Korean J. Math. **32** (2024), No. 4, pp. 673–682
https://dx.doi.org/10.11568/kjm.2024.32.4.673

A STUDY OF NEGATIVE ARITHMETIC MATRIX WITH FIBONACCI NUMBERS

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Abstract. In this work the Pascal matrix P and the negative Pascal matrix Q are studied by means of certain polynomials. We investigate an LU-factorization of Q by P, and express the powers Q^m by Fibonacci numbers.

1. Introduction

The arithmetic tables of $(x+1)^k$ and $(x+1)^{-k}$ for $k > 0$ are the Pascal matrix $P = [p_{i,j}]$ and the negative Pascal matrix $Q = [q_{i,j}]$ $(i, j \geq 1)$, respectively. Clearly $p_{i,j} = \binom{i-1}{j-1}$ and $q_{i,j} = \binom{-i}{j-1} = (-1)^{j-1}\binom{i+j-2}{j-1} = (-1)^{j-1}p_{i+j-1,j}$. Let $\tilde{P} = \begin{bmatrix} 1\,1\,1\ \ 1 \dots \\ 1\,2\,3\ \ 4 \dots \\ 1\,3\,6\,10 \dots \\ \dots \end{bmatrix}$ be the symmetric matrix form of P ([9]). Many research articles including [1], [2] and [5] have been devoted to investigating properties of the matrices P, $\tilde{P}$ and Q, and their interrelationships. The inverse and power matrices, as well as decompositions of these matrices into lower and upper triangular matrices were studied in [7]. Most of these were proved by means of binomial coefficients or certain recurrence relations over the arithmetic matrices ([6]). In particular, we remark that in [4], the Pascal matrix P was studied by Fibonacci numbers.

Just as the matrices P, $\tilde{P}$ and Q were made from $(x+1)^{\pm k}$, we let $P(m)$, $\widetilde{P(m)}$ and $Q(m)$ be the arithmetic matrices of the binomial polynomial $(mx+1)^{\pm k}$ respectively, for any $m > 0$. A purpose of the work is to investigate powers and inverse of the matrices $P(m)$, $\widetilde{P(m)}$ and $Q(m)$. We explore interrelationships of the matrices by means of polynomials and Fibonacci numbers. Indeed, LU-factorizations of $Q(m)$ by P are obtained in Theorem 4, and powers Q^m in terms of Fibonacci numbers are discussed in Theorem 7 and Theorem 9.

Our notations in this work are as follows. Given a matrix A, A^T is the transpose matrix, $A^B = B^{-1}AB$ is a conjugate of A by a matrix B, and A_n indicates the n square matrix. Let $r_i(A)$ and $c_j(A)$ be the i^{th}row and j^{th}column of A, respectively. And $[\{0\}_t; r_i(A)]$ denotes a row matrix of t zeros followed by $r_i(A)$, while $\begin{bmatrix} \{0\}_t \\ c_j(A) \end{bmatrix}$ is a column matrix of t zeros followed by $c_j(A)$. Let $\text{di}(b_1, b_2, b_3, \dots)$ be a diagonal

Received December 15, 2023. Revised October 13, 2024. Accepted October 15, 2024.
2010 Mathematics Subject Classification: 05A10, 11R11.
Key words and phrases: Arithmetic matrix, negative Pascal matrix, Fibonacci number.

matrix having diagonal entries $b_1, b_2, b_3, \dots$, in particular di$[a^i]$ is a diagonal matrix di$(1, a, a^2, \dots)$. We define a multiplication $\cdot$ of two row matrices by $(a_1, a_2, a_3, \dots) \cdot (b_1, b_2, b_3, \dots) = (a_1, a_2, a_3, \dots)\text{di}(b_1, b_2, b_3, \dots) = (a_1b_1, a_2b_2, a_3b_3, \dots)$.

2. Negative Pascal Matrix via polynomial

The arithmetic matrix $P(m)$ of $(mx+1)^k$ depends on whether the expansions are in ascending or descending order in x, so we write $P(m)^{\uparrow} = \begin{bmatrix} 1 \\ 1\ m \\ 1\,2m\,m^2 \\ \cdots \end{bmatrix}$ and $P(m)^{\downarrow} = \begin{bmatrix} 1 \\ m\ \ 1 \\ m^2\,2m\,1 \\ \cdots \end{bmatrix}$ according to ascending or descending order expansion in x, respectively. We study the arithmetic matrices $P(m)$ and $Q(m)$ by means of polynomials.

Theorem 1. *$P^m = P(m)^{\downarrow}$ and $r_i(P^m) = r_i(P) \cdot (m^{i-1}, \dots, m, 1)$. Moreover $\widetilde{P(m)^{\downarrow}} = P(m)^{\downarrow}\, P(m)^{\uparrow T}$.*

Proof. Let $v = [1, x, x^2, \dots]^T$ be a column matrix with an indeterminate variable x. Then

$$Pv = \begin{bmatrix} 1 & \cdots \\ 11 & \cdots \\ 121 & \cdots \\ & \cdots \end{bmatrix} \begin{bmatrix} 1 \\ x \\ x^2 \\ \cdots \end{bmatrix} = \begin{bmatrix} 1 \\ 1+x \\ (1+x)^2 \\ \cdots \end{bmatrix} \text{ and } P^2v = P\begin{bmatrix} 1 \\ 1+x \\ (1+x)^2 \\ \cdots \end{bmatrix} = \begin{bmatrix} 1 \\ 2+x \\ (2+x)^2 \\ \cdots \end{bmatrix},$$

so

$$P^m v = \begin{bmatrix} 1 \\ m+x \\ (m+x)^2 \\ \cdots \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & \cdots \\ m & 1 & 0 & \cdots \\ m^2 & 2m & 1 & \cdots \\ & \cdots \end{bmatrix} \begin{bmatrix} 1 \\ x \\ x^2 \\ \cdots \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & \cdots \\ m & 1 & 0 & \cdots \\ m^2 & 2m & 1 & \cdots \\ & \cdots \end{bmatrix} v \quad \text{for any } m.$$

Thus $P^m = \begin{bmatrix} 1 & 0 & 0 & \cdots \\ m & 1 & 0 & \cdots \\ m^2 & 2m & 1 & \cdots \\ & \cdots \end{bmatrix} = P(m)^{\downarrow}$. Moreover $P(m)^{\downarrow} = \begin{bmatrix} r_1(P)\cdot(1) \\ r_2(P)\cdot(m,1) \\ r_3(P)\cdot(m^2,m,1) \\ \cdots \end{bmatrix}$ shows $r_i(P^m) = r_i(P) \cdot (m^{i-1}, m^{i-2}, \dots, 1)$ for all $i \geq 1$.

Now $P(m)^{\downarrow}v = \begin{bmatrix} 1 & 0 & 0 & \cdots \\ m & 1 & 0 & \cdots \\ m^2 & 2m & 1 & \cdots \\ & \cdots \end{bmatrix} v = \begin{bmatrix} 1 \\ (m+x) \\ (m+x)^2 \\ \cdots \end{bmatrix}$ and

$$\widetilde{P(m)^{\downarrow}}v = \begin{bmatrix} 1 & 1 & 1 & 1 \cdots \\ m & 2m & 3m & 4m \cdots \\ m^2 & 3m^2 & 6m^2 & 10m^2 \cdots \\ & & & \cdots \end{bmatrix} \begin{bmatrix} 1 \\ x \\ x^2 \\ \cdots \end{bmatrix} = \begin{bmatrix} 1 + x + x^2 + x^3 + \dots \\ m(1 + 2x + 3x^2 + 4x^3 + \dots) \\ m^2(1 + 3x + 6x^2 + 10x^3 + \dots) \\ \cdots \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{1-x} \\ \frac{m}{(1-x)^2} \\ \frac{m^2}{(1-x)^3} \\ \cdots \end{bmatrix} = \begin{bmatrix} \alpha \\ m\alpha^2 \\ m^2\alpha^3 \\ \cdots \end{bmatrix}$$

where $\alpha = \frac{1}{1-x}$. On the other hand, $P(m)^{\uparrow} = \begin{bmatrix} 1 \\ 1\ m \\ 1\,2m\ \ m^2 \\ 1\,3m\,3m^2\,m^3 \\ \cdots \end{bmatrix}$ shows

$$P(m)^{\uparrow T} v = \begin{bmatrix} 1 & 1 & 1 & 1 \\ & m & 2m & 3m \\ & & m^2 & 3m^2 \\ & & & m^3 \\ & & & \cdots \end{bmatrix} v = \begin{bmatrix} 1 + x + x^2 + x^3 + \dots \\ mx(1 + 2x + 3x^2 + \dots) \\ m^2x^2(1 + 3x + \dots) \\ m^3x^3(1 + \dots) \\ \cdots \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{1-x} \\ \frac{mx}{(1-x)^2} \\ \frac{(mx)^2}{(1-x)^3} \\ \cdots \end{bmatrix} = \frac{1}{1-x} \begin{bmatrix} 1 \\ \frac{mx}{1-x} \\ (\frac{mx}{1-x})^2 \\ \cdots \end{bmatrix} = \alpha \begin{bmatrix} 1 \\ \beta \\ \beta^2 \\ \cdots \end{bmatrix}$$

where $\beta = \frac{mx}{1-x}$. But since $\alpha(m+\beta)^i = m^i \alpha^{i+1}$ for all i, we have

$$P(m)^{\downarrow} P(m)^{\uparrow T} v = P(m)^{\downarrow} \alpha \begin{bmatrix} 1 \\ \beta \\ \beta^2 \\ \cdots \end{bmatrix} = \alpha \begin{bmatrix} 1 \\ (m+\beta) \\ (m+\beta)^2 \\ \cdots \end{bmatrix} = \begin{bmatrix} \alpha \\ \alpha(m+\beta) \\ \alpha(m+\beta)^2 \\ \cdots \end{bmatrix}$$

$$= \begin{bmatrix} \alpha \\ m\alpha^2 \\ m^2\alpha^3 \\ \cdots \end{bmatrix} = \widetilde{P(m)^{\downarrow}} v.$$

Thus $\widetilde{P(m)^{\downarrow}} = P(m)^{\downarrow} P(m)^{\uparrow T}$. □

In particular Theorem 1 shows $\tilde{P} = PP^T$ if $m = 1$ (refer to [3]). Moreover $P(ms)^{\downarrow} = P(m)^{\downarrow^s} = P(s)^{\downarrow^m}$ for any $m, s > 0$. And the next corollary follows immediately from Theorem 1.

Corollary 2. *Let $D = di[(-1)^i]$ be a diagonal matrix. Then $P^{-m} = (P^m)^D$, in particular $P^{-1} = P(-1)^{\downarrow} = P^D$.*

Now for the negative Pascal matrix $Q = [q_{i,j}] = \begin{bmatrix} 1 & -1 & 1 & -1 & \dots \\ 1 & -2 & 3 & -4 & \dots \\ 1 & -3 & 6 & -10 & \dots \\ 1 & -4 & 10 & -20 & \dots \\ & & & & \dots \end{bmatrix}$ of $(x+1)^{-k}$, let $\hat{Q}$ be a matrix such that each j^{th}column $c_j(\hat{Q})$ equals $\begin{bmatrix} \{0\}_{j-1} \\ c_j(Q^T) \end{bmatrix}$ for all $j \geq 1$. Then the inverse of Pascal matrix $P = [p_{i,j}]$ is $P^{-1} = [(-1)^{(i+j)} p_{i,j}] = \hat{Q}$ (see [7]). As an analog of $\hat{Q}$ with respect to Q, let us consider $\widehat{Q(m)}$ with respect to $Q(m)$ for any $m > 0$. Note that $c_j(\widehat{Q(m)}) = \begin{bmatrix} \{0\}_{j-1} \\ c_j(Q(m)^T) \end{bmatrix}$ for all $j \geq 1$. The next theorem shows a relation of P^{-m} and $\widehat{Q(m)}$.

Theorem 3. *$Q(m) = Qdi[m^i]$ and $P^{-m} = \widehat{Q(m)}$ for all $m > 0$.*

Proof. Consider a column matrix $v = [1, x, x^2, \dots]^T$. Then

$$(mx+1)^{-1} = ((1,-1,1,-1,\dots)\cdot(1,m,m^2,m^3,\dots))v$$
$$= (r_1(Q)\cdot(1,m,m^2,m^3,\dots))v$$

and $(mx+1)^{-i} = (r_i(Q)\cdot(1,m,m^2,\dots))v$ for all $i \geq 1$. Thus

$$Q(m) = \begin{bmatrix} r_1(Q)\cdot(1,m,m^2,\dots) \\ r_2(Q)\cdot(1,m,m^2,\dots) \\ \cdots \\ r_i(Q)\cdot(1,m,m^2,\dots) \\ \cdots \end{bmatrix} = Q\ \mathrm{di}[m^i].$$

Therefore in $Q(m)$, each j^{th}column $c_j(\widehat{Q(m)}) = \begin{bmatrix} \{0\}_{j-1} \\ c_j(Q(m)^T) \end{bmatrix}$ with $c_j(Q(m)^T) = (r_i(Q)\cdot(1,m,m^2,\dots))^T$ yields

$$\widehat{Q(m)} = \begin{bmatrix} 1 & 0 & 0 & 0 \dots \\ -m & 1 & 0 & 0 \dots \\ m^2 & -2m & 1 & 0 \dots \\ -m^3 & 3m^2 & -3m & 1 \dots \\ m^4 & -4m^3 & 6m^2 & -4m \dots \end{bmatrix}.$$

On the other hand, owing to Corollary 2 we have

$$P^{-m} = (P^m)^D = P(m)^{\downarrow D}$$
$$= D\begin{bmatrix} r_1(P)\cdot(1) \\ r_2(P)\cdot(m,1) \\ \cdots \\ r_i(P)\cdot(m^{i-1},\dots,m,1) \\ \cdots \end{bmatrix} D = \begin{bmatrix} 1 & 0 & 0 & 0\dots \\ -m & 1 & 0 & 0\dots \\ m^2 & -2m & 1 & 0\dots \\ -m^3 & 3m^2 & -3m & 1\dots \\ m^4 & -4m^3 & 6m^2 & -4m\dots \end{bmatrix}.$$

Hence it proves $P^{-m} = \widehat{Q(m)}$. □

In fact $Q(3) = Q\ \mathrm{di}[3^i] = \begin{bmatrix} 1 & -1 & 1 & -1\dots \\ 1 & -2 & 3 & -4\dots \\ 1 & -3 & 6 & -10\dots \\ 1 & -4 & 10 & -20\dots \\ & \dots \end{bmatrix}\begin{bmatrix} 1 & & & \\ & 3 & & \\ & & 3^2 & \\ & & & 3^3 \\ & & & \dots \end{bmatrix} = \begin{bmatrix} 1 & -3 & 9 & -27\dots \\ 1 & -6 & 27 & -108\dots \\ 1 & -9 & 54 & -270\dots \\ 1 & -12 & 90 & -540\dots \\ & \dots \end{bmatrix}$ and

$$\widehat{Q(3)}_4 = \begin{bmatrix} 1 & & & \\ -3 & 1 & & \\ 9 & -6 & 1 & \\ -27 & 27 & -9 & 1 \end{bmatrix} = \begin{bmatrix} 1 & & & \\ -1 & 1 & & \\ 1 & -2 & 1 & \\ -1 & 3 & -3 & 1 \end{bmatrix}^3 = \begin{bmatrix} 1 & & & \\ 3 & 1 & & \\ 9 & 6 & 1 & \\ 27 & 27 & 9 & 1 \end{bmatrix}^{-1} = P_4^{-3} = P(3)_4^{\downarrow -1}.$$

Let $G = [g_{i,j}]$ be a matrix with $g_{1,1} = g_{i,i-1} = 1$, $g_{i,i} = -1$ $(i > 1)$, and $H = [h_{i,j}]$ be with $h_{i,i} = h_{i,i+1} = 1$ $(i \geq 1)$ and the other entries in G and H are zeros. An LU-factorization of Q is investigated in the next theorem.

THEOREM 4. *With the above matrices G and H, let L [resp. U] be a lower [resp. upper] triangular matrix satisfying* $GL = \left[\begin{array}{c|c} 1 & 0 \\ \hline 0 & L_{n-1} \end{array}\right]$, $UH = \left[\begin{array}{c|c} 1 & 0 \\ \hline 0 & U_{n-1} \end{array}\right]$ *for all $n > 1$, and $L_1 = U_1 = [1]$. Then $Q = LU = PY$ where $Y = [(-1)^{j-1}p_{j,i}]$.*

Proof. With $v = [1, x, x^2, \dots]^T$, we have

$$Yv = \begin{bmatrix} 1 & -1 & 1 & -1\dots \\ & -1 & 2 & -3\dots \\ & & 1 & -3\dots \\ & & \dots & \end{bmatrix}\begin{bmatrix} 1 \\ x \\ x^2 \\ \dots \end{bmatrix} = \begin{bmatrix} 1 - x + x^2 - x^3 + \dots \\ -x + 2x^2 - 3x^3 + \dots \\ x^2 - 3x^3 + \dots \\ \dots \end{bmatrix} = \begin{bmatrix} \frac{1}{1+x} \\ \frac{-x}{(1+x)^2} \\ \frac{x^2}{(1+x)^3} \\ \dots \end{bmatrix},$$

so $Yv = \gamma\begin{bmatrix} 1 \\ \delta \\ \delta^2 \\ \dots \end{bmatrix}$ with $\gamma = \frac{1}{1+x}$, $\delta = \frac{-x}{1+x}$. Since $Pv = \begin{bmatrix} 1 \\ (1+x) \\ (1+x)^2 \\ \dots \end{bmatrix}$, we have

$$PYv = P\gamma\begin{bmatrix} 1 \\ \delta \\ \delta^2 \\ \dots \end{bmatrix} = \gamma\begin{bmatrix} 1 \\ (1+\delta) \\ (1+\delta)^2 \\ \dots \end{bmatrix} = \gamma\begin{bmatrix} 1 \\ \gamma \\ \gamma^2 \\ \dots \end{bmatrix} = \begin{bmatrix} \frac{1}{1+x} \\ \frac{1}{(1+x)^2} \\ \frac{1}{(1+x)^3} \\ \dots \end{bmatrix}$$

because $1 + \delta = \gamma$. On the other hand, since

$$Qv = \begin{bmatrix} 1 & -1 & 1 & -1\dots \\ 1 & -2 & 3 & -4\dots \\ 1 & -3 & 6 & -10\dots \\ & \dots & & \end{bmatrix}\begin{bmatrix} 1 \\ x \\ x^2 \\ \dots \end{bmatrix} = \begin{bmatrix} 1 - x + x^2 - x^3 + \dots \\ 1 - 2x + 3x^2 - 4x^3 + \dots \\ 1 - 3x + 6x^2 - 10x^3 + \dots \\ \dots \end{bmatrix} = \begin{bmatrix} \frac{1}{1+x} \\ \frac{1}{(1+x)^2} \\ \frac{1}{(1+x)^3} \\ \dots \end{bmatrix}, \tag{1}$$

it follows that $Qv = PYv$, i.e., $Q = PY$.

The constructions of G and H show, for instance $G_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix} = \left[\begin{array}{c|c} G_3 & 0 \\ \hline 001 & -1 \end{array}\right]$,

$H_4 = \begin{bmatrix} 1100 \\ 0110 \\ 0011 \\ 0001 \end{bmatrix} = \left[\begin{array}{c|c} H_3 & \begin{smallmatrix} 0 \\ 0 \\ 1 \end{smallmatrix} \\ \hline 000 & 1 \end{array}\right]$, and $G = \left[\begin{array}{c|c} G_{n-1} & 0 \\ \hline 0\dots 01 & -1 \end{array}\right]$ and $H = \left[\begin{array}{c|c} H_{n-1} & \begin{smallmatrix} 0 \\ \dots \\ 0 \\ 1 \end{smallmatrix} \\ \hline 0\dots 0 & 1 \end{array}\right]$ for all $n > 1$.

Moreover $GL = \left[\begin{array}{c|c} 1 & 0 \\ \hline 0 & L_{n-1}\end{array}\right]$ and $UH = \left[\begin{array}{c|c} 1 & 0 \\ \hline 0 & U_{n-1}\end{array}\right]$ yield $L_4 = \begin{bmatrix} 1 & & & \\ 1 & -1 & & \\ 1 & -2 & 1 & \\ 1 & -3 & 3 & -1 \end{bmatrix}$ and $U_4 = \begin{bmatrix} 1 & -1 & 1 & -1 \\ & 1 & -2 & 3 \\ & & 1 & -3 \\ & & & 1 \end{bmatrix}$, so $L = \left[\begin{array}{c|c} L_{n-1} & 0 \\ \hline p_{n,1}, \dots (-1)^{i-1}p_{n,i}, \dots (-1)^{n}p_{n,n-1} & (-1)^{n-1}p_{n,n}\end{array}\right]$ and

$U = \left[\begin{array}{c|c} & (-1)^{n-1}p_{n,1} \\ & \dots \\ U_{n-1} & (-1)^{n-i}p_{n,i} \\ & \dots \\ & -p_{n,n-1} \\ \hline 0 & p_{n,n}\end{array}\right]$ for $n > 1$. Therefore we have

$$Lv = \begin{bmatrix} 1 \\ 1-x \\ (1-x)^2 \\ \dots \end{bmatrix} \text{ and } Uv = \begin{bmatrix} 1 - x + x^2 - x^3 - \dots \\ x - 2x^2 + 3x^3 + \dots \\ x^2 - 3x^3 + 6x^4 - \dots \\ \dots \end{bmatrix} = \begin{bmatrix} \frac{1}{1+x} \\ \frac{x}{(1+x)^2} \\ \frac{x^2}{(1+x)^3} \\ \dots \end{bmatrix},$$

thus due to (1) it follows that

$$LUv = L\tfrac{1}{1+x}\begin{bmatrix} 1 \\ \frac{x}{1+x} \\ (\frac{x}{1+x})^2 \\ \dots \end{bmatrix} = \tfrac{1}{1+x}\begin{bmatrix} 1 \\ 1 - \frac{x}{1+x} \\ (1-\frac{x}{1+x})^2 \\ \dots \end{bmatrix} = \begin{bmatrix} \frac{1}{1+x} \\ \frac{1}{(1+x)^2} \\ \frac{1}{(1+x)^3} \\ \dots \end{bmatrix} = Qv.$$

It proves $Q = LU$. □

Clearly $L_4U_4 = \begin{bmatrix} 1 & -1 & 1 & -1 \\ 1 & -2 & 3 & -4 \\ 1 & -3 & 6 & -10 \\ 1 & -4 & 10 & -20 \end{bmatrix} = Q_4$. We also notice $L = \mathrm{di}[(-1)^i]P^{-1}$, $U = (P^{-1})^T$ and $Y = P^T\mathrm{di}[(-1)^i]$, so Theorem 4 shows

$Q = \mathrm{di}[(-1)^i]P^{-1}(P^{-1})^T = \mathrm{di}[(-1)^i](P^TP)^{-1}$

and, Theorem 1 yields $Q = PP^T\mathrm{di}[(-1)^i] = \tilde{P}\ \mathrm{di}[(-1)^i]$.

Theorem 5. *For any $m > 0$, the power Q^m of negative Pascal matrix satisfies $Q^mv = z_1 \dots z_m[1, z_m, z_m^2, \dots]^T$ where $z_1 = \frac{1}{1+x}$ and $z_{m+1} = \frac{1}{1+z_m}$.*

Proof. Clearly $Qv = \begin{bmatrix} \frac{1}{1+x} \\ \frac{1}{(1+x)^2} \\ \frac{1}{(1+x)^3} \\ \dots \end{bmatrix} = z_1\begin{bmatrix} 1 \\ z_1 \\ z_1^2 \\ \dots \end{bmatrix}$ by (1). And $Q^2v = z_1Q\begin{bmatrix} 1 \\ z_1 \\ z_1^2 \\ \dots \end{bmatrix} = z_1z_2\begin{bmatrix} 1 \\ z_2 \\ z_2^2 \\ \dots \end{bmatrix}$ and $Q^3v = z_1z_2z_3\begin{bmatrix} 1 \\ z_3 \\ z_3^2 \\ \dots \end{bmatrix}$ with $z_2 = \frac{1}{1+z_1}$, $z_3 = \frac{1}{1+z_2}$. Hence for some $m > 0$, if we assume $Q^mv = z_1 \dots z_m[1, z_m, z_m^2, \ \dots]^T$ with $z_m = \frac{1}{1+z_{m-1}}$ then

$$Q^{m+1}v = Q(Q^mv) = z_1 \dots z_mQ\begin{bmatrix} 1 \\ z_m \\ z_m^2 \\ \dots \end{bmatrix} = z_1 \dots z_m\begin{bmatrix} \frac{1}{1+z_m} \\ \frac{1}{(1+z_m)^2} \\ \frac{1}{(1+z_m)^3} \\ \dots \end{bmatrix}$$

$$= z_1 \dots z_m\begin{bmatrix} z_{m+1} \\ z_{m+1}^2 \\ z_{m+1}^3 \\ \dots \end{bmatrix} = z_1 \dots z_{m+1}[1, z_{m+1}, z_{m+1}^2, z_{m+1}^3, \dots]^T.$$ □

3. Negative Pascal matrix via Fibonacci numbers

Let $\{f_i | i \geq 1\} = \{1, 1, 2, 3, 5, \dots\}$ be the set of Fibonacci numbers.

THEOREM 6. *$Q^m v = \left[\frac{1}{f_{m+1}+f_m x}, \frac{f_m+f_{m-1}x}{(f_{m+1}+f_m x)^2}, \dots, \frac{(f_m+f_{m-1}x)^{i-1}}{(f_{m+1}+f_m x)^i}, \dots\right]^T$ for $m>0$.*

Proof. With $z_1 = \frac{1}{1+x}$ and $z_1^i = \frac{1}{(1+x)^i}$ in Theorem 5, we have

$z_2 = \frac{1}{1+z_1} = \frac{1+x}{2+x}$, $z_1 z_2 = \frac{1}{2+x}$ and $z_1 z_2^i = \frac{(1+x)^{i-1}}{(2+x)^i}$,

$z_3 = \frac{1}{1+z_2} = \frac{2+x}{3+2x}$, $z_1 z_2 z_3 = \frac{1}{3+2x}$ and $z_1 z_2 z_3^i = \frac{(2+x)^{i-1}}{(3+2x)^i}$,

and $z_4 = \frac{1}{1+z_3} = \frac{3+2x}{5+3x}$, $z_1 z_2 z_3 z_4 = \frac{1}{5+3x}$ and $z_1 z_2 z_3 z_4^i = \frac{(3+2x)^{i-1}}{(5+3x)^i}$.

Then by means of Fibonacci numbers f_i, it shows

$z_4 = \frac{f_4+f_3 x}{f_5+f_4 x}$, $z_1 z_2 z_3 z_4 = \frac{1}{f_5+f_4 x}$ and $z_1 z_2 z_3 z_4^i = \frac{(f_4+f_3 x)^{i-1}}{(f_5+f_4 x)^i}$.

Now for some $j>0$, we assume the identities $z_j = \frac{f_j+f_{j-1}x}{f_{j+1}+f_j x}$, $z_1 \dots z_j = \frac{1}{f_{j+1}+f_j x}$ and $z_1 \dots z_{j-1} z_j^i = \frac{(f_j+f_{j-1}x)^{i-1}}{(f_{j+1}+f_j x)^i}$ as inductive hypothesis. Then

$z_{j+1} = \frac{1}{1+z_j} = \frac{1}{\frac{f_{j+1}+f_j x+f_j+f_{j-1}x}{f_{j+1}+f_j x}} = \frac{f_{j+1}+f_j x}{f_{j+2}+f_{j+1}x}$,

$z_1 \dots z_{j+1} = z_1 \dots z_j z_{j+1} = \frac{1}{f_{j+1}+f_j x} \frac{f_{j+1}+f_j x}{f_{j+2}+f_{j+1}x} = \frac{1}{f_{j+2}+f_{j+1}x}$,

and

$z_1 \dots z_j z_{j+1}^i = \frac{1}{f_{j+1}+f_j x} \frac{(f_{j+1}+f_j x)^i}{(f_{j+2}+f_{j+1}x)^i} = \frac{(f_{j+1}+f_j x)^i}{(f_{j+2}+f_{j+1}x)^{i+1}}$,

hence due to Theorem 5 we have

$$Q^m v = z_1 \dots z_m \begin{bmatrix} 1 \\ z_m \\ z_m^2 \\ z_m^3 \\ \dots \end{bmatrix} = \left[\frac{1}{(f_{m+1}+f_m x)}, \frac{(f_m+f_{m-1}x)}{(f_{m+1}+f_m x)^2}, \dots, \frac{(f_m+f_{m-1}x)^{i-1}}{(f_{m+1}+f_m x)^i}, \dots\right]^T. \qquad \square$$

Hence due to Theorem 5 and Theorem 6, we are able to express the power Q^m by Fibonacci numbers explicitly.

THEOREM 7. *Let $Q^m = [q_{i,j}^{(m)}]$. Then the i,j^{th}element $q_{i,j}^{(m)}$ in Q^m is equal to $\sum_{h=0}^{j-1} \binom{-i}{l}\binom{i-1}{j-1-h} f_{m-1}^{j-1-h} f_m^{i-j+2h} f_{m+1}^{-i-h}$. In particular $Q^2 = \left[\sum_{h=0}^{j-1} \binom{-i}{h}\binom{i-1}{j-1-h} 2^{-i-h}\right]$ and $Q^3 = \left[\sum_{h=0}^{j-1} \binom{-i}{h}\binom{i-1}{j-1-h} 2^{-i-j+2h} 3^{-i-h}\right]$ for $i,j \ge 1$.*

Proof. The $Q^2 v = \left[\frac{1}{f_3+f_2 x}, \frac{f_2+f_1 x}{(f_3+f_2 x)^2}, \frac{(f_2+f_1 x)^2}{(f_3+f_2 x)^3}, \dots\right]^T$ in Theorem 6 shows that the i^{th}row $r_i(Q^2 v)$ is composed of the coefficients of expansion of $\frac{(f_2+f_1 x)^{i-1}}{(f_3+f_2 x)^i} = (1+x)^{i-1}(2+x)^{-i}$. Hence the i,j^{th} component $q_{i,j}^{(2)}$ in Q^2 is the coefficient of x^{j-1} in the expansion of $(1+x)^{i-1}(2+x)^{-i}$. But since

$$\begin{aligned}
&(1+x)^{i-1}(2+x)^{-i} \\
&= \left(\tbinom{i-1}{0} + \tbinom{i-1}{1}x + \tbinom{i-1}{2}x^2 + \dots\right)\left(\tbinom{-i}{0}2^{-i} + \tbinom{-i}{1}2^{-i-1}x + \tbinom{-i}{2}2^{-i-2}x^2 + \dots\right) \\
&= \tbinom{i-1}{0}\tbinom{-i}{0}2^{-i} + \left(\tbinom{i-1}{0}\tbinom{-i}{1}2^{-i-1} + \tbinom{i-1}{1}\tbinom{-i}{0}2^{-i}\right)x \\
&\quad + \left(\tbinom{i-1}{0}\tbinom{-i}{2}2^{-i-2} + \tbinom{i-1}{1}\tbinom{-i}{1}2^{-i-1} + \tbinom{i-1}{2}\tbinom{-i}{0}2^{-i}\right)x^2 \\
&\quad + \left(\tbinom{i-1}{0}\tbinom{-i}{3}2^{-i-3} + \tbinom{i-1}{1}\tbinom{-i}{2}2^{-i-2} + \tbinom{i-1}{2}\tbinom{-i}{1}2^{-i-1} + \tbinom{i-1}{3}\tbinom{-i}{0}2^{-i}\right)x^3 \\
&\quad + \dots,
\end{aligned}$$

the coefficient of x^{j-1} equals $\sum_{t=0}^{j-1} \binom{-i}{t}\binom{i-1}{j-1-t} 2^{-i-t}$. So $Q^2 = \left[\sum_{h=0}^{j-1} \binom{-i}{h}\binom{i-1}{j-1-h} 2^{-i-h}\right]$.

Similarly the i^{th}row $r_i(Q^3 v)$ is generated by the expansion of

$$\begin{aligned}
&\tfrac{(f_3+f_2x)^{i-1}}{(f_4+f_3x)^i}\\
&= (2+x)^{i-1}(3+2x)^{-i}\\
&= \tbinom{-i}{0}\tbinom{i-1}{0}2^{i-1}3^{-i} \quad + \left(\tbinom{-i}{0}\tbinom{i-1}{1}2^{i-2}3^{-i} + \tbinom{-i}{1}\tbinom{i-1}{0}2^{i}3^{-i-1}\right)x\\
&\quad + \left(\tbinom{-i}{0}\tbinom{i-1}{2}2^{i-3}3^{-i} + \tbinom{-i}{1}\tbinom{i-1}{1}2^{i-1}3^{-i-1} + \tbinom{-i}{2}\tbinom{i-1}{0}2^{i+1}3^{-i-2}\right)x^2\\
&\quad + \left(\tbinom{-i}{0}\tbinom{i-1}{3}2^{i-4}3^{-i} + \tbinom{-i}{1}\tbinom{i-1}{2}2^{i-2}3^{-i-1} + \tbinom{-i}{3}\tbinom{i-1}{0}2^{i+2}3^{-i-3}\right)x^3\\
&\quad + \ldots.
\end{aligned}$$

Thus the coefficient of x^{j-1} equals $\sum_{h=0}^{j-1}\binom{-i}{h}\binom{i-1}{j-1-h}2^{-i-j+2h}3^{-i-h}$, so it follows that $Q^3 = \left[\sum_{h=0}^{j-1}\binom{-i}{h}\binom{i-1}{j-1-h}2^{-i-j+2h}3^{-i-h}\right]$.

Now let us look at $Q^m v = \left[\frac{1}{(f_{m+1}+f_m x)}, \frac{(f_m+f_{m-1}x)}{(f_{m+1}+f_m x)^2}, \frac{(f_m+f_{m-1}x)^2}{(f_{m+1}+f_m x)^3}, \ldots\right]^T$ with $Q^m = [q_{i,j}^{(m)}]$. From the expansions

$(f_m + f_{m-1}x)^{i-1} = \binom{i-1}{0}{f_m}^{i-1} + \binom{i-1}{1}{f_m}^{i-2}f_{m-1}x + \binom{i-1}{2}{f_m}^{i-3}f_{m-1}^2x^2 + \ldots$

and

$(f_{m+1} + f_m x)^i = \binom{-i}{0}f_{m+1}^{-i} + \binom{-i}{1}f_{m+1}^{-i-1}f_m x + \binom{-i}{2}f_{m+1}^{-i-2}f_m^2x^2 + \ldots,$

the coefficient $q_{i,j}^{(m)}$ of x^{j-1} in the expansion of $(f_m + f_{m-1}x)^{i-1}(f_{m+1} + f_m x)^i$ is equal to $q_{i,j}^{(m)} = \sum_{h=0}^{j-1}\binom{-i}{h}\binom{i-1}{j-1-h}f_{m-1}^{j-1-h}f_m^{i-j+2h}f_{m+1}^{-i-h}$. Thus we conclude that $Q^m = \left[\sum_{h=0}^{j-1}\binom{-i}{h}\binom{i-1}{j-1-h}f_{m-1}^{j-1-h}f_m^{i-j+2h}f_{m+1}^{-i-h}\right]$. □

In fact $Q^3v = \begin{bmatrix}\frac{1}{3+2x}\\ \frac{(2+x)}{(3+2x)^2}\\ \frac{(2+x)^2}{(3+2x)^3}\\ \cdots\end{bmatrix} = \begin{bmatrix}\frac{1}{3} & -\frac{2}{9} & \frac{4}{27} & -\frac{8}{81} & \cdots\\ \frac{2}{9} & -\frac{5}{27} & \frac{4}{27} & -\frac{28}{243} & \cdots\\ \frac{4}{27} & -\frac{4}{27} & \frac{11}{81} & -\frac{86}{729} & \cdots\\ \frac{8}{81} & -\frac{28}{243} & \frac{86}{729} & -\frac{245}{2187} & \cdots\end{bmatrix}v$, and for instance we observe $q_{5,5}^{(3)} = \sum_{h=0}^{4}\binom{-5}{h}\binom{4}{4-h}2^{2h}3^{-5-h} = \frac{1921}{19683}$. Moreover the i^{th}row $r_i(Q^4v)$ comes from $(3 + 2x)^{i-1}(5 + 3x)^{-i}$. But since

$$\begin{aligned}
&(3+2x)^{i-1}(5+3x)^{-i}\\
&= \tbinom{-i}{0}\tbinom{i-1}{0}3^{i-1}5^{-i} + \left(\tbinom{-i}{0}\tbinom{i-1}{1}2^1 3^{i-2}5^{-i} + \tbinom{-i}{1}\tbinom{i-1}{0}3^{i}5^{-i-1}\right)x\\
&+ \left(\tbinom{-i}{0}\tbinom{i-1}{2}2^2 3^{i-3}5^{-i} + \tbinom{-i}{1}\tbinom{i-1}{1}2^1 3^{i-1}5^{-i-1} + \tbinom{-i}{2}\tbinom{i-1}{0}3^{i+1}5^{-i-2}\right)x^2\\
&+ \left(\tbinom{-i}{0}\tbinom{i-1}{3}2^3 3^{i-4}5^{-i} + \tbinom{-i}{1}\tbinom{i-1}{2}2^2 3^{i-2}5^{-i-1} + \cdots + \tbinom{-i}{3}\tbinom{i-1}{0}3^{i+2}5^{-i-3}\right)x^3 + \ldots,
\end{aligned}$$

the coefficient of x^{j-1} in $(3 + 2x)^{i-1}(5 + 3x)^{-i}$ is $\sum_{h=0}^{j-1}\binom{-i}{h}\binom{i-1}{j-1-h}2^{j-1-h}3^{i-j+2h}5^{-i-h}$.

Therefore $Q^4v = \begin{bmatrix}\frac{1}{5+3x}\\ \frac{(3+2x)}{(5+3x)^2}\\ \frac{(3+2x)^2}{(5+3x)^3}\\ \cdots\end{bmatrix} = \begin{bmatrix}\frac{1}{5} & -\frac{3}{25} & \frac{9}{125} & -\frac{27}{625}\\ \frac{3}{25} & -\frac{8}{125} & \frac{21}{625} & -\frac{54}{3125}\\ \frac{9}{125} & -\frac{21}{625} & \frac{46}{3125} & -\frac{90}{51625}\\ \frac{27}{625} & -\frac{54}{3125} & \frac{90}{51625} & -\frac{450}{78125}\\ & \cdots & & \end{bmatrix}v$ and

$Q^4 = \left[\sum_{h=0}^{j-1}\binom{-i}{h}\binom{i-1}{j-1-h}2^{j-1-h}3^{i-j+2h}5^{-i-h}\right].$

On the other hand from $P=[p_{i,j}]=\left[\binom{i-1}{j-1}\right]$ and $Q=[q_{i,j}]=[\binom{-i}{j-1}]$ for $i,j\geq 1$, it follows $Q^m=\left[\sum_{h=0}^{j-1} q_{i,h+1}p_{i,j-h}f_{m-1}^{j-1-h}f_m^{i-j+2h}f_{m+1}^{-i-h}\right]$. The next theorem gives an expression of Q^m by Fibonacci numbers f_i together with Q and P.

THEOREM 8. *Let $Q^m=[q_{i,j}^{(m)}]$. Then for all $i,j\geq 1$, we have*
$$q_{i,j}^{(m)}=(f_{m-1}^{j-1},f_{m-1}^{j-2},\ldots,f_{m-1}^{0})\cdot f_{m+1}^{-i}(q_{i,1},f_{m+1}^{-1}q_{i,2},f_{m+1}^{-2}q_{i,3},\ldots,f_{m+1}^{-(j-1)}q_{i,j})$$
$$\cdot f_m^i(f_m^{-j}p_{i,j},f_m^{-j+2}p_{i,j-1},\ldots,f_m^{-j+2h}p_{i,j-h},\ldots,f_m^{j-2}p_{i,1}).$$

Proof. The all entries $q_{i,j}^{(3)}$ at i^{th}row of Q^3 are
$$q_{i,1}^{(3)}=q_{i,1}p_{i,1}2^{i-1}3^{-i}=(3^{-i}q_{i,1})(2^{i-1}p_{i,1}),$$
$$\begin{aligned}q_{i,2}^{(3)}&=q_{i,1}p_{i,2}2^{i-1}3^{-i}+q_{i,2}p_{i,1}2^{i}3^{-i-1}\\&=(3^{-i}q_{i,1},3^{-i-1}q_{i,2})\cdot(2^{i-2}p_{i,2},2^{i}p_{i,1})=3^{-i}(q_{i,1},3^{-1}q_{i,2})\cdot 2^i(2^{-2}p_{i,2},p_{i,1}),\end{aligned}$$
$$\begin{aligned}q_{i,3}^{(3)}&=q_{i,1}p_{i,3}2^{-i-3}3^{-i}+q_{i,2}p_{i,2}2^{i-1}3^{-i-1}+q_{i,3}p_{i,1}2^{i+1}3^{-i-2}\\&=(3^{-i}q_{i,1},3^{-i-1}q_{i,2},3^{-i-2}q_{i,3})\cdot(2^{i-3}p_{i,3},2^{i-1}p_{i,2},2^{i+1}p_{i,1})\\&=3^{-i}(q_{i,1},3^{-1}q_{i,2},3^{-2}q_{i,3})\cdot 2^i(2^{-3}p_{i,3},2^{-1}p_{i,2},2^{1}p_{i,1}),\end{aligned}$$
and for any $j\geq 1$, we have
$$\begin{aligned}q_{i,j}^{(3)}&=(3^{-i}q_{i,1},3^{-i-1}q_{i,2},\ldots,3^{-i-(h-1)}q_{i,h},\ldots,3^{-i-j+1}q_{i,j})\\&\quad\cdot(2^{i-j}p_{i,j},2^{i-j+2}p_{i,j-1},\ldots,2^{i-j+2h}p_{i,j-h},\ldots,2^{i+j-2}p_{i,1})\\&=3^{-i}(q_{i,1},3^{-1}q_{i,2},\ldots,3^{-(h-1)}q_{i,h},\ldots,3^{-(j-1)}q_{i,j})\\&\quad\cdot 2^i(2^{-j}p_{i,j},2^{-j+2}p_{i,j-1},\ldots,2^{-j+2h}p_{i,j-h},\ldots,2^{j-2}p_{i,1}).\end{aligned}$$
Similarly all entries $q_{i,j}^{(4)}$ at i^{th}row of Q^4 are
$$\begin{aligned}q_{i,2}^{(4)}&=q_{i,1}p_{i,2}2^{1}3^{i-1}5^{-i}+q_{i,2}p_{i,1}3^{i}5^{-i-1}\\&=(2^1,1)\cdot 5^{-i}(q_{i,1},5^{-1}q_{i,2})\cdot 3^i(3^{-2}p_{i,2},p_{i,1}),\end{aligned}$$
$$\begin{aligned}q_{i,3}^{(4)}&=q_{i,1}p_{i,3}2^{2}3^{i-3}5^{-i}+q_{i,2}p_{i,2}2^{1}3^{i-1}5^{-i-1}+q_{i,3}p_{i,1}3^{i+1}5^{-i-2}\\&=(2^2,2^1,1)\cdot 5^{-i}(q_{i,1},5^{-1}q_{i,2},5^{-2}q_{i,3})\cdot 3^i(3^{-3}p_{i,3},3^{-1}p_{i,2},3^{1}p_{i,1}),\end{aligned}$$
and for any $j\geq 1$, we have
$$\begin{aligned}q_{i,j}^{(4)}&=(2^{j-1},2^{j-2},\ldots,2^1,1)\cdot 5^{-i}(q_{i,1},5^{-1}q_{i,2},5^{-2}q_{i,3},\ldots,5^{-(j-1)}q_{i,j})\\&\quad\cdot 3^i(3^{-j}p_{i,j},3^{-j+2}p_{i,j-1},\ldots,3^{-j+2h}p_{i,j-h},\ldots,3^{j-2}p_{i,1}).\end{aligned}$$
But since $Q^4v=\begin{bmatrix}\frac{1}{5+3x}\\ \frac{3+2x}{(5+3x)^2}\\ \frac{(3+2x)^2}{(5+3x)^3}\\ \cdots\end{bmatrix}$ and $Q^4=\left[\sum_{h=0}^{j-1}q_{i,h+1}p_{i,j-h}f_3^{j-1-h}f_4^{i-j+2h}f_5^{-i-h}\right]$, the i,j^{th} component in Q^4 can be written as
$$\begin{aligned}q_{i,j}^{(4)}&=(f_3^{j-1},f_3^{j-2},\ldots,f_3^1,f_3^0)\cdot f_5^{-i}(q_{i,1},f_5^{-1}q_{i,2},\ldots,f_5^{-(j-1)}q_{i,j})\\&\quad\cdot f_4^i(f_4^{-j}p_{i,j},f_4^{-j+2}p_{i,j-1},\ldots,f_4^{-j+2h}p_{i,j-h},\ldots,f_4^{j-2}p_{i,1}).\end{aligned}$$
Thus $Q^mv=\begin{bmatrix}\frac{1}{f_{m+1}+f_mx}\\ \frac{f_m+f_{m-1}x}{(f_{m+1}+f_mx)^2}\\ \frac{(f_m+f_{m-1}x)^2}{(f_{m+1}+f_mx)^3}\\ \cdots\end{bmatrix}$ and $Q^m=\left[\sum_{h=0}^{j-1}q_{i,h+1}p_{i,j-h}f_{m-1}^{j-1-h}f_m^{i-j+2h}f_{m+1}^{-i-h}\right]$ for any m imply
$$\begin{aligned}q_{i,j}^{(m)}&=(f_{m-1}^{j-1},f_{m-1}^{j-2},\ldots,f_{m-1}^1,f_{m-1}^0)\cdot f_{m+1}^{-i}(q_{i,1},f_{m+1}^{-1}q_{i,2},\ldots,f_{m+1}^{-(j-1)}q_{i,j})\\&\quad\cdot f_m^i(f_m^{-j}p_{i,j},f_m^{-j+2}p_{i,j-1},\ldots,f_m^{-j+2h}p_{i,j-h},\ldots,f_m^{j-2}p_{i,1}).\end{aligned}$$
□

Note that $\begin{bmatrix} \frac{1}{2+x} \\ \frac{1}{(2+x)^2} \\ \frac{1}{(2+x)^3} \\ \cdots \end{bmatrix} = Rv$ where $R = \begin{bmatrix} \frac{1}{2} & -\frac{1}{4} & \frac{1}{8} & -\frac{1}{16} & \cdots \\ \frac{1}{4} & -\frac{1}{4} & \frac{3}{16} & -\frac{1}{8} & \cdots \\ \frac{1}{8} & -\frac{3}{16} & \frac{3}{16} & -\frac{5}{32} & \cdots \\ & & \cdots & & \end{bmatrix}$. Let A be a matrix such that $Av = \begin{bmatrix} 1 \\ 1-x \\ (1-x)^2 \\ \cdots \end{bmatrix}$. Then due to Theorem 7, we have

$$ARv = \tfrac{1}{2+x} A \begin{bmatrix} 1 \\ \frac{1}{2+x} \\ \frac{1}{(2+x)^2} \\ \cdots \end{bmatrix} = \tfrac{1}{2+x} \begin{bmatrix} 1 \\ 1 - \frac{1}{2+x} \\ (1 - \frac{1}{2+x})^2 \\ \cdots \end{bmatrix} = \begin{bmatrix} \frac{1}{2+x} \\ \frac{1+x}{(2+x)^2} \\ \frac{(1+x)^2}{(2+x)^3} \\ \cdots \end{bmatrix} = Q^2 v. \tag{2}$$

Thus $Q^2 = AR$. Indeed $Q^2 = \begin{bmatrix} 1 & & \\ 1 & -1 & \\ 1 & -2 & 1 \\ & \cdots & \end{bmatrix} \begin{bmatrix} \frac{1}{2} & -\frac{1}{4} & \frac{1}{8} & -\frac{1}{16} & \cdots \\ \frac{1}{4} & -\frac{1}{4} & \frac{3}{16} & -\frac{1}{8} & \cdots \\ \frac{1}{8} & -\frac{3}{16} & \frac{3}{16} & -\frac{5}{32} & \cdots \\ & & \cdots & & \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & -\frac{1}{4} & \frac{1}{8} & -\frac{1}{16} & \cdots \\ \frac{1}{4} & 0 & -\frac{1}{16} & \frac{1}{16} & \cdots \\ \frac{1}{8} & \frac{1}{16} & -\frac{1}{16} & \frac{1}{32} & \cdots \\ & & \cdots & & \end{bmatrix}$. This can be generalized to Q^m as follows.

THEOREM 9. *For $m > 0$, let $R_{[m]}$ and $A_{[m]}$ be matrices such that* $R_{[m]}v = \begin{bmatrix} \frac{1}{f_{m+1}+f_m x} \\ \frac{1}{(f_{m+1}+f_m x)^2} \\ \frac{1}{(f_{m+1}+f_m x)^3} \\ \cdots \end{bmatrix}$ *and* $A_{[m]}v = \begin{bmatrix} 1 \\ \frac{f_{m-1}+(-1)^{m-1}x}{f_m} \\ (\frac{f_{m-1}+(-1)^{m-1}x}{f_m})^2 \\ (\frac{f_{m-1}+(-1)^{m-1}x}{f_m})^3 \\ \cdots \end{bmatrix}$. *Then* $Q^m = A_{[m]}R_{[m]}$.

Proof. Clearly $Q^2 = A_{[2]}R_{[2]}$ by (2). When $m = 3$, $Q^3 v = \begin{bmatrix} \frac{1}{3+2x} \\ \frac{(2+x)}{(3+2x)^2} \\ \frac{(2+x)^2}{(3+2x)^3} \\ \cdots \end{bmatrix} = u \begin{bmatrix} 1 \\ \frac{2+x}{3+2x} \\ (\frac{2+x}{3+2x})^2 \\ \cdots \end{bmatrix}$ for $u = \frac{1}{3+2x}$ by Theorem 7. Thus with $R_{[3]}v = \begin{bmatrix} \frac{1}{3+2x} \\ \frac{1}{(3+2x)^2} \\ \frac{1}{(3+2x)^3} \\ \cdots \end{bmatrix} = u \begin{bmatrix} 1 \\ u \\ u^2 \\ \cdots \end{bmatrix}$, the matrix $A_{[3]}$ such that $A_{[3]}v = \begin{bmatrix} 1 \\ \frac{x+1}{2} \\ (\frac{x+1}{2})^2 \\ \cdots \end{bmatrix}$ satisfies

$$A_{[3]}R_{[3]}v = uA_{[3]} \begin{bmatrix} 1 \\ u \\ u^2 \\ \cdots \end{bmatrix} = u \begin{bmatrix} 1 \\ \frac{u+1}{2} \\ (\frac{u+1}{2})^2 \\ \cdots \end{bmatrix} = \frac{1}{3+2x} \begin{bmatrix} 1 \\ \frac{2+x}{3+2x} \\ (\frac{2+x}{3+2x})^2 \\ \cdots \end{bmatrix} = Q^3 v,$$

so we have $Q^3 = A_{[3]}R_{[3]}$. Now let $u = \frac{1}{f_{m+1}+f_m x}$ for $m > 0$. Then

$$Q^m v = \begin{bmatrix} \frac{1}{(f_{m+1}+f_m x)} \\ \frac{(f_m+f_{m-1}x)}{(f_{m+1}+f_m x)^2} \\ \frac{(f_m+f_{m-1}x)^2}{(f_{m+1}+f_m x)^3} \\ \cdots \end{bmatrix} = u \begin{bmatrix} 1 \\ \frac{f_m+f_{m-1}x}{f_{m+1}+f_m x} \\ (\frac{f_m+f_{m-1}x}{f_{m+1}+f_m x})^2 \\ \cdots \end{bmatrix} \text{ and } R_{[m]}v = \begin{bmatrix} \frac{1}{f_{m+1}+f_m} \\ \frac{1}{(f_{m+1}+f_m x)^2} \\ \frac{1}{(f_{m+1}+f_m x)^3} \\ \cdots \end{bmatrix} = u \begin{bmatrix} 1 \\ u \\ u^2 \\ \cdots \end{bmatrix}.$$

Moreover let $A_{[m]}v = \begin{bmatrix} 1 \\ \frac{f_{m-1}+(-1)^{m-1}x}{f_m} \\ (\frac{f_{m-1}+(-1)^{m-1}x}{f_m})^2 \\ \cdots \end{bmatrix}$. Owing to the identity $f_{m-1}f_{m+1} + (-1)^{m-1} = f_m^2$ (see [8]), we have

$\frac{f_{m-1}+(-1)^{m-1}u}{f_m} = \frac{1}{f_m}\frac{f_m^2+f_{m-1}f_m x}{f_{m+1}+f_m x} = \frac{f_m+f_{m-1}x}{f_{m+1}+f_m x}$,
so it follows that

$$A_{[m]}R_{[m]}v = uA_{[m]}\begin{bmatrix}1\\ u\\ u^2\\ \cdots\end{bmatrix} = u\begin{bmatrix}1\\ \frac{f_{m-1}+(-1)^{m-1}u}{f_m}\\ (\frac{f_{m-1}+(-1)^{m-1}u}{f_m})^2\\ \cdots\end{bmatrix} = u\begin{bmatrix}1\\ \frac{f_m+f_{m-1}x}{f_{m+1}+f_m x}\\ (\frac{f_m+f_{m-1}x}{f_{m+1}+f_m x})^2\\ \cdots\end{bmatrix} = Q^m v.$$

This completes the proof $Q^m = A_{[m]}R_{[m]}$. □

Indeed $R_{[3]}v = \begin{bmatrix}\frac{1}{3+2x}\\ \frac{1}{(3+2x)^2}\\ \frac{1}{(3+2x)^3}\\ \cdots\end{bmatrix} = \begin{bmatrix}\frac{1}{3} & -\frac{2}{9} & \frac{4}{27} & \cdots\\ \frac{1}{9} & -\frac{4}{27} & \frac{4}{27} & \cdots\\ \frac{1}{27} & -\frac{2}{27} & \frac{8}{81} & \cdots\\ \cdots\end{bmatrix}v$ and $A_{[3]}v = \begin{bmatrix}1\\ \frac{x+1}{2}\\ (\frac{x+1}{2})^2\\ \cdots\end{bmatrix} = \begin{bmatrix}1 & & \\ \frac{1}{2} & \frac{1}{2} & \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4}\\ \cdots\end{bmatrix}v$,

hence $A_{[3]}R_{[3]} = \begin{bmatrix}\frac{1}{3} & -\frac{2}{9} & \frac{4}{27} & \cdots\\ \frac{2}{9} & -\frac{5}{27} & \frac{4}{27} & \cdots\\ \frac{4}{27} & -\frac{4}{27} & \frac{11}{81} & \cdots\\ \cdots\end{bmatrix} = Q^3$.

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