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CLASSIFICATION OF FOUR DIMENSIONAL BARIC ALGEBRAS SATISFYING POLYNOMIAL IDENTITY OF DEGREE SIX

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Abstract. In this paper, we proceeded to the classification of four dimensional baric algebras strictly satisfying a polynomial identity of degree six. After some results on the structure of such algebras, we show that the type of an algebra of the studied class is an invariant under change of idempotent in the Peirce decomposition. This last result plays a major role in our classification.

1. Introduction

Bernstein (see [3], [4]) is one of the artisans of the use of mathematics to model genetics. He gave a mathematical demonstration of the principle of stationarity, that is to say the conditions required to ensure the equilibrium of a population after one generation. Later, Etherington [5], Schafer [14] and more authors (see [8], [16]) raised the interest of non-associative algebras in genetic modeling. They thus explored the structure of several classes of not necessarily associative algebras that underlie the theory of the algebras of population genetics. Based on these ideas, Holgate [9] revisited the study of Bernstein's problem and this inspired several authors (see [1], [2], [12], [13], [15]) who studied other classes of baric algebras satisfying polynomial identities of degree less than five strongly related to Bernstein algebras. In the same dynamic, in [10], the authors studied the structure of baric algebras satisfying the identity

$$2x^2x^4 = \omega(x)^2x^4 + \omega(x)^4x^2 \tag{1}$$

and it will be for us in this paper to proceed to the classification of this class of algebras in dimension four. The identity (1) is a special case of the identity $2x^ix^j = \omega(x)^ix^j + \omega(x)^jx^i$, $\forall i, j \geq 2$ (see [16]) which itself follows from the identity $x^2x^2 = \omega(x)^2x^2$ which characterizes the Bernstein algebras. If for $\omega(x) = 1$, the Bernstein identity models a population whose genetic background stabilizes in the second generation, in identity (1) we consider the model of a population whose random crossing of two respective individuals of the second and fourth generation gives individuals whose genetic background is the arithmetic mean of the two genetic

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backgrounds. We begin in the preliminaries, by recalling the notions of baric algebras, train algebras and Peirce decomposition. In this paper, Peirce decomposition technique is fully used. For an in-depth understanding of this technique, reader could consult the following publications: [7], [6]. In the classification given in this paper, we are interested in algebras strictly verifying identity (1). After showing that such algebras are of dimension at least 4, we determined the possible types of these algebras in dimension 4. This allowed us to find algebras not yet present in the literature.

2. Preliminaries

Let K be a commutative field and A a commutative not necessarily associative K-algebra. For any element x of A we define the *principal powers* of x by $x^1 = x$, $x^{k+1} = xx^k$ for any integer $k \geq 1$.

A nonzero element e of A satisfying $e^2 = e$ is called *idempotent* of A. The idempotents considered here are non-zero.

Definition 2.1. We will say that the commutative not necessarily associative K-algebra A is baric algebra if there exists a nonzero morphism of algebras $\omega : A \to K$. The weight of an element x of A is the scalar $\omega(x)$.

Definition 2.2. A baric K-algebra(A, ω) is a train algebra of rank $n \geq 2$ if there exist scalars $\gamma_i \in K$ such that $x^n + \gamma_1\omega(x)x^{n-1} + ... + \gamma_{n-1}\omega(x)^{n-1}x = 0$, $\forall x \in A$ where n is the smallest integer satisfying this property.

Definition 2.3. We recall that a baric K-algebra (A, ω) is a Bernstein algebra if for all x in A, $(x^2)^2 = \omega(x)^2x^2$.

A baric K-algebra (A, ω) strictly satisfies (1), if A satisfies the polynomial identity (1) and does not verify a polynomial identity of degree at most five.

in the rest of the paper, $K = \mathbb{C}$ is the field of complex numbers and (A, ω) a baric K-algebra of finite dimension on K satisfying the identity (1). In this paper, we strongly use the Peirce decomposition whose main results are given in the following theorem ([10]). Let e be a non-zero idempotent of A. It is clear that $\omega(e) = 1$ and $A = Ke \oplus ker\omega$. A partial linearization of (1) is given by the identity

$4x^2(x(x(xy))) + 2x^2(x(x^2y)) + 2x^2(x^3y) + 4x^4(xy) = \omega(x)^2[2x(x(xy)) + x(x^2y) + x^3y] + 2\omega(xy)x^4 + 4\omega(x^3y)x^2 + 2\omega(x)^4(xy)$.

For $x = e$ and y in $ker\omega$ in this identity, we have $4\ell_e^4 + 5\ell_e^2 - 3\ell_e = P(\ell_e) = 0$ where $\ell_e = L_e/ker(\omega)$, $L_e : A \longrightarrow A, x \mapsto ex$ and $P(X) = 4X^4 + 5X^2 - 3X = 4X(X - \frac{1}{2})(X - \lambda)(X - \bar{\lambda})$ (with $\lambda = \frac{-1-i\sqrt{23}}{4}$ and $\bar{\lambda} = \frac{-1+i\sqrt{23}}{4}$). According to a well-known linear algebra theorem, it follows that $Ker\omega = kerP(\ell_e) = ker\ell_e \oplus ker(\ell_e - \frac{1}{2}i_d) \oplus ker(\ell_e - \lambda i_d) \oplus ker(\ell_e - \bar{\lambda} i_d)$. By setting $A_\alpha = ker(\ell_e - \alpha i_d)$, with $\alpha \in \{0; \frac{1}{2}; \lambda = \frac{-1-i\sqrt{23}}{4}; \bar{\lambda} = \frac{-1+i\sqrt{23}}{4}\}$ and $i_d : ker\omega \longrightarrow ker\omega, x \mapsto x$, we obtain the following equality, called the Peirce decomposition of A and given in the theorem below: $A = Ke \oplus A_0 \oplus A_{\frac{1}{2}} \oplus A_\lambda \oplus A_{\bar{\lambda}}$.

Theorem 2.4. *[10] Let (A, ω) be a K-algebra satisfying (1) and e an idempotent of A, then A admits a Peirce decomposition relatively to e: $A = Ke \oplus A_0 \oplus A_{\frac{1}{2}} \oplus A_\lambda \oplus A_{\bar{\lambda}}$ where $A_\alpha = \{y \in Ker(\omega), ey = \alpha y\}$, with $\alpha \in \{0, \frac{1}{2}, \lambda = \frac{-1-i\sqrt{23}}{4}, \bar{\lambda} = \frac{-1+i\sqrt{23}}{4}\}$ and the following relations hold:*

i): $A_0A_0 \subset A_{\frac{1}{2}}$;
ii): $A_{\frac{1}{2}}A_{\frac{1}{2}} \subset A_0 \oplus A_\lambda \oplus A_{\bar{\lambda}}$;
iii): $A_\lambda A_{\bar{\lambda}} = A_{\bar{\lambda}}A_{\bar{\lambda}} = A_\lambda A_\lambda = 0$;
iv): $A_0A_{\frac{1}{2}} \subset A_{\frac{1}{2}} \oplus A_\lambda \oplus A_{\bar{\lambda}}$;
v): $A_\lambda A_{\frac{1}{2}} \subset A_{\frac{1}{2}} \oplus A_0 \oplus A_{\bar{\lambda}}$;
vi): $A_{\bar{\lambda}}A_{\frac{1}{2}} \subset A_{\frac{1}{2}} \oplus A_0 \oplus A_\lambda$;
vii): $A_0A_\lambda \subset A_{\frac{1}{2}}$;
viii): $A_0A_{\bar{\lambda}} \subset A_{\frac{1}{2}}$.

The following corollary, established in [11], will be very useful for our classification.

Corollary 2.5. *If $A_{\bar{\alpha}} = 0$, with $\alpha \in \{\lambda, \bar{\lambda}\}$ then the following identities are verified:*
i): $2ex_0^3 - x_0^3 = 0$;
ii): $[12(x_{\frac{1}{2}}(x_{\frac{1}{2}}^2)_0) + 3(\alpha+1)(x_{\frac{1}{2}}(x_{\frac{1}{2}}^2)_\alpha)]_{\frac{1}{2}} = 0$;
iii): $[(\alpha+1)(x_{\frac{1}{2}}(x_0^2)_{\frac{1}{2}}) + 8(x_0(x_0x_{\frac{1}{2}})_{\frac{1}{2}})]_\alpha = 0$;
iv): $[(-4\alpha-2)(x_{\frac{1}{2}}(x_0x_{\frac{1}{2}})_\alpha) + 12(x_{\frac{1}{2}}(x_0x_{\frac{1}{2}})_{\frac{1}{2}})]_0 = 0$;
v): $[(6\alpha+13)(x_{\frac{1}{2}}(x_\alpha x_{\frac{1}{2}})_0) + (2\alpha+12)(x_{\frac{1}{2}}(x_\alpha x_{\frac{1}{2}})_{\frac{1}{2}})]_\alpha = 0$;
vi): $[(7\alpha+3)(x_\alpha(x_\alpha x_{\frac{1}{2}})_{\frac{1}{2}}) + 8\alpha(x_\alpha(x_\alpha x_{\frac{1}{2}})_0)]_{\frac{1}{2}} = 0$;
vii): $[x_{\frac{1}{2}}(x_0^2)_{\frac{1}{2}}]_0 = [x_\alpha(x_0^2)_{\frac{1}{2}}]_0 = [x_\alpha(x_0x_\alpha)_{\frac{1}{2}}]_0 = [x_\alpha(x_{\frac{1}{2}}x_\alpha)_{\frac{1}{2}}]_0 = [x_\alpha(x_{\frac{1}{2}}x_0)_{\frac{1}{2}}]_0 = 0$;
viii): $[x_\alpha(x_0^2)_{\frac{1}{2}}]_{\frac{1}{2}} = [x_\alpha(x_0x_\alpha)_{\frac{1}{2}}]_{\frac{1}{2}} = [x_\alpha(x_{\frac{1}{2}}^2)_0]_{\frac{1}{2}} = 0$;
ix): $[x_0(x_0x_\alpha)_{\frac{1}{2}}]_\alpha = [x_{\frac{1}{2}}(x_0x_\alpha)_{\frac{1}{2}}]_\alpha = 0$.

3. Some results on the structure

Definition 3.1. Let $A = Ke \oplus A_0 \oplus A_{\frac{1}{2}} \oplus A_\lambda \oplus A_{\bar{\lambda}}$ be a Peirce decomposition of a finite dimensional algebra satisfying the identity (1). If the dimension of A is n, we call type of A the quadruplet $(1+r, s, t_1, t_2)$ such that $1+r+s+t_1+t_2 = n$ where $r = dimA_{\frac{1}{2}}$, $s = dimA_0$, $t_1 = dimA_\lambda$, $t_2 = dimA_{\bar{\lambda}}$.

Remark 3.2. The type of A is an invariant in dimension 4 according to the proposition 4.4, which facilitates the classification in dimension 4.

The following proposition gives us the necessary conditions allowing us to rule out certain cases in the classification in dimension 4 of the algebras strictly satisfying the identity (1). This proposition therefore makes it possible to find algebras not yet present in the literature.

Proposition 3.3. *Let $A = Ke \oplus A_0 \oplus A_{\frac{1}{2}} \oplus A_\lambda \oplus A_{\bar{\lambda}}$ the Peirce decomposition relatively to an idempotent e of an algebra strictly satisfying the identity* (1).
i): *the dimension of A is strictly greater than* 3;
ii): *if A has dimension* 4 *then the possible types of A are* $(2,1,1,0)$, $(2,0,1,1)$ *and* $(2,1,0,1)$.

Proof. *i*) Consider the cases where A has dimension less than or equal to 3. If A has dimension 1, we have $A = Ke$ and $x^2 = \omega(x)x$ for all x in, which is impossible, hence the dimension of A is different from 1.

In dimension 2, the possible types of A are $(1,1,0,0)$, $(1,0,1,0)$, $(1,0,0,1)$ and $(2,0,0,0)$:

- ✓: the type $(2,0,0,0)$ implies that the algebra checks the identity $x^2 = \omega(x)x$,
- ✓: the type $(1,1,0,0)$ implies that $A_\lambda = A_{\bar{\lambda}} = 0$, thus A is a Bernstein algebra according to Theorem 4.1 of [10],
- ✓: Referring to Proposition 3 of [11], the types $(1,0,1,0)$ and $(1,0,0,1)$ lead us to train algebras of rank 3 because we have respectively $A_{\bar{\lambda}} = A_0 = 0$ and $A_\lambda = A_0 = 0$.

Thus, all possible types of A in dimension 2 are not admissible so the dimension of A is different from 2.
Suppose A has dimension 3; for $A_{\frac{1}{2}} = 0$, we know from Proposition 2 of [11] that A is a train algebra of rank at most 5, so cannot verify strictly a polynomial identity of degree 6.
Possible types of A would be $(2,1,0,0)$, $(2,0,1,0)$, $(2,0,0,1)$ and $(3,0,0,0)$ in case $A_{\frac{1}{2}} \neq 0$: Now the type $(3,0,0,0)$ implies that the algebra checks the identity $x^2 = \omega(x)x$, and therefore not admissible and the types $(2,1,0,0)$, $(2,0,1,0)$ and $(2,0,0,1)$ imply respectively $A_\lambda = A_{\bar{\lambda}} = 0$, $A_0 = A_{\bar{\lambda}} = 0$, $A_\lambda = A_0 = 0$; then according to Theorem 4.1 of [10] and Proposition 3 of [11], we had respectively a Bernstein algebra and train algebras of rank 3, so these types are also not admissible and consequently the dimension of A is different from 3.
ii) Now suppose A of dimension 4. If the type of A is $(2,2,0,0)$ or $(3,1,0,0)$, then $A_\lambda = A_{\bar{\lambda}} = 0$ and according to theorem 4.1 of [10], A is a Bernstein algebra and therefore satisfies a polynomial identity of degree at most 4.
Assuming the type of A is $(2,0,2,0)$ or $(3,0,1,0)$, we have $A_0 = A_{\bar{\lambda}} = 0$, or according to the assertion *i*) of Proposition 3 of [11]; A satisfies a polynomial identity of degree less than 6, which is nonsense. Thus the type of A is different from $(2,0,2,0)$ and $(3,0,1,0)$.
The spaces A_λ and $A_{\bar{\lambda}}$ having analogous properties, we proceed as the previous proof to justify that the type of A is different from (2, 0,0.2) and $(3,0,0,1)$. □

4. Classification in dimension four

This classification is given according to the type of the algebra and according to the proposition 3.3 the possible types in dimension 4 are $(2,1,1,0)$, $(2,0,1,1)$, $(2,1,0,1)$. Let (A,ω) be a baric algebra verifying a polynomial identity $P(x) = 0$. For the polynomial identity $P(x) = 0$ to be verified by A, it is necessary and sufficient that it be verified for all $x \in A$ of weight 1. This is justified by the fact that the set of elements of weight 1 is dense in A by the Zariski topology.

Proposition 4.1. *Let A be an algebra strictly satisfying the identity $2x^2x^4 = \omega(x)^2x^4 + \omega(x)^4x^2$ of Peirce's decomposition $A = Ke \oplus A_0 \oplus A_{\frac{1}{2}} \oplus A_\lambda \oplus A_{\bar{\lambda}}$ relative to an idempotent e. If the type of A is $(2,0,1,1)$ then A is an algebra satisfying the equation $(x^2)^2 - 2\omega(x)x^3 + \omega(x)^2x^2 = 0$.*

Proof. Let $A = Ke \oplus < e_1, e_2, e_3 >= Ke \oplus A_0 \oplus A_{\frac{1}{2}} \oplus A_\lambda \oplus A_{\bar{\lambda}}$ a polynomial algebra of degree 6 and dimension 4. The type of A being $(2,0,1,1)$, we have $A_0 = 0$. Setting $A_{\frac{1}{2}} =< e_1 >$, $A_\lambda =< e_2 >$, $A_{\bar{\lambda}} =< e_3 >$ we have the following multiplication table: $e^2 = e$, $ee_1 = \frac{1}{2}e_1$, $ee_2 = \lambda e_2$, $ee_3 = \bar{\lambda}e_3$, $e_1^2 = ae_2 + be_3$, $e_1e_2 = ce_1 + de_3$, $e_1e_3 = fe_1 + ge_2$, $e_2^2 = 0$, $e_2e_3 = 0$, $e_3^2 = 0$. Using the identities of Corollary 1 of [11], we have $f = c = dg = 0$ and the multiplication table of A becomes: $e^2 = e$, $ee_1 = \frac{1}{2}e_1$, $ee_2 = \lambda e_2$, $ee_3 = \bar{\lambda}e_3$, $e_1^2 = ae_2 + be_3$, $e_1e_2 = de_3$, $e_1e_3 = ge_2$, $e_2^2 = 0$, $e_2e_3 = 0$, $e_3^2 = 0$.

Let $x = e + \alpha_1e_1 + \alpha_2e_2 + \alpha_3e_3$ be an element of A such that $\omega(x) = 1$. we have $x^2 = e + \alpha_1e_1 + (2\lambda\alpha_2 + \alpha_1^2a + 2g\alpha_1\alpha_3)e_2 + (2\bar{\lambda}\alpha_3 + 2d\alpha_1\alpha_2 + \alpha_1^2b)e_3$ and $x^2 - x = (2\lambda\alpha_2 + \alpha_1^2a + 2g\alpha_1\alpha_3 - \alpha_2)e_2 + (2\bar{\lambda}\alpha_3 + 2d\alpha_1\alpha_2 + \alpha_1^2b - \alpha_3)e_3$. We have $0 = (x^2 - x)^2 = (x^2)^2 - 2x^3 + x^2$ for any $x \in A$ such that $\omega(x) = 1$. Thus A satisfies the polynomial identity $(x^2)^2 - 2\omega(x)x^3 + \omega(x)^2x^2 = 0$.

□

Remark 4.2. The Proposition 4.1 shows A is not of type $(2, 0, 1, 1)$ because it satisfies a polynomial identity of degree less than 6. Thus the classification will be made through the two types $(2, 1, 1, 0)$ and $(2, 1, 0, 1)$.

This allows us to state the main result of this paper.

Theorem 4.3. *Let A be a K-algebra satisfying the identity $2x^2x^4 = \omega(x)^2x^4 + \omega(x)^4x^2$ but not satisfying an identity of degree less than six. We have the following multiplication tables according to the type of A. Products not mentioned being zero.*

i): *type of* $A = (2, 1, 1, 0)$

- $A_1 :: e^2 = e,\ ee_1 = \frac{1}{2}e_1,\ ee_3 = \lambda e_3,\ e_2e_3 = e_1;$
- $A_2 :: e^2 = e,\ ee_1 = \frac{1}{2}e_1,\ ee_3 = \lambda e_3,\ e_2^2 = e_1,\ e_2e_3 = e_1;$
- $A_3 :: e^2 = e,\ ee_1 = \frac{1}{2}e_1,\ ee_3 = \lambda e_3,\ e_1e_2 = e_1,\ e_2e_3 = e_1;$
- $A_4 :: e^2 = e,\ ee_1 = \frac{1}{2}e_1,\ ee_3 = \lambda e_3,\ e_2^2 = e_1,\ e_1e_2 = e_1,\ e_2e_3 = e_1;$

ii): *type of* $A = (2, 1, 0, 1)$

- $A_5 :: e^2 = e,\ ee_1 = \frac{1}{2}e_1,\ ee_3 = \bar{\lambda}e_3,\ e_2e_3 = e_1;$
- $A_6 :: e^2 = e,\ ee_1 = \frac{1}{2}e_1,\ ee_3 = \bar{\lambda}e_3,\ e_2^2 = e_1,\ e_2e_3 = e_1;$
- $A_7 :: e^2 = e,\ ee_1 = \frac{1}{2}e_1,\ ee_3 = \bar{\lambda}e_3,\ e_1e_2 = e_1,\ e_2e_3 = e_1;$
- $A_8 :: e^2 = e,\ ee_1 = \frac{1}{2}e_1,\ ee_3 = \bar{\lambda}e_3,\ e_2^2 = e_1,\ e_1e_2 = e_1,\ e_2e_3 = e_1;$

Proof. **i):** Let $A = (2, 1, 1, 0) = Ke\oplus < e_1, e_2, e_3 >= Ke \oplus A_{\frac{1}{2}} \oplus A_0 \oplus A_\lambda \oplus A_{\bar{\lambda}}$ an algebra verifying the identity $2x^2x^4 = \omega(x)^2x^4 + \omega(x)^4x^2$ of dimension 4. By setting $A_{\frac{1}{2}} =< e_1 >$, $A_0 =< e_2 >$, $A_\lambda =< e_3 >$ we have the following table: $e^2 = e$, $ee_1 = \frac{1}{2}e_1$, $ee_2 = 0$, $ee_3 = \lambda e_3$, $e_1^2 = ae_2 + be_3$, $e_1e_2 = ce_1 + de_3$, $e_1e_3 = fe_1 + ge_2$, $e_2^2 = he_1$, $e_2e_3 = ke_1$, $e_3^2 = 0$. with $(a, b, c, d, e, f, g, h, k) \in K^9$.

Using the relations of the Proposition 2.5 we obtain:
$(\lambda + 1)bh + 8cd = f = hd = ha = bh = hg = kg = kd = kb = ka = ac = ad = bg = dg = bc = gc = 0$.

Let's discuss following the value of k:

1): If $k \neq 0$ we have $a = b = d = g = 0$ and the table becomes $e^2 = e$, $ee_1 = \frac{1}{2}e_1$, $ee_2 = 0$, $ee_3 = \lambda e_3$, $e_1^2 = 0$, $e_1e_2 = ce_1$, $e_1e_3 = 0$, $e_2^2 = he_1$, $e_2e_3 = ke_1$, $e_3^2 = 0$.

1.1): For $h = c = 0$ the multiplication table becomes: $e^2 = e$, $ee_1 = \frac{1}{2}e_1$, $ee_3 = \lambda e_3$, $e_2e_3 = e_1$ (replacing e_1 with $e_1' = k^{-1}e_1$) the products not mentioned being zero.

1.2): For $h \neq 0$ and $c = 0$ the multiplication table becomes: $e^2 = e$, $ee_1 = \frac{1}{2}e_1$, $ee_3 = \lambda e_3$, $e_2^2 = e_1$, $e_2e_3 = e_1$ (replacing e_1 with $e_1' = h^{-1}e_1$ and e_3 with $e_3' = kh^{-1}e_3$) products not mentioned being zero.

1.3): For $h = 0$ and $c \neq 0$ the multiplication table becomes: $e^2 = e$, $ee_1 = \frac{1}{2}e_1$, $ee_3 = \lambda e_3$, $e_1e_2 = e_1$, $e_2e_3 = e_1$(replacing e_2 with $e_2^{'} = ce_2$ and e_3 with $e_3^{'} = kc^{-1}e_3$) products not mentioned being void.

1.4): For $h \neq 0$ and $c \neq 0$ the multiplication table becomes: $e^2 = e$, $ee_1 = \frac{1}{2}e_1$, $ee_3 = \lambda e_3$, $e_2^2 = e_1$, $e_1e_2 = e_1$, $e_2e_3 = e_1$ (replacing e_1 with $e_1^{'} = c^2h^{-1}e_1$, e_2 by $e_2^{'} = ce_2$ and e_3 by $e_3^{'} = ckh^{-1}e_3$) products not mentioned being zero.

2): Suppose $k = 0$ and discuss according to the value of the parameter h.

2.1): For $h \neq 0$, we immediately have $a = d = g = b = 0$ and the table looks like this: $e^2 = e$, $ee_1 = \frac{1}{2}e_1$, $ee_2 = 0$, $ee_3 = \lambda e_3$, $e_1^2 = 0$, $e_1e_2 = ce_1$, $e_1e_3 = 0$, $e_2^2 = he_1$, $e_2e_3 = 0$, $e_3^2 = 0$.
Let $x = e+\alpha_1e_1+\alpha_2e_2+\alpha_3e_3$, we have $x^2 = e+(\alpha_1+\alpha_2^2h+2c\alpha_1\alpha_2)e_1+2\lambda\alpha_3e_3$, $(x^2)^2 = e+(\alpha_1+\alpha_2^2h+2c\alpha_1\alpha_2)e_1+4\lambda^2\alpha_3e_3$v, $(x^2)^2-x^2 = (4\lambda^2-2\lambda)\alpha_3e_3$ and $x((x^2)^2 - x^2) = \lambda((x^2)^2 - x^2)$. Therefore Then the algebra A satisfies the equation: $x(x^2)^2 - \lambda\omega(x)(x^2)^2 - \omega(x)^2x^3 + \lambda\omega(x)^3x^2 = 0$.

2.2): Suppose $h = 0$. We have the following different cases:
2.2.1) If $g \neq 0$, we have: $b = c = d$ and the table becomes $e^2 = e$, $ee_1 = \frac{1}{2}e_1$, $ee_2 = 0$, $ee_3 = \lambda e_3$, $e_1^2 = ae_2$, $e_1e_2 = 0$, $e_1e_3 = ge_2$, $e_2^2 = 0$, $e_2e_3 = 0$, $e_3^2 = 0$.
Let $x = e+\alpha_1e_1+\alpha_2e_2+\alpha_3e_3$, we have $x^2 = e+\alpha_1e_1+(\alpha_1^2a+2\alpha_1\alpha_3g)e_2+2\lambda\alpha_3e_3$, $x^2-x = (\alpha_1^2a+2\alpha_1\alpha_3g-\alpha_2)e_2+(2\lambda-1)\alpha_3e_3$ and $(x^2-x)^2 = 0$. Therefore Then the algebra A satisfies the equation $(x^2)^2 - 2\omega(x)x^3 + \omega(x)^2x^2 = 0$.
2.2.2) If $g = 0$, we have the following table $e^2 = e$, $ee_1 = \frac{1}{2}e_1$, $ee_2 = 0$, $ee_3 = \lambda e_3$, $e_1^2 = ae_2 + be_3$, $e_1e_2 = ce_1 + de_3$, $e_1e_3 = 0$, $e_2^2 = 0$, $e_2e_3 = 0$, $e_3^2 = 0$.
If $c \neq 0$, we have $a = b = d = 0$ the multiplication table becomes: $e^2 = e$, $ee_1 = \frac{1}{2}e_1$, $ee_3 = \lambda e_3$, $e_1e_2 = ce_1$, the products not mentioned being zero. We then show that the algebra satisfies the identity $x(x^2)^2 - \lambda\omega(x)(x^2)^2 - \omega(x)^2x^3 + \lambda\omega(x)^3x^2 = 0$.
When $c = 0$, we show that the algebra satisfies the equation $(x^2)^2 - 2\omega(x)x^3 + \omega(x)^2x^2 = 0$.

ii): The proof is similar to that of the assertion i).

□

The following proposition proves that the type of an algebra of dimension 4 satisfying the identity (1), is an invariant of the algebra.

Proposition 4.4. *Let A be an algebra strictly satisfying the identity $2x^2x^4 = \omega(x)^2x^4 + \omega(x)^4x^2$, of type $(2, 1, 1, 0)$ and Peirce decomposition $A = Ke \oplus A_0 \oplus A_{\frac{1}{2}} \oplus A_\lambda \oplus A_{\bar{\lambda}}$ relative to an idempotent e. The dimensions of the spaces A_0, $A_{\frac{1}{2}}$, and A_λ are independent of the chosen idempotent e.*

Proof. Let $A = Ke\oplus < e_1, e_2, e_3 >= Ke \oplus A_{\frac{1}{2}} \oplus A_0 \oplus A_\lambda \oplus A_{\bar{\lambda}}$ a four-dimensional algebra of type $(2, 1, 1, 0)$ verifying the identity $2x^2x^4 = \omega(x)^2x^4 + \omega(x)^4x^2$. Setting $A_{\frac{1}{2}} =< e_1 >$, $A_0 =< e_2 >$, $A_\lambda =< e_3 >$ we have the following table: $e^2 = e$, $ee_1 = \frac{1}{2}e_1$, $ee_3 = \lambda e_3$, $e_1^2 = ae_2 + be_3$, $e_2^2 = he_1$, $e_1e_2 = ce_1 + de_3$, $e_1e_3 = fe_1 + ge_2$, $e_2e_3 = ke_1$, the other products being zero.

Let $e^{'}=e+a_1e_1+a_2e_2+a_3e_3$ another idempotent, we have: $e^{'2}=e+a_1e_1+(ha_2^2+2a_1a_2c+2a_1a_3f+2a_2a_3k)e_1+(aa_1^2+2a_1a_3g)e_2+(ba_1^2+2\lambda a_3+2a_1a_2d)e_3$ and $e^{'}=e^{'2}$ implies that

$$(2)\qquad \begin{cases} ha_2^2+2a_1a_2c+2a_1a_3f+2a_2a_3k=0\\ aa_1^2+2a_1a_3g=a_2\\ ba_1^2+2\lambda a_3+2a_1a_2d=a_3\end{cases}$$

•According to the proof of Theorem4.3, if $k\neq 0$, we have $a=b=d=g=f=0$, so $a_2=a_3=0$ and $e^{'}=e+a_1e_1$.
Let $x=\alpha_1e_1+\alpha_2e_2+\alpha_3e_3$ be an element of A, then $e^{'}x=(\frac{1}{2}\alpha_1+a_1\alpha_2c)e_1+\lambda\alpha_3e_3$.
For $x\in A_{\frac{1}{2}}(e^{'})=\{x\in A/e^{'}x=\frac{1}{2}x\}$, we have by identification $a_1\alpha_2c=0$, $\alpha_2=0$ and $\alpha_3=\frac{1}{2}\alpha_3$. This implies that $\alpha_2=\alpha_3=0$ and $x=\alpha_1e_1$ hence $A_{\frac{1}{2}}(e^{'})=A_{\frac{1}{2}}(e)$ therefore $dimA_{\frac{1}{2}}(e^{'})=dimA_{\frac{1}{2}}(e)=1$. For $x\in A_0(e^{'})=\{x\in A/e^{'}x=0\}$, we have by identification $\frac{1}{2}\alpha_1+a_1\alpha_2c=0$ and $\alpha_3=0$. Therefore $\alpha_1=-2a_1\alpha_2c$ and $\alpha_3=0$, so $x=-2a_1\alpha_2ce_1+\alpha_2e_2=\alpha_2(-2a_1ce_1+e_2)$ and $A_0(e^{'})=<e_2-2a_1ce_1>$, then $dimA_0(e^{'})=dimA_0(e)=1$.
We deduce that $dimA_\lambda(e^{'})=dimA_\lambda(e)=1$
•If $k=0$, the system (2) becomes

$$(3)\qquad \begin{cases} ha_2^2+2a_1a_2c=0\\ aa_1^2+2a_1a_3g=a_2\\ ba_1^2+2\lambda a_3+2a_1a_2d=a_3\end{cases}$$

and the system (3) implies

$$(4)\qquad \begin{cases} ha_2^2+2a_1a_2c=0\\ haa_1^2+2a_1a_3gh=ha_2\\ hba_1^2+2\lambda ha_3+2a_1a_2hd=ha_3\end{cases}$$

But according to the proof of Theorem4.3, $ah=gh=0$ and the system (4) becomes

$$(5)\qquad \begin{cases} a_1a_2c=0\\ ha_2=0\\ hba_1^2+2\lambda ha_3=ha_3\end{cases}$$

✓ If $h\neq 0$ we have $a_2=0$, which implies that $a_3=\frac{ba_1^2}{1-2\lambda}$ and $e^{'}=e+a_1e_1+\frac{ba_1^2}{1-2\lambda}e_3$.
Indeed $e^{'}x=(\frac{1}{2}\alpha_1+a_1\alpha_2c)e_1+\lambda\alpha_3e_3$
For $x\in A_{\frac{1}{2}}(e^{'})=\{x\in A/e^{'}x=\frac{1}{2}x\}$, we have by identification $a_1\alpha_2c=0$, $\alpha_2=0$ and $\lambda\alpha_3=\frac{1}{2}\alpha_3$, which implies that $\alpha_2=\alpha_3=0$ and $x=\alpha_1e_1$, hence $A_{\frac{1}{2}}(e^{'})=A_{\frac{1}{2}}(e)$ and $dimA_{\frac{1}{2}}(e^{'})=dimA_{\frac{1}{2}}(e)=1$
For $x\in A_0(e^{'})=\{x\in A/e^{'}x=0\}$, we have by identification $\frac{1}{2}\alpha_1+a_1\alpha_2c=0$ and $\alpha_3=0$, thus $\alpha_1=-2a_1\alpha_2c$ and $\alpha_3=0$, so $x=-2a_1\alpha_2ce_1+\alpha_2e_2=\alpha_2(-2a_1ce_1+e_2)$.
We have then $A_0(e^{'})=<e_2-2a_1ce_1>$ so $dimA_0(e^{'})=dimA_0(e)=1$.
We deduce that $dimA_\lambda(e^{'})=dimA_\lambda(e)=1$

✓ If $h = 0$ the system (2) becomes

$$\begin{cases} 2a_1a_2c + 2a_1a_3f = 0 \\ aa_1^2 + 2a_1a_3g = a_2 \\ ba_1^2 + 2\lambda a_3 + 2a_1a_2d = a_3 \end{cases} \tag{6}$$

1) Suppose $g \neq 0$, from the proof of the Theorem4.3, we have $c = b = d = f = 0$ and then $a_2 = aa_1^2$ and $a_3 = 0$, so $e^{'} = e + a_1e_1 + aa_1^2e_2$.
For $x \in A_{\frac{1}{2}}(e^{'}) = \{x \in A/e^{'}x = \frac{1}{2}x\}$, we have by identification $a_1\alpha_1a + a_1\alpha_3g = \frac{1}{2}\alpha_2$ and $\lambda\alpha_3 = \frac{1}{2}\alpha_3$. This implies that $\alpha_3 = 0$, $\alpha_2 = 2a_1\alpha_1a$ and $x = \alpha_1e_1 + 2a_1\alpha_1a\alpha_2 = \alpha_1(e_1 + 2a_1a\alpha_2)$ hence $A_{\frac{1}{2}}(e^{'}) =< e_1 + 2a_1a\alpha_2 >$ therefore $dimA_{\frac{1}{2}}(e^{'}) = dimA_{\frac{1}{2}}(e) = 1$
For $x \in A_0(e^{'}) = \{x \in A/e^{'}x = 0\}$, we have by identification we have $\alpha_1 = \alpha_3 = 0$, so $x = \alpha_2e_2$ and $A_0(e^{'}) = A_0(e)$, therefore $dimA_0(e^{'}) = dimA_0(e) = 1$.
We deduce that $dimA_\lambda(e^{'}) = dimA_\lambda(e) = 1$
2) Suppose $g = 0$, the system (2) becomes

$$\begin{cases} 2a_1a_2c + 2a_1a_3f = 0 \\ aa_1^2 = a_2 \\ ba_1^2 + 2\lambda a_3 + 2a_1a_2d = a_3 \end{cases} \tag{7}$$

Assuming $c \neq 0$, we have from the proof of Theorem4.3, $a = b = d = f = 0$, so $a_2 = a_3 = 0$ and $e^{'} = e + a_1e_1$. Thus, $e^{'}x = (\frac{1}{2}\alpha_1 + a_1\alpha_2c)e_1 + \lambda\alpha_3e_3$
✓ For $x \in A_{\frac{1}{2}}(e^{'}) = \{x \in A/e^{'}x = \frac{1}{2}x\}$, we have by identification $a_1\alpha_2c = 0$, $\frac{1}{2}\alpha_2 = 0$ and $\lambda\alpha_3 = \frac{1}{2}\alpha_3$. This implies that $\alpha_3 = \alpha_2 = 0$ and $x = \alpha_1e_1$ hence $A_{\frac{1}{2}}(e^{'}) = A_{\frac{1}{2}}(e)$ therefore $dimA_{\frac{1}{2}}(e^{'}) = dimA_{\frac{1}{2}}(e) = 1$
✓ For $x \in A_0(e^{'}) = \{x \in A/e^{'}x = 0\}$, we have by identification $\alpha_1 = -2a_1\alpha_2c$ and $\alpha_3 = 0$, so $x = \alpha_2e_2 - 2a_1\alpha_2ce_1 = \alpha_2(e_2 - 2a_1ce_1)$ and $A_0(e^{'}) =< e_2 - 2a_1ce_1 >$ therefore $dimA_0(e^{'}) = dimA_0(e) = 1$.
We deduce that $dimA_\lambda(e^{'}) = dimA_\lambda(e) = 1$.

□

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