epsilon) = \sup \{ |Q(\mu_2, \psi) - Q(\mu_1, \psi)| : \mu_1, \mu_2 \in [0, L], \psi \in [-\varkappa, \varkappa], |\mu_2 - \mu_1| \leq \epsilon \}$,

$\omega(\delta, \epsilon) = \sup \{ |\delta(\mu_2, \psi) - \delta(\mu_1, \psi)| : \mu_1, \mu_2 \in [0, L], \psi \in [-\varkappa, \varkappa], |\mu_2 - \mu_1| \leq \epsilon \}$.

Now,

$$\begin{aligned}
\left|(T\psi)(\mu_1)\right| &= \left| \frac{1}{\Gamma(\lambda)} \int_0^{\mu_1} \vartheta^{\beta-1} \left( \frac{\mu_1^\beta - \vartheta^\beta}{\beta} \right)^{\lambda-1} \mathcal{K}_1(\mu_1, \vartheta) \mathcal{F}_1(\vartheta, \psi(\vartheta)) d\vartheta \right| \\
&\leq \frac{l_1 \Omega_1(\|\psi\|)}{\Gamma(\lambda)} \int_0^{\mu_1} \vartheta^{\beta-1} \left( \frac{\mu_1^\beta - \vartheta^\beta}{\beta} \right)^{\lambda-1} d\vartheta \\
&\leq \frac{l_1 \Omega_1(\varkappa)}{\Gamma(\lambda)} \int_0^{\mu_1} \vartheta^{\beta-1} \left( \frac{\mu_1^\beta - \vartheta^\beta}{\beta} \right)^{\lambda-1} d\vartheta \\
&\leq \frac{l_1 \Omega_1(\varkappa)}{\Gamma(\lambda+1)} \beta^{-\lambda} L^{\beta\lambda}, \qquad (25)
\end{aligned}$$

and

$$\begin{aligned}
\left|(G\psi)(\mu_1)\right| &= \left| \frac{1}{\Gamma(\lambda)} \int_0^{L} \vartheta^{\beta-1} \left( \frac{L^\beta - \vartheta^\beta}{\beta} \right)^{\lambda-1} \mathcal{K}_2(\mu_1, \vartheta) \mathcal{F}_2(\vartheta, \psi(\vartheta)) d\vartheta \right| \\
&\leq \frac{l_2 \Omega_2(\|\psi\|)}{\Gamma(\lambda)} \int_0^{L} \vartheta^{\beta-1} \left( \frac{L^\beta - \vartheta^\beta}{\beta} \right)^{\lambda-1} d\vartheta \\
&\leq \frac{l_2 \Omega_2(\varkappa)}{\Gamma(\lambda+1)} \beta^{-\lambda} L^{\beta\lambda}. \qquad (26)
\end{aligned}$$

By using (25) and (26), we get from (24) as follows:

$$\begin{aligned}
\left|(\mathcal{U}\psi)(\mu_2) - (\mathcal{U}\psi)(\mu_1)\right| &\leq M_1 \omega(\psi, \epsilon) + \omega(Q, \epsilon) + (d_1 \varkappa + M_2) \cdot \left|(T\psi)(\mu_2) - (T\psi)(\mu_1)\right| \\
&\quad + (d_1 \omega(\psi, \epsilon) + \omega(\delta, \epsilon)) \cdot \frac{l_1 \Omega_1(\varkappa)}{\Gamma(\lambda+1)} \beta^{-\lambda} L^{\beta\lambda} \\
&\quad + (d_2 \varkappa + M_3) \cdot \left|(G\psi)(\mu_2) - (G\psi)(\mu_1)\right| \\
&\quad + (d_2 \omega(\psi, \epsilon) + \omega(\xi, \epsilon)) \cdot \frac{l_2 \Omega_2(\varkappa)}{\Gamma(\lambda+1)} \beta^{-\lambda} L^{\beta\lambda}, \qquad (27)
\end{aligned}$$

i.e.,

$$\begin{aligned}\omega(\mathcal{U}\psi,\epsilon) \le{}& M_1\omega(\psi,\epsilon)+\omega(Q,\epsilon)+(d_1\varkappa+M_2)\cdot\left|(T\psi)(\mu_2)-(T\psi)(\mu_1)\right|\\ &+(d_1\omega(\psi,\epsilon)+\omega(\delta,\epsilon))\cdot\frac{l_1\Omega_1(\varkappa)}{\Gamma(\lambda+1)}\beta^{-\lambda}L^{\beta\lambda}\\ &+(d_2\varkappa+M_3)\cdot\left|(G\psi)(\mu_2)-(G\psi)(\mu_1)\right|\\ &+(d_2\omega(\psi,\epsilon)+\omega(\xi,\epsilon))\cdot\frac{l_2\Omega_2(\varkappa)}{\Gamma(\lambda+1)}\beta^{-\lambda}L^{\beta\lambda}. \end{aligned}\tag{28}$$

Thus, if $|\mu_2-\mu_1|\le\epsilon$, and $\epsilon\to 0$. Then as $\mu_2\to\mu_1$ and by following the inequality (13) to (22), and (23), we obtain $\left|(T\psi)(\mu_2)-(T\psi)(\mu_1)\right|\to 0$, and $\left|(G\psi)(\mu_2)-(G\psi)(\mu_1)\right|\to 0$.

Again, by the uniform continuity of the functions $Q$, $\delta$, $\xi$ on $[0,L]\times[-\varkappa,\varkappa]$, we obtain $\omega(Q,\epsilon)\to 0$, $\omega(\delta,\epsilon)\to 0$, and $\omega(\xi,\epsilon)\to 0$ as $\epsilon\to 0$.

Then we get

$$\omega_0(\mathcal{U}\mathcal{X})\le\left(M_1+\frac{d_1l_1\Omega_1(\varkappa)}{\Gamma(\lambda+1)}\beta^{-\lambda}L^{\beta\lambda}+\frac{d_2l_2\Omega_2(\varkappa)}{\Gamma(\lambda+1)}\beta^{-\lambda}L^{\beta\lambda}\right)\omega_0(\mathcal{X}).\tag{29}$$

From the condition $(A_5)$, we observe that $\left(M_1+\frac{d_1l_1\Omega_1(\varkappa)}{\Gamma(\lambda+1)}\beta^{-\lambda}L^{\beta\lambda}+\frac{d_2l_2\Omega_2(\varkappa)}{\Gamma(\lambda+1)}\beta^{-\lambda}L^{\beta\lambda}\right)<1$.

Thus the Theorem 2.3 allows us to deduce that Eq. (4) has a solution in $C([0,L],\mathbb{R})$. □

## 4. Method for approximate solutions

In this section, we will introduce a method that is based on Fibonacci wavelets and collocation technique to obtain the approximate solutions of Eq. (4).

**4.1. Approximation of function.** An arbitrary function $\psi(\mu)\in C([0,1),\mathbb{R})$ can be approximately expanded in terms of the Fibonacci wavelets as follows:

$$\psi(\mu)\simeq\sum_{\sigma=1}^{2^{k-1}}\sum_{\eta=0}^{M-1}g_{\sigma,\eta}\Phi_{\sigma,\eta}(\mu)=G^T\Phi(\mu),\tag{30}$$

where $G$ and $\Phi(\mu)$ are given by

$$G=[g_{1,0},g_{1,1},\ldots,g_{1,M-1},g_{2,0},g_{2,1},\ldots,g_{2,M-1},\ldots,g_{2^{k-1},0},\ldots,g_{2^{k-1},M-1}]^T,\tag{31}$$

$$\Phi(\mu)=[\Phi_{1,0}(\mu),\ldots,\Phi_{1,M-1}(\mu),\Phi_{2,0}(\mu),\ldots,\Phi_{2,M-1}(\mu),\ldots,\Phi_{2^{k-1},0}(\mu),\ldots,\Phi_{2^{k-1},M-1}(\mu)]^T.\tag{32}$$

**4.2. Computational steps.** To establish the method, we rewriting the considered equation Eq. (4) as follows:

(33)
$$\psi(\mu) = Q(\mu,\psi(\mu)) + \frac{\delta(\mu,\psi(\mu))}{\Gamma(\lambda)} \int_0^\mu \vartheta^{\beta-1}\left(\frac{\mu^\beta-\vartheta^\beta}{\beta}\right)^{\lambda-1} \mathcal{K}_1(\mu,\vartheta)\mathcal{F}_1(\vartheta,\psi(\vartheta))d\vartheta$$
$$+\frac{\xi(\mu,\psi(\mu))}{\Gamma(\lambda)} \int_0^1 \vartheta^{\beta-1}\left(\frac{1-\vartheta^\beta}{\beta}\right)^{\lambda-1} \mathcal{K}_2(\mu,\vartheta)\mathcal{F}_2(\vartheta,\psi(\vartheta))d\vartheta.$$

Therefore, the method is as follows:

**Step 1.** Choose the values of $k$ and $M$ for the approximate function

$$\psi(\mu) \simeq \sum_{\sigma=1}^{2^{k-1}} \sum_{\eta=0}^{M-1} g_{\sigma,\eta}\Phi_{\sigma,\eta}(\mu) = G^T\Phi(\mu), \tag{34}$$

where $G$ and $\Phi(\mu)$ are $\widehat{k}\times 1$ ($\widehat{k} = 2^{k-1}M$) vectors given in Eq (31) and Eq. (32), respectively and then obtain the corresponding Fibonacci wavelets $\Phi_{\sigma,\eta}(\mu)$, which is defined in Eq. (8).

**Step 2.** Remember that, we have to find the solution of Eq. (33), that is the unknown function $\psi(\mu)$ appears in Eq. (33). Thus, in this step, substitute Eq. (34) into Eq. (33) and we get

$$G^T\Phi(\mu)$$
$$= Q(\mu,G^T\Phi(\mu)) + \frac{\delta\left(\mu,G^T\Phi(\mu)\right)}{\Gamma(\lambda)} \int_0^\mu \vartheta^{\beta-1}\left(\frac{\mu^\beta-\vartheta^\beta}{\beta}\right)^{\lambda-1} \mathcal{K}_1(\mu,\vartheta)\mathcal{F}_1\left(\vartheta,G^T\Phi(\vartheta)\right)d\vartheta$$

(35)
$$+\frac{\xi\left(\mu,G^T\Phi(\mu)\right)}{\Gamma(\lambda)} \int_0^1 \vartheta^{\beta-1}\left(\frac{1-\vartheta^\beta}{\beta}\right)^{\lambda-1} \mathcal{K}_2(\mu,\vartheta)\mathcal{F}_2\left(\vartheta,G^T\Phi(\vartheta)\right)d\vartheta.$$

**Step 3.** Now to obtain the unknown coefficients vector $G = [g_{1,0}, g_{1,1},\ldots,g_{1,M-1}, g_{2,0}, g_{2,1},\ldots,g_{2,M-1},\ldots,g_{2^{k-1},0},\ldots,g_{2^{k-1},M-1}]^T$, we consider the collocation points as $\mu_i = \frac{(i-0.5)}{2^{k-1}M}$, $i = 1,2,\ldots,2^{k-1}M$. Then by substituting these collocation points in Eq. (35), we get a system of $2^{k-1}M$ algebraic equations. That is, we get

$$G^T\Phi(\mu_i)$$
$$= Q(\mu_i,G^T\Phi(\mu_i)) + \frac{\delta\left(\mu_i,G^T\Phi(\mu_i)\right)}{\Gamma(\lambda)} \int_0^{\mu_i} \vartheta^{\beta-1}\left(\frac{\mu_i^\beta-\vartheta^\beta}{\beta}\right)^{\lambda-1} \mathcal{K}_1(\mu_i,\vartheta)\mathcal{F}_1\left(\vartheta,G^T\Phi(\vartheta)\right)d\vartheta$$

(36)
$$+\frac{\xi\left(\mu_i,G^T\Phi(\mu_i)\right)}{\Gamma(\lambda)} \int_0^1 \vartheta^{\beta-1}\left(\frac{1-\vartheta^\beta}{\beta}\right)^{\lambda-1} \mathcal{K}_2(\mu_i,\vartheta)\mathcal{F}_2\left(\vartheta,G^T\Phi(\vartheta)\right)d\vartheta.$$

**Step 4.** Therefore by solving these algebraic equations by any classical method or the fsolve command in the MATLAB program, we can obtain the unknown coefficients $G$.

**Step 5.** Now, plug the obtained values of $G$ in Eq.(34) to get the approximate solution of Eq. (33).

Remark 4.1. It follows from Remark 1.1 that, our presented computational method is also applicable for Eqs. (1), (2) and (3) to get the approximate numerical solutions.

## 5. Error analysis

It is important to establish the accuracy of the obtained solutions. In this section we will study on the convergence results and error bounds for the presented Fibonacci wavelets method.

THEOREM 5.1. *Let $\psi_{\widehat{k}}(\mu) = G^T\Phi(\mu)$ be the Fibonacci wavelet expansion of any sufficiently smooth function $\psi(\mu) \in C^M([0,1),\mathbb{R})$. Then, we have $\|\psi - \psi_{\widehat{k}}\|_2 \leq \frac{\mathcal{R}}{M!\sqrt{2M+1}}$, where $\mathcal{R} = \max_{\mu\in[0,1)} \left|\psi^{(M)}(\mu)\right|$.*

*Proof.* See [37]. □

THEOREM 5.2. *Let $\psi(\mu) \in L^2[0,1)$ be a continuous bounded function with bound $\bar{M}$. Then, the Fibonacci wavelets expansion given by Eq. (30) converges uniformly to $\psi(\mu)$, where the coefficients $g_{\sigma,\eta}$ can be obtained by $g_{\sigma,\eta} = \langle\psi, \Phi_{\sigma,\eta}\rangle$.*

*Proof.* See [40]. □

**5.1. Accuracy of solutions.** Since Eq. (35) has the following form given by Eq. (37), i.e.,

$$G^T\Phi(\mu) = Q(\mu, G^T\Phi(\mu)) + \frac{\delta\left(\mu, G^T\Phi(\mu)\right)}{\Gamma(\lambda)} \int_0^{\mu} \vartheta^{\beta-1}\left(\frac{\mu^\beta - \vartheta^\beta}{\beta}\right)^{\lambda-1} \mathcal{K}_1(\mu,\vartheta)\mathcal{F}_1\left(\vartheta, G^T\Phi(\vartheta)\right) d\vartheta$$

$$+ \frac{\xi\left(\mu, G^T\Phi(\mu)\right)}{\Gamma(\lambda)} \int_0^{1} \vartheta^{\beta-1}\left(\frac{1 - \vartheta^\beta}{\beta}\right)^{\lambda-1} \mathcal{K}_2(\mu,\vartheta)\mathcal{F}_2\left(\vartheta, G^T\Phi(\vartheta)\right) d\vartheta. \tag{37}$$

Also, from Eq. (34), we have

$$\psi(\mu) \simeq \sum_{\sigma=1}^{2^{k-1}} \sum_{\eta=0}^{M-1} g_{\sigma,\eta}\Phi_{\sigma,\eta}(\mu) = G^T\Phi(\mu), \tag{38}$$

and the unknown coefficients $G$ were obtained from Eq. (36). Thus, Eq. (38) is the approximate solution of Eq. (33), and is substituted into Eq. (33).

Now, assume that $\mu = \mu_r \in [0,1)$, $r = 1,2,3,\ldots$, then $\exists$ a positive integer $N_q$ such that

$$\mathcal{E}(\mu_r) = \Bigg| G^T\Phi(\mu_r) - Q(\mu_r, G^T\Phi(\mu_r)) - \frac{\delta\left(\mu_r, G^T\Phi(\mu_r)\right)}{\Gamma(\lambda)} \int_0^{\mu_r} \vartheta^{\beta-1}\left(\frac{\mu_r^\beta - \vartheta^\beta}{\beta}\right)^{\lambda-1} \mathcal{K}_1(\mu_r,\vartheta)\mathcal{F}_1\left(\vartheta, G^T\Phi(\vartheta)\right) d\vartheta - \frac{\xi\left(\mu_r, G^T\Phi(\mu_r)\right)}{\Gamma(\lambda)} \int_0^{1} \vartheta^{\beta-1}\left(\frac{1 - \vartheta^\beta}{\beta}\right)^{\lambda-1} \mathcal{K}_2(\mu_r,\vartheta)\mathcal{F}_2\left(\vartheta, G^T\Phi(\vartheta)\right) d\vartheta \Bigg| \cong 0,$$

and $\mathcal{E}(\mu_r) \leq 10^{-N_q}$. If $\max 10^{-N_q} = 10^{-N}$, is prescribed, then $\widehat{k} = 2^{k-1}M$ is increased until the difference $\mathcal{E}(\mu_r)$ at each of the points becomes smaller than the prescribed $10^{-N}$, where N is a positive integer. For $\max 10^{-N_q} \neq 10^{-N}$, the error can be estimated by the following function:

$$\mathcal{E}_{\widehat{k}}(\mu) = \Bigg| G^T\Phi(\mu) - Q(\mu, G^T\Phi(\mu)) - \frac{\delta\left(\mu, G^T\Phi(\mu)\right)}{\Gamma(\lambda)} \int_0^{\mu} \vartheta^{\beta-1} \left(\frac{\mu^\beta - \vartheta^\beta}{\beta}\right)^{\lambda-1} \mathcal{K}_1(\mu,\vartheta)\mathcal{F}_1\left(\vartheta, G^T\Phi(\vartheta)\right) d\vartheta - \frac{\xi\left(\mu, G^T\Phi(\mu)\right)}{\Gamma(\lambda)} \int_0^{1} \vartheta^{\beta-1} \left(\frac{1 - \vartheta^\beta}{\beta}\right)^{\lambda-1} \mathcal{K}_2(\mu,\vartheta)\mathcal{F}_2\left(\vartheta, G^T\Phi(\vartheta)\right) d\vartheta \Bigg|.$$

If $\mathcal{E}_{\widehat{k}}(\mu) \to 0$, for sufficiently large $\widehat{k}$, then the error decreases.

## 6. Applications and discussions

Two examples have been given to demonstrate the efficiency and applicability of the presented method. The convergence of solutions and the convergence of absolute errors have been shown in the graphs. Here, absolute errors are the values of $|\psi(\mu) - \psi_{\widehat{k}}(\mu)|$ at selected points, where $\psi_{\widehat{k}}(\mu)$ is the approximate solution and the exact solution is $\psi(\mu)$.

Example 6.1. Consider the following example:

$$\psi(\mu) = Q(\mu, \psi(\mu)) + \frac{\sin(\psi(\mu))}{\Gamma(3)} \int_0^{\mu} \vartheta^{\frac{1}{2}} \left(\frac{\mu^{\frac{3}{2}} - \vartheta^{\frac{3}{2}}}{\frac{3}{2}}\right)^2 (1 + \mu\vartheta)\, \psi(\vartheta) d\vartheta + \frac{\psi(\mu)}{\Gamma(3)} \int_0^{1} \vartheta^{\frac{1}{2}} \left(\frac{1 - \vartheta^{\frac{3}{2}}}{\frac{3}{2}}\right)^2 (\mu + \vartheta)(\vartheta + \psi(\vartheta)) d\vartheta, \tag{39}$$

where $Q(\mu, \psi(\mu)) = \left(\frac{\mu - \mu^3}{90}\right) - \frac{\sin(\psi(\mu))}{90}\left[\frac{\mu^{\frac{11}{2}}}{55} + \frac{142\mu^{\frac{15}{2}}}{36855} - \frac{4\mu^{\frac{19}{2}}}{1309}\right] - \frac{\psi(\mu)}{90}\left[\frac{7349\mu}{4455} + \frac{5216}{6545}\right]$, and the exact solution is $\psi(\mu) = \frac{\mu - \mu^3}{90}$.

Now, comparing Eq. (39) with the Eq. (4), we get

$$\delta(\mu, \psi(\mu)) = \sin(\psi(\mu)),\ \ \xi(\mu, \psi(\mu)) = \psi(\mu),\ \ \mathcal{K}_1(\mu,\vartheta) = 1 + \mu\vartheta,\ \ \mathcal{K}_2(\mu,\vartheta) = \mu + \vartheta,$$

$$\mathcal{F}_1(\vartheta, \psi(\vartheta)) = \psi(\vartheta), \mathcal{F}_2(\vartheta, \psi(\vartheta)) = \vartheta + \psi(\vartheta),\ \ L = 1,\ \ \lambda = 3,\ \ \beta = \frac{3}{2}.$$

It can be observed that the function $Q$ satisfies the condition $(A_1)$ with $M_1 = 0.0274$, and $Q^* = 0.0043$. Condition $(A_2)$ is satisfied by the functions $\delta$ and $\xi$ with $d_1 = 1$, $d_2 = 1$, $M_2 = 0$ and $M_3 = 0$. Condition $(A_3)$ is satisfied by the functions $\mathcal{K}_1$ and $\mathcal{K}_2$ with $l_1 = 2$ and $l_2 = 2$, respectively. Condition $(A_4)$ is satisfied by the functions $\mathcal{F}_1$ and $\mathcal{F}_2$ with $\Omega_1(|\psi|) = |\psi|$ and $\Omega_2(|\psi|) = 1 + |\psi|$. Then, the inequalities appearing in condition $(A_5)$ becomes as

$$(M_1\varkappa + Q^*) + \frac{l_1\left(d_1\varkappa + M_2\right)\Omega_1(\varkappa)\beta^{-\lambda}}{\Gamma(\lambda+1)} L^{\beta\lambda} + \frac{l_2\left(d_2\varkappa + M_3\right)\Omega_2(\varkappa)\beta^{-\lambda}}{\Gamma(\lambda+1)} L^{\beta\lambda} \leq \varkappa,$$

i.e.,

$$0.0274\varkappa + 0.0043 + \frac{\frac{16}{27}\varkappa^2}{\Gamma(4)} + \frac{\varkappa(1+\varkappa)\frac{16}{27}}{\Gamma(4)} \leq \varkappa, \tag{40}$$

and

$$\left(M_1+\frac{d_1 l_1 \Omega_1(\varkappa)}{\Gamma(\lambda+1)}\beta^{-\lambda}L^{\beta\lambda}+\frac{d_2 l_2 \Omega_2(\varkappa)}{\Gamma(\lambda+1)}\beta^{-\lambda}L^{\beta\lambda}\right)<1,$$

i.e.,

$$0.0274+\frac{\frac{16}{27}\varkappa}{\Gamma(4)}+\frac{(1+\varkappa)\frac{16}{27}}{\Gamma(4)}<1. \tag{41}$$

Thus, it is clear that the condition $(A_5)$ is satisfied for $\varkappa = 1$. So, by Theorem 3.1, Eq. (39) has at least one solution in $C([0,1],\mathbb{R})$.

To obtain the approximate solutions to this problem, we are going to apply the presented computational method, i.e., discussed in Section 4. For this purpose, choosing different values of $k$ and $M$, so that $\widehat{k}$ is increasing, where $\widehat{k} = 2^{k-1}M$. All calculations have been carried out using the MATLAB program on a computer. For this computational purpose we are using the collocation points as $\mu_i = \frac{(i-0.5)}{2^{k-1}M}$. The variations of absolute errors and maximum absolute errors for some values of $k$ and $M$ are shown in Table 1 and Table 2, respectively. Figure 2 shows the solution convergence graph for $k = 2, M = 4$, and Figure 3 shows the absolute error convergence graph.

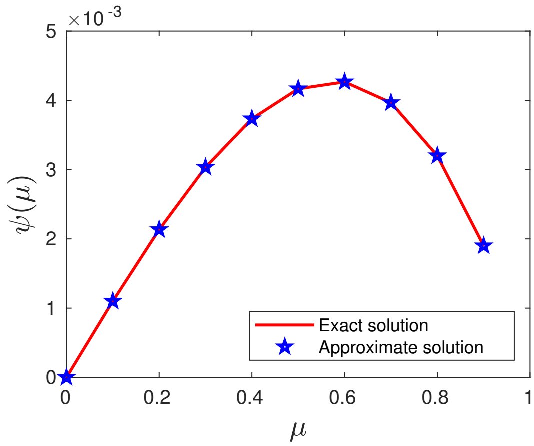


FIGURE 2. Solution convergence graph for Example 6.1.

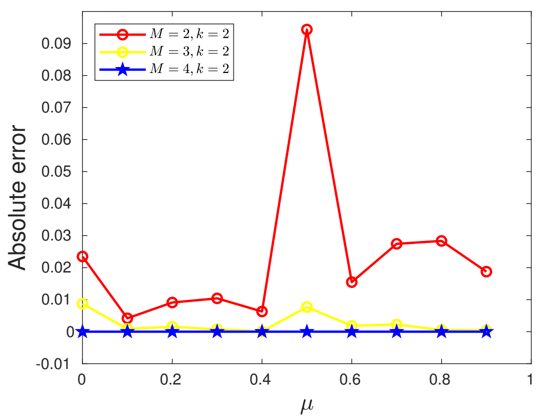


FIGURE 3. Absolute error convergence graph for Example 6.1.

TABLE 1. Absolute errors variation for Example 6.1.

| $\mu$ | $k=2, M=2$ | $k=2, M=3$ | $k=2, M=4$ |
|---|---|---|---|
| 0 | 2.3476e-02 | 8.7202e-03 | 3.3152e-08 |
| 0.1 | 4.1823e-03 | 8.7153e-04 | 2.1469e-08 |
| 0.2 | 9.1113e-03 | 1.5330e-03 | 6.1319e-08 |
| 0.3 | 1.0405e-02 | 7.3586e-04 | 3.1205e-08 |
| 0.4 | 6.3015e-03 | 6.5064e-05 | 4.2004e-07 |
| 0.5 | 9.4377e-02 | 7.6896e-03 | 1.0012e-08 |
| 0.6 | 1.5469e-02 | 1.8435e-03 | 4.1239e-08 |
| 0.7 | 2.7439e-02 | 2.2381e-03 | 1.3135e-09 |
| 0.8 | 2.8348e-02 | 5.0588e-04 | 2.1254e-08 |
| 0.9 | 1.8744e-02 | 3.8843e-04 | 1.1239e-08 |

TABLE 2. Comparison of maximum absolute errors for Example 6.1.

| $k$ | $M$ | $\widehat{k}=2^{k-1}M$ | Maximum absolute error |
|---|---|---|---|
| 2 | 2 | 4 | 9.4377e-02 |
| 2 | 3 | 6 | 8.7202e-03 |
| 2 | 4 | 8 | 4.2004e-07 |

EXAMPLE 6.2. Consider the following example:

(42)
$$\psi(\mu)=Q(\mu,\psi(\mu))+\frac{\psi(\mu)}{\Gamma(\frac{5}{2})}\int_0^\mu\vartheta\left(\frac{\mu^2-\vartheta^2}{2}\right)^{\frac{3}{2}}\psi^2(\vartheta)d\vartheta+\frac{\sin(\mu)}{\Gamma(\frac{5}{2})}\int_0^1\vartheta\left(\frac{1-\vartheta^2}{2}\right)^{\frac{3}{2}}\vartheta\psi(\vartheta)d\vartheta,$$

where $Q(\mu,\psi(\mu))=\frac{\mu^2}{20}-\frac{\mu^9\sqrt{2}\psi(\mu)}{63000\,\Gamma(\frac{5}{2})}-\frac{\pi\sin(\mu)\sqrt{18}}{20480\,\Gamma(\frac{5}{2})}$, and $\psi(\mu)=\frac{\mu^2}{20}$ is the exact solution.

Now, comparing Eq. (39) with the Eq. (4), we get

$$\delta(\mu,\psi(\mu))=\psi(\mu),\ \ \xi(\mu,\psi(\mu))=\sin(\mu),\ \ \mathcal{K}_1(\mu,\vartheta)=1,\ \ \mathcal{K}_2(\mu,\vartheta)=\vartheta,\ \ \mathcal{F}_1(\vartheta,\psi(\vartheta))=\psi^2(\vartheta),$$
$$\mathcal{F}_2(\vartheta,\psi(\vartheta))=\psi(\vartheta),\ \ L=1,\ \ \lambda=\frac{5}{2},\ \ \beta=2.$$

It can be observed that the function $Q$ satisfies the condition $(A_1)$ with $M_1=0.000017$, and $Q^*=0.0505$. Condition $(A_2)$ is satisfied by the functions $\delta$ and $\xi$ with $d_1=1$, $d_2=0$, $M_2=0$ and $M_3=1$. Condition $(A_3)$ is satisfied by the functions $\mathcal{K}_1$ and $\mathcal{K}_2$ with $l_1=1$ and $l_2=1$, respectively. Condition $(A_4)$ is satisfied by the functions $\mathcal{F}_1$ and $\mathcal{F}_2$ with $\Omega_1(|\psi|)=|\psi|^2$ and $\Omega_2(|\psi|)=|\psi|$. Then, the inequalities appearing in condition $(A_5)$ becomes as

$$(M_1\varkappa+Q^*)+\frac{l_1\,(d_1\varkappa+M_2)\,\Omega_1(\varkappa)\beta^{-\lambda}}{\Gamma(\lambda+1)}L^{\beta\lambda}+\frac{l_2\,(d_2\varkappa+M_3)\,\Omega_2(\varkappa)\beta^{-\lambda}}{\Gamma(\lambda+1)}L^{\beta\lambda}\le\varkappa,$$

i.e.,

(43)
$$0.000017\varkappa+0.0505+\frac{(\varkappa+0)0.1768\varkappa^2}{\Gamma(\frac{7}{2})}+\frac{(0+1)0.1768\varkappa}{\Gamma(\frac{7}{2})}\le\varkappa,$$

and

$$\left(M_1 + \frac{d_1 l_1 \Omega_1(\varkappa)}{\Gamma(\lambda+1)}\beta^{-\lambda}L^{\beta\lambda} + \frac{d_2 l_2 \Omega_2(\varkappa)}{\Gamma(\lambda+1)}\beta^{-\lambda}L^{\beta\lambda}\right) < 1,$$

i.e.,

$$0.000017 + \frac{0.1768\varkappa^2}{\Gamma(\frac{7}{2})} + 0 < 1. \tag{44}$$

Thus, it is clear that the condition $(A_5)$ is satisfied for $\varkappa = 1$. So, by Theorem 3.1, Eq. (42) has at least one solution in $C([0,1],\mathbb{R})$.

To obtain the approximate solutions to this problem, we are going to apply the proposed computational method, i.e., discussed in Section 4. For this purpose, choosing different values of $k$ and $M$, so that $\widehat{k}$ is increasing, where $\widehat{k} = 2^{k-1}M$. All calculations have been carried out using the MATLAB program on a computer. For this computational purpose we are using the collocation points as $\mu_i = \frac{(i-0.5)}{2^{k-1}M}$. The variations of absolute errors and maximum absolute errors for some values of $k$ and $M$ are shown in Table 3 and Table 4, respectively. Figure 4 shows the solution convergence graph for $k = 2, M = 4$, and Figure 5 shows the absolute error convergence graph.

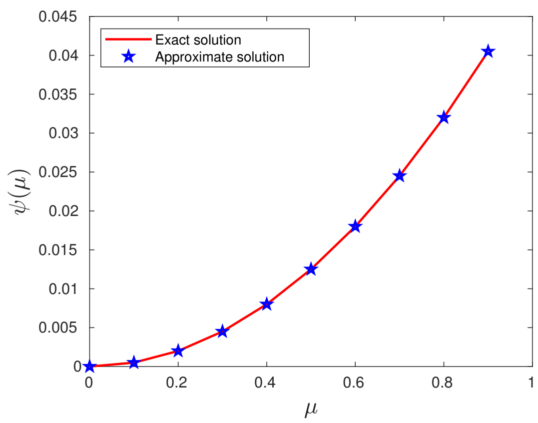


FIGURE 4. Solution convergence graph for Example 6.2.

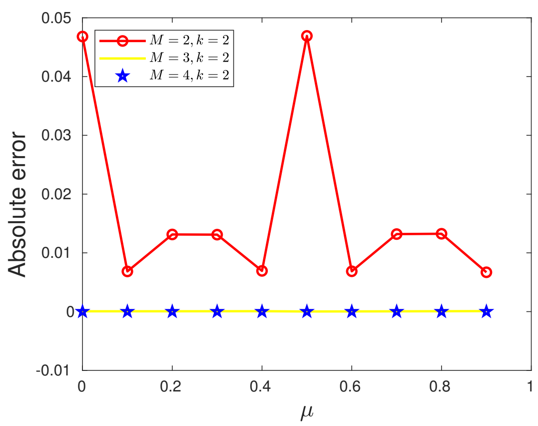


FIGURE 5. Absolute error convergence graph for Example 6.2.

TABLE 3. Absolute errors variation for Example 6.2

| $\mu$ | $k=2, M=2$ | $k=2, M=3$ | $k=2, M=4$ |
|---|---|---|---|
| 0 | 4.6810e-02 | 5.6709e-05 | 1.1310e-09 |
| 0.1 | 6.8409e-03 | 5.7659e-05 | 3.7818e-09 |
| 0.2 | 1.3129e-02 | 6.0509e-05 | 2.5614e-09 |
| 0.3 | 1.3098e-02 | 6.5259e-05 | 3.4309e-09 |
| 0.4 | 6.9321e-03 | 7.1910e-05 | 4.1517e-09 |
| 0.5 | 4.6919e-02 | 2.3751e-05 | 1.0816e-10 |
| 0.6 | 6.8632e-03 | 3.6530e-05 | 2.1817e-09 |
| 0.7 | 1.3193e-02 | 5.1209e-05 | 2.5109e-09 |
| 0.8 | 1.3248e-02 | 6.7788e-05 | 4.0891e-09 |
| 0.9 | 6.6960e-03 | 8.6267e-05 | 6.1253e-08 |

TABLE 4. Comparison of maximum absolute errors for Example 6.2.

| $k$ | $M$ | $\widehat{k}=2^{k-1}M$ | Maximum absolute error |
|---|---|---|---|
| 2 | 2 | 4 | 4.6919e-02 |
| 2 | 3 | 6 | 8.6267e-05 |
| 2 | 4 | 8 | 6.1253e-08 |

## 7. Conclusions and future work

In this study, we considered Eq. (4), involving the Katugampola fractional integral of order $\lambda > 0$ and with the parameter $\beta > 0$. We have stated some requirements for the existence of solutions. The concepts of the fixed point theorem and the measure of noncompactness have been used to prove the existence result. Furthermore, a computational method based on the Fibonacci wavelets and collocation technique has been presented to obtain the approximate solutions of Eq. (4). By this method, Eq. (4) has been reduced to a system of algebraic equations with unknown Fibonacci coefficients, and then solved by the MATLAB program. To evaluate the efficiency and applicability of the method, we provided two examples along with error estimates. Absolute error convergence and solution convergence graphs for Examples 6.1 and 6.2 have been given in computational tables and figures. It can be observed from Remark 1.1 that our suggested method is also applicable for Eqs. (1), (2) and (3). By observation of computational results and relevant figures, we have seen that the approximate solutions are in strong agreement with those of the exact solutions, and as a result, we draw the conclusion that the presented method is efficient, accurate, and effective.

In the future, one can extend the concepts presented here for the existence of solutions and approximate solutions to nonlinear V-F fractional integro-differential equations and also for stochastic integral equations, or by considering some generalized fractional integral equations, satisfy criteria different from those executed in this work.

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**Supriya Kumar Paul**
Department of Mathematics, School of Advanced Sciences,
Vellore Institute of Technology, Vellore 632 014, Tamil Nadu, India
*E-mail*: rgumathsupriya@gmail.com

**Lakshmi Narayan Mishra**
Department of Mathematics, School of Advanced Sciences,
Vellore Institute of Technology, Vellore 632 014, Tamil Nadu, India
*E-mail*: lakshminarayanmishra04@gmail.com, lakshminarayan.mishra@vit.ac.in