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MULTIVALUED FIXED POINT THEOREM INVOLVING HYBRID CONTRACTION OF THE JAGGI-SUZUKI TYPE

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Abstract. In this manuscript, a new multi-valued contraction is defined from a combination of Jaggi-type hybrid contraction and Suzuku-type hybrid contraction. Conditions for the existence of fixed points for such contractions in metric space are investigated. Moreover, some consequences are highlighted and discussed to indicate the significance of our proposed ideas. An example is given to support the assumptions of our theorems.

1. Introduction

Almost a century ago, Banach [1] initiated the metric fixed point theory with a magnificently simple but enormously useful result, known as the Banach contraction principle. It is one of the useful theorems for solving differential and integral equations to guarantee both the existence and uniqueness of the solution. The main result of Banach has been generalized in different directions by many researchers. For some interesting results, see, e.g., [15–17,22]. Jaggi [4] proved the theorem satisfying a contractive condition of a rational type. Very recently in 2018, Karapinar [8] obtained a new type of contraction from the well known Kannan contraction by adopting an interpolative approach. In [9], a common fixed point of the interpolative Kannan contraction was considered. In 2008, Suzuki [6] published one of the most comprehensive generalizations of Banach's and Edelstein's basic results. When all of the domain's points do not meet the necessary contractive condition, this is known as Suzuki contraction.

The existence and uniqueness of fixed points of maps satisfying a Suzuki type contraction has been extensively studied. In this direction, Popescu [7] modified the nonexpansiveness situation with the weaker C-condition presented by Suzuki [6]. Recently, Mitrovic et al. [19] used interpolation contraction and Reich contraction together and combined these two contractions in b-metric spaces. The combination of these two types of contractions has been called the new hybrid-type contraction. In the last years, inspired by the result in [8] and [13], Maha and Seher [21] introduced a

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single-valued version of hybrid-type contraction that combines Jaggi hybrid-type contractions and Suzuki hybrid-type contractions in the framework of a complete metric spaces. Moreover, Kamaleldin et. al [20] introduced hybrid contractions on Branciari type distance spaces and proved the existence of fixed point of such operators. On the other hand, the concept of multivalued contraction was introduced by Nadler [14] and the corresponding fixed point result was proposed therein. Moreover, Shagari et. al [18] introduced the concepts of Jaggi and Dass-Gupta type bilateral multi-valued contractions and under some suitable conditions, the existence of fixed points for such mappings are established. Following the existing literature, we note that fixed point theorem of multivalued contraction using Jaggi and Suzuki-type contraction is not sufficiently investigated. On this background information, this paper presents new multivalued fixed point results via Jaggi-Suzuki-type contractive inequality. Unlike the main ideas of [21], the key concept herein is presented by dropping the notion of w-orbital admissibility. Therefore, some consequences of our results in the setting of single-valued mappings which improve a few corresponding concepts are pointed out and analyzed.

2. Preliminaries

In the sequel, we record some basic concepts/results that are needed in the sequel. Throughout this paper, we denote the set of natural numbers, nonnegative reals and real numbers by $\mathbb{N}$, $\mathbb{R}_+$ and $\mathbb{R}$, respectively.
Let (X, d) be a metric space, $K(X)$ be the class of nonempty compact subsets of X. For $A, B \in K(X)$. The Hausdorff metric H on $K(X)$ induced by the metric d is given by

$$H(A, B) = \max\{\sup_{a\in A} D(a, B), \sup_{b\in B} D(b, A)\},$$

where

$$D(a, B) = \inf_{b\in B}\{d(a, b)\}.$$

It is known that H is a metric on $K(X)$ and H is called the Hausdorff metric or Pompeiu-Hausdorff metric induced by d.

Definition 2.1. Let (X, d) be a complete metric space and $T : X \longrightarrow X$.

- (i) (Jaggi [5]) There exists $\lambda_1, \lambda_2 \in [0, 1)$ with $\lambda_1 + \lambda_2 < 1$ such that $d(Tx, Ty) \leq \lambda_1 d(x, y) + \lambda_2 \frac{d(x,Tx)d(y,Ty)}{d(x,y)}$
- (ii) (Dass and Gupta [4]) There exists $\lambda_1, \lambda_2 \in [0, 1)$ with $\lambda_1 + \lambda_2 < 1$ such that $d(Tx, Ty) \leq \lambda_1 d(x, y) + \lambda_2 \frac{[1+d(x,Tx)]d(y,Ty)}{[1+d(x,y)]}$
- (iii) (Ciric [3]) There exists a constant λ, $0 \leq \lambda < 1$, such that, for each $x, y \in X$, $d(Tx, Ty) \leq \lambda \max\{d(x, y), d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)\}$.

Not long ago, Chen et al. [11] introduced the notion of Bilateral contraction in the following manner:

Definition 2.2. [11] Let (X, d) be a complete metric space. A self mapping $T : X \longrightarrow X$ is called a Jaggi type bilateral contraction if there is $\vartheta : X \longrightarrow [0, \infty)$ such that, $d(x, Tx) > 0$ implies

$$d(Tx, Ty) \leq [\vartheta(x) - \vartheta(Tx)]R_T(x, y),$$

for all distinct $x, y \in X$, where

$$R_T(x,y) = \max\{d(x,y), \frac{d(x,Tx)d(y,Ty)}{d(x,y)}\}.$$

DEFINITION 2.3. [11] Let (X,d) be a complete metric space. A self mapping $T : X \longrightarrow X$ is called a Dass-Gupta type bilateral contraction if there is $\vartheta : X \longrightarrow [0,\infty)$ such that, $d(x,Tx) > 0$ implies

$$d(Tx,Ty) \leq [\vartheta(x) - \vartheta(Tx)]Q_T(x,y),$$

for all $x, y \in X$, where

$$Q_T(x,y) = \max\{d(x,y), \frac{(1+d(x,Tx))d(y,Ty)}{1+d(x,y)}\}.$$

DEFINITION 2.4. [10, Definition 2.1] Let T be a self mapping on a complete metric space (X,d). If there exists $\xi \in Z$ and $\vartheta : X \longrightarrow [0,\infty)$ such that, $d(x,Tx) > 0$ implies

$$\xi(d(Tx,Ty), (\vartheta(x) - \vartheta(Tx))C_T(x,y)) \geq 0,$$

in which

$$C_T(x,y) = \max\{d(x,y), d(x,Tx), d(y,Ty), \frac{d(x,Tx)+d(y,Ty)}{2}\},$$

for all $x, y \in X$, then T is called a bilateral contraction of Ciric-Caristi.

DEFINITION 2.5. [12] Let (X,d) be a complete metric space. A self mapping $T : X \longrightarrow X$ is called a Ciric - Caristi type contraction if there is a mapping $\vartheta : X \longrightarrow \mathbb{R}_+$ such that, $d(x,Tx) > 0$ implies

$$d(Tx,Ty) \leq [\vartheta(x) - \vartheta(Tx)]N(x,y),$$

for all $x, y \in X$, where

$$N(x,y) = \max\{d(x,y), d(x,Tx), d(y,Ty), d(x,Ty), d(y,Tx)\}.$$

LEMMA 2.6. *[2] Let (X,d) be a metric space and $A, B \in K(X)$. If $a \in A$, then there exists $b \in B$ such that*

$$d(a,b) \leq H(A,B).$$

We will require the following class of functions in the sequel. Let Ψ denote the family of all functions $\psi : [0,1] \to [0,\infty)$ such that ψ is continuous at zero with $\psi(0) = 0$. In this section, we study fixed point results of multivalued Jaggi-Suzuki-type contractions.

DEFINITION 2.7. Let (X,d) be a metric space. A mapping $T : X \longrightarrow K(X)$ is called a multivalued Jaggi-Suzuki-type hybrid contraction if there exists $\psi \in \Psi$ such that $\frac{1}{2}D(x,Tx) \leq d(x,y)$ implies

$$H(Tx,Ty) \leq \psi(J_T^s(x,y)), \tag{1}$$

for all $x, y \in X$, where $s \geq 0$ and $\delta_i \geq 0$, $i = 1, 2, ...$, such that $\delta_1 + \delta_2 = 1$,

$$\psi(J_T^s(x,y)) = \begin{cases} [\delta_1 \left(\frac{D(x,Tx)D(y,Ty)}{1+d(x,y)}\right)^s + \delta_2(d(x,y))^s]^{\frac{1}{s}} & \text{if } s > 0, x, y \in X, x \neq y, \\ (D(x,Tx))^{\delta_1}(D(y,Ty))^{\delta_2} & \text{if } s = 0, x, y \in X \setminus F_T(X), \end{cases}$$

where $F_T(X) = \{x \in X : x \in Tx\}$.

THEOREM 2.8. *Let (X,d) be a complete metric space and $T : X \longrightarrow K(X)$ be a multivalued Jaggi-Suzuki-type hybrid contraction mapping. Then, T has a fixed point in X.*

Proof. Let $x_0 \in X$ be arbitrary. Then, by hypothesis $Tx_0 \in K(X)$. Choose $x_1 \in Tx_0$, then for this $x_1 \in X$, Tx_1 is a nonempty compact subset of X. Hence, we can find $x_2 \in Tx_1$ such that $d(x_1,x_2) = d(x_1,Tx_1) = D(Tx_0,Tx_1)$. Continuing in this manner, we construct a sequence (x_n) in X such that

$$x_{n+1} \in Tx_n \quad \text{and} \quad d(x_n,x_{n+1}) = D(x_n,Tx_n), \text{ for all n} \geq 0.$$

Now,

$$\begin{aligned} \frac{1}{2}D(x_n,Tx_n) &= \frac{1}{2}d(x_n,x_{n+1}) \leq d(x_n,x_{n+1}) \\ \Rightarrow \quad H(Tx_n,Tx_{n+1}) &\leq \psi(J_T^s(x_n,x_{n+1})) \qquad (2) \end{aligned}$$

We shall prove the claim by examining two cases: $s = 0$ and $s > 0$
Case I: If $s > 0$, then by Lemma (2.6) and inequality (2) we have

$$\begin{aligned} d(x_{n+1},x_{n+2}) &\leq H(Tx_n,Tx_{n+1}) \\ &\leq \psi(J_T^s(x_n,x_{n+1})) \\ &\leq \psi[\delta_1(\frac{D(x_n,Tx_n)D(x_{n+1},Tx_{n+1})}{1+d(x_n,x_{n+1})})^s + \delta_2(d(x_n,x_{n+1}))^s]^{\frac{1}{s}} \\ &\leq \psi[\delta_1(\frac{d(x_n,x_{n+1})d(x_{n+1},x_{n+2})}{1+d(x_n,x_{n+1})})^s + \delta_2(d(x_n,x_{n+1}))^s]^{\frac{1}{s}} \\ &\leq \psi[\delta_1(\frac{d(x_n,x_{n+1})d(x_{n+1},x_{n+2})}{d(x_n,x_{n+1})})^s + \delta_2(d(x_n,x_{n+1}))^s]^{\frac{1}{s}} \\ &= \psi[\delta_1(d(x_{n+1},x_{n+2}))^s + \delta_2(d(x_n,x_{n+1}))^s]^{\frac{1}{s}}. \qquad (3) \end{aligned}$$

Suppose that $d(x_{n+1},x_{n+2}) \geq d(x_n,x_{n+1})$, then inequality (3) becomes

$$\begin{aligned} d(x_{n+1},x_{n+2}) &\leq \psi[\delta_1(d(x_{n+1},x_{n+2}))^s + \delta_2(d(x_n,x_{n+1}))^s]^{\frac{1}{s}} \\ &\leq \psi[\delta_1(d(x_{n+1},x_{n+2}))^s + \delta_2(d(x_{n+1},x_{n+2}))^s]^{\frac{1}{s}} \\ &= \psi[(\delta_1+\delta_2)(d(x_{n+1},x_{n+2}))^s]^{\frac{1}{s}} \\ &= \psi[(d(x_{n+1},x_{n+2}))^s]^{\frac{1}{s}} \\ &= \psi(d(x_{n+1},x_{n+2})) \\ &< d(x_{n+1},x_{n+2}), \end{aligned}$$

a contradiction. This implies that

$$d(x_{n+1},x_{n+2}) < d(x_n,x_{n+1}).$$

So we have

$$\begin{aligned} d(x_{n+1},x_{n+2}) &\leq \psi(d(x_n,x_{n+1})) \\ &< d(x_n,x_{n+1}). \end{aligned}$$

Again,

$$\begin{aligned}
d(x_{n+2}, x_{n+3}) &\leq H(Tx_{n+1}, Tx_{n+2}) \\
&\leq \psi(J_T^s(x_{n+1}, x_{n+2})) \\
&\leq \psi[\delta_1(\frac{D(x_{n+1}, Tx_{n+1})D(x_{n+2}, Tx_{n+2})}{1 + d(x_{n+1}, x_{n+2})})^s + \delta_2(d(x_{n+1}, x_{n+2}))^s]^{\frac{1}{s}} \\
&\leq \psi[\delta_1(\frac{d(x_{n+1}, x_{n+2})d(x_{n+2}, x_{n+3})}{1 + d(x_{n+1}, x_{n+2})})^s + \delta_2(d(x_{n+1}, x_{n+2}))^s]^{\frac{1}{s}} \\
&\leq \psi[\delta_1(\frac{d(x_{n+1}, x_{n+2})d(x_{n+2}, x_{n+3})}{d(x_{n+1}, x_{n+2})})^s + \delta_2(d(x_{n+1}, x_{n+2}))^s]^{\frac{1}{s}} \\
&= \psi[\delta_1(d(x_{n+2}, x_{n+3}))^s + \delta_2(d(x_{n+1}, x_{n+2}))^s]^{\frac{1}{s}}. \qquad (4)
\end{aligned}$$

Suppose that $d(x_{n+2}, x_{n+3}) \geq d(x_{n+1}, x_{n+2})$, then inequality (4) becomes

$$\begin{aligned}
d(x_{n+2}, x_{n+3}) &\leq \psi[\delta_1(d(x_{n+2}, x_{n+3}))^s + \delta_2(d(x_{n+1}, x_{n+2}))^s]^{\frac{1}{s}} \\
&\leq \psi[\delta_1(d(x_{n+2}, x_{n+3}))^s + \delta_2(d(x_{n+2}, x_{n+3}))^s]^{\frac{1}{s}} \\
&= \psi[(\delta_1 + \delta_2)(d(x_{n+2}, x_{n+3}))^s]^{\frac{1}{s}} \\
&= \psi[(d(x_{n+2}, x_{n+3}))^s]^{\frac{1}{s}} \\
&= \psi(d(x_{n+2}, x_{n+3})) \\
&< d(x_{n+2}, x_{n+3}),
\end{aligned}$$

a contradiction.This implies that

$$d(x_{n+2}, x_{n+3}) < d(x_{n+1}, x_{n+2}).$$

So we have

$$\begin{aligned}
d(x_{n+2}, x_{n+3}) &\leq \psi(d(x_{n+1}, x_{n+2})) \\
&\leq \psi(\psi(d(x_n, x_{n+1}))) \\
&= \psi^2(d(x_n, x_{n+1})).
\end{aligned}$$

Continuing in this manner, we get

$$d(x_n, x_{n+1}) \leq \psi^n(d(x_0, x_1)).$$

Now, we show that the sequence (x_n) in X is a Cauchy sequence.
Let $m, n \in \aleph$ with $n \leq m$, then

$$\begin{aligned}
d(x_n, x_m) &\leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + ... + d(x_{m-1}, x_{m+1}) \\
&\leq \psi^n(d(x_0, x_1)) + \psi^{n+1}(d(x_0, x_1)) + ... + \psi^{m-1}(d(x_0, x_1)) \\
&\leq (\psi^n + \psi^{n+1} + ... + \psi^{m-1})d(x_0, x_1) \\
&\leq (\psi^n + \psi^{n+1} + ... + \psi^{m-1})d(x_0, x_1).
\end{aligned}$$

$\longrightarrow 0$ as n, m $\longrightarrow \infty$.
Hence, (x_n) in X is Cauchy sequence and since (X, d) is complete, there exists $u \in X$ such that (x_n) converges to u.

$$x_n \longrightarrow u \quad \text{as} \quad n \longrightarrow \infty.$$

So we have

$$\begin{aligned} D(x_{n+1}, Tu) &\leq d(x_{n+1}, x_n) + D(x_n, Tu) \\ &\leq d(x_{n+1}, x_n) + H(Tx_{n-1}, Tu) \\ &\leq d(x_{n+1}, x_n) + \psi(J_T^s(x_{n-1}, u)) \\ &\leq d(x_{n+1}, x_n) + \psi[\delta_1 \left(\frac{D(x_{n-1}, Tx_{n-1}) D(u, Tu)}{1 + d(x_{n-1}, u)}\right)^s + \delta_2 (d(x_{n-1}, u))^s]^{\frac{1}{s}} \\ &\leq d(x_{n+1}, x_n) + \psi[\delta_1 \left(\frac{d(x_{n-1}, x_n) D(u, Tu)}{1 + d(x_{n-1}, u)}\right)^s + \delta_2 (d(x_{n-1}, u))^s]^{\frac{1}{s}}. \end{aligned} \tag{5}$$

Taking $n \longrightarrow \infty$ in inequality (5) give $D(u, Tu) \leq 0$.
That is $D(u, Tu) = 0$. This implies that $u \in Tu$.
Case II: If $s = 0$, then by Lemma (2.6) and inequality (2) we have

$$\begin{aligned} d(x_{n+1}, x_{n+2}) &\leq H(Tx_n, Tx_{n+1}) \\ &\leq \psi(J_T^s(x_n, x_{n+1})) \\ &\leq \psi[(D(x_n, Tx_n))^{\delta_1} (D(x_{n+1}, Tx_{n+1}))^{\delta_2}] \\ &= \psi[(d(x_n, x_{n+1}))^{\delta_1} (d(x_{n+1}, x_{n+2}))^{\delta_2}]. \end{aligned} \tag{6}$$

Suppose that $d(x_{n+1}, x_{n+2}) \geq d(x_n, x_{n+1})$, then inequality (6) becomes

$$\begin{aligned} d(x_{n+1}, x_{n+2}) &\leq \psi[(d(x_n, x_{n+1}))^{\delta_1} (d(x_{n+1}, x_{n+2}))^{\delta_2}] \\ &\leq \psi[(d(x_{n+1}, x_{n+2}))^{\delta_1} (d(x_{n+1}, x_{n+2}))^{\delta_2}] \\ &= \psi[(d(x_{n+1}, x_{n+2}))^{(\delta_1 + \delta_2)}] \\ &= \psi[(d(x_{n+1}, x_{n+2}))] \\ &= \psi(d(x_{n+1}, x_{n+2})) \\ &< d(x_{n+1}, x_{n+2}), \end{aligned}$$

a contradiction. This implies that

$$d(x_{n+1}, x_{n+2}) < d(x_n, x_{n+1}).$$

So we have

$$\begin{aligned} d(x_{n+1}, x_{n+2}) &\leq \psi(d(x_n, x_{n+1})) \\ &< d(x_n, x_{n+1}). \end{aligned}$$

Again,

$$\begin{aligned} d(x_{n+2}, x_{n+3}) &\leq H(Tx_{n+1}, Tx_{n+2}) \\ &\leq \psi(J_T^s(x_{n+1}, x_{n+2})) \\ &\leq \psi[(D(x_{n+1}, Tx_{n+1}))^{\delta_1} (D(x_{n+2}, Tx_{n+2}))^{\delta_2}] \\ &= \psi[(d(x_{n+1}, x_{n+2}))^{\delta_1} (d(x_{n+2}, x_{n+3}))^{\delta_2}]. \end{aligned} \tag{7}$$

Suppose that $d(x_{n+2}, x_{n+3}) \geq d(x_{n+1}, x_{n+2})$, then inequality (7) becomes

$$\begin{aligned} d(x_{n+2}, x_{n+3}) &\leq \psi[(d(x_{n+1}, x_{n+2}))^{\delta_1}(d(x_{n+2}, x_{n+3}))^{\delta_2}] \\ &\leq \psi[(d(x_{n+2}, x_{n+3}))^{\delta_1}(d(x_{n+2}, x_{n+3}))^{\delta_2}] \\ &= \psi[(d(x_{n+2}, x_{n+3}))^{(\delta_1+\delta_2)}] \\ &= \psi(d(x_{n+2}, x_{n+3})) \\ &< d(x_{n+2}, x_{n+3}), \end{aligned}$$

a contradiction. This implies that

$$d(x_{n+2}, x_{n+3}) < d(x_{n+1}, x_{n+2}).$$

So we have

$$\begin{aligned} d(x_{n+2}, x_{n+3}) &\leq \psi(d(x_{n+1}, x_{n+2})) \\ &\leq \psi(\psi(d(x_n, x_{n+1}))) \\ &= \psi^2(d(x_n, x_{n+1})). \end{aligned}$$

Continuing in this manner, we get

$$d(x_n, x_{n+1}) \leq \psi^n(d(x_0, x_1)).$$

By using the same argument as in the case of $s > 0$, we know that the sequence (x_n) in X forms a Cauchy sequence in complete metric space. Subsequently, there exists $u \in X$ such that $u \in Tu$. □

THEOREM 2.9. *Suppose that (X, d) is a complete metric space and $T : X \longrightarrow K(X)$ a multivalued mapping if there exists $\psi \in \Psi$ such that* $\frac{1}{2}D(x, Tx) \leq d(x, y)$ *implies*

$$H(Tx, Ty) \leq [\delta_1 \left(\frac{D(x, Tx)D(y, Ty)}{1 + d(x, y)}\right)^s + \delta_2(d(x, y))^s]^{\frac{1}{s}}, \tag{8}$$

for all $x, y \in X$, where $s \geq 0$ and $\delta_i \geq 0$, $i = 1, 2, ...$, such that $\delta_1 + \delta_2 = 1$. Then, T has a fixed point in X.

Proof. Let $x_0 \in X$ be arbitrary. Then, by hypothesis $Tx_0 \in K(X)$. Choose $x_1 \in Tx_0$, then for this $x_1 \in X$, Tx_1 is a nonempty compact subset of X. Hence, we can find $x_2 \in Tx_1$ such that $d(x_1, x_2) = d(x_1, Tx_1) = D(Tx_0, Tx_1)$. Continuing in this manner, we construct a sequence (x_n) in X such that

$$x_{n+1} \in Tx_n \quad \text{and} \quad d(x_n, x_{n+1}) = D(x_n, Tx_n), \text{ for all n} \geq 0.$$

Now,

$$\frac{1}{2}D(x_n, Tx_n) = \frac{1}{2}d(x_n, x_{n+1}) \leq d(x_n, x_{n+1})$$

$$\Rightarrow \quad H(Tx_n, Tx_{n+1}) \leq \psi(J_T^s(x_n, x_{n+1})) \tag{9}$$

We shall prove the claim only as $s > 0$ then by Lemma (2.6) and inequality (9) we have

$$\begin{aligned} d(x_{n+1}, x_{n+2}) &\leq H(Tx_n, Tx_{n+1}) \\ &\leq \psi(J_T^s(x_n, x_{n+1})) \\ &\leq \psi[\delta_1(\frac{D(x_n, Tx_n)D(x_{n+1}, Tx_{n+1})}{1 + d(x_n, x_{n+1})})^s + \delta_2(d(x_n, x_{n+1}))^s]^{\frac{1}{s}} \\ &\leq \psi[\delta_1(\frac{d(x_n, x_{n+1})d(x_{n+1}, x_{n+2})}{1 + d(x_n, x_{n+1})})^s + \delta_2(d(x_n, x_{n+1}))^s]^{\frac{1}{s}} \\ &\leq \psi[\delta_1(\frac{d(x_n, x_{n+1})d(x_{n+1}, x_{n+2})}{d(x_n, x_{n+1})})^s + \delta_2(d(x_n, x_{n+1}))^s]^{\frac{1}{s}} \\ &= \psi[\delta_1(d(x_{n+1}, x_{n+2}))^s + \delta_2(d(x_n, x_{n+1}))^s]^{\frac{1}{s}}. \end{aligned} \tag{10}$$

Suppose that $d(x_{n+1}, x_{n+2}) \geq d(x_n, x_{n+1})$, then inequality (10) becomes

$$\begin{aligned} d(x_{n+1}, x_{n+2}) &\leq \psi[\delta_1(d(x_{n+1}, x_{n+2}))^s + \delta_2(d(x_n, x_{n+1}))^s]^{\frac{1}{s}} \\ &\leq \psi[\delta_1(d(x_{n+1}, x_{n+2}))^s + \delta_2(d(x_{n+1}, x_{n+2}))^s]^{\frac{1}{s}} \\ &= \psi[(\delta_1 + \delta_2)(d(x_{n+1}, x_{n+2}))^s]^{\frac{1}{s}} \\ &= \psi[(d(x_{n+1}, x_{n+2}))^s]^{\frac{1}{s}} \\ &= \psi(d(x_{n+1}, x_{n+2})) \\ &< d(x_{n+1}, x_{n+2}), \end{aligned}$$

a contradiction. This implies that

$$d(x_{n+1}, x_{n+2}) < d(x_n, x_{n+1}).$$

So we have

$$\begin{aligned} d(x_{n+1}, x_{n+2}) &\leq \psi(d(x_n, x_{n+1})) \\ &< d(x_n, x_{n+1}). \end{aligned}$$

Again,

$$\begin{aligned} d(x_{n+2}, x_{n+3}) &\leq H(Tx_{n+1}, Tx_{n+2}) \\ &\leq \psi(J_T^s(x_{n+1}, x_{n+2})) \\ &\leq \psi[\delta_1(\frac{D(x_{n+1}, Tx_{n+1})D(x_{n+2}, Tx_{n+2})}{1 + d(x_{n+1}, x_{n+2})})^s + \delta_2(d(x_{n+1}, x_{n+2}))^s]^{\frac{1}{s}} \\ &\leq \psi[\delta_1(\frac{d(x_{n+1}, x_{n+2})d(x_{n+2}, x_{n+3})}{1 + d(x_{n+1}, x_{n+2})})^s + \delta_2(d(x_{n+1}, x_{n+2}))^s]^{\frac{1}{s}} \\ &\leq \psi[\delta_1(\frac{d(x_{n+1}, x_{n+2})d(x_{n+2}, x_{n+3})}{d(x_{n+1}, x_{n+2})})^s + \delta_2(d(x_{n+1}, x_{n+2}))^s]^{\frac{1}{s}} \\ &= \psi[\delta_1(d(x_{n+2}, x_{n+3}))^s + \delta_2(d(x_{n+1}, x_{n+2}))^s]^{\frac{1}{s}}. \end{aligned} \tag{11}$$

Suppose that $d(x_{n+2}, x_{n+3}) \geq d(x_{n+1}, x_{n+2})$, then inequality (11) becomes

$$\begin{aligned} d(x_{n+2}, x_{n+3}) &\leq \psi[\delta_1(d(x_{n+2}, x_{n+3}))^s + \delta_2(d(x_{n+1}, x_{n+2}))^s]^{\frac{1}{s}} \\ &\leq \psi[\delta_1(d(x_{n+2}, x_{n+3}))^s + \delta_2(d(x_{n+2}, x_{n+3}))^s]^{\frac{1}{s}} \\ &= \psi[(\delta_1 + \delta_2)(d(x_{n+2}, x_{n+3}))^s]^{\frac{1}{s}} \\ &= \psi[(d(x_{n+2}, x_{n+3}))^s]^{\frac{1}{s}} \\ &= \psi(d(x_{n+2}, x_{n+3})) \\ &< d(x_{n+2}, x_{n+3}), \end{aligned}$$

a contradiction.This implies that

$$d(x_{n+2}, x_{n+3}) < d(x_{n+1}, x_{n+2}).$$

So we have

$$\begin{aligned} d(x_{n+2}, x_{n+3}) &\leq \psi(d(x_{n+1}, x_{n+2})) \\ &\leq \psi(\psi(d(x_n, x_{n+1}))) \\ &= \psi^2(d(x_n, x_{n+1})). \end{aligned}$$

Continuing in this manner, we get

$$d(x_n, x_{n+1}) \leq \psi^n(d(x_0, x_1)).$$

Now, we show that the sequence (x_n) in X is a Cauchy sequence.
Let $m, n \in \mathbb{N}$ with $n \leq m$, then

$$\begin{aligned} d(x_n, x_m) &\leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + ... + d(x_{m-1}, x_{m+1}) \\ &\leq \psi^n(d(x_0, x_1)) + \psi^{n+1}(d(x_0, x_1)) + ... + \psi^{m-1}(d(x_0, x_1)) \\ &\leq (\psi^n + \psi^{n+1} + ... + \psi^{m-1})d(x_0, x_1) \\ &\leq (\psi^n + \psi^{n+1} + ... + \psi^{m-1})d(x_0, x_1). \end{aligned}$$

$\longrightarrow 0$ as n, m $\longrightarrow \infty$.
Hence, (x_n) in X is Cauchy sequence and since (X, d) is complete, there exists $u \in X$ such that (x_n) converges to u.

$$x_n \longrightarrow u \quad \text{as} \quad n \longrightarrow \infty.$$

So we have

$$\begin{aligned} D(x_{n+1}, Tu) &\leq d(x_{n+1}, x_n) + D(x_n, Tu) \\ &\leq d(x_{n+1}, x_n) + H(Tx_{n-1}, Tu) \\ &\leq d(x_{n+1}, x_n) + \psi(J_T^s(x_{n-1}, u)) \\ &\leq d(x_{n+1}, x_n) + \psi[\delta_1 \left(\frac{D(x_{n-1}, Tx_{n-1})D(u, Tu)}{1 + d(x_{n-1}, u)}\right)^s + \delta_2(d(x_{n-1}, u))^s]^{\frac{1}{s}} \\ &\leq d(x_{n+1}, x_n) + \psi[\delta_1 \left(\frac{d(x_{n-1}, x_n)D(u, Tu)}{1 + d(x_{n-1}, u)}\right)^s + \delta_2(d(x_{n-1}, u))^s]^{\frac{1}{s}}. \qquad (12) \end{aligned}$$

Taking $n \longrightarrow \infty$ in inequality (12) give $D(u, Tu) \leq 0$.
That is $D(u, Tu) = 0$. This implies that $u \in Tu$.

□

In what follows, a comparative example is constructed to support the hypotheses of Theorem 2.8.

EXAMPLE 2.10. Let $X = \{(0,2),(3,4),(5,5)\}$ and $d : X \times X \longrightarrow \mathbb{R}_+$ be given by

$$d((x_1,x_2),(y_1,y_2)) = \max\{|x_1 - y_1|, |x_2 - y_2|\}.$$

Clearly, (X,d) is a complete metric space. Defined a multivalued mapping $T : X \longrightarrow K(X)$ as follows:

$$Tx = \begin{cases} \{(0,2),(3,4)\} & \text{if } x = (0,2), \\ \{(5,5)\} & \text{if } x \neq (0,2). \end{cases}$$

Also, defined $\vartheta(t) = \frac{3t}{4}$. Now, we examine two cases:
Case I: for $x = (0,2)$, we have

$$\begin{aligned} \frac{1}{2}D((0,2),T(0,2)) &= \frac{1}{2}\inf\{d((0,2),y) : y \in T(0,2)\} \\ &= \frac{1}{2}d((0,2),(0,2),(3,4)) \\ &= \frac{1}{2}d((0,2),(3,4)) \\ &= \frac{3}{2}. \end{aligned}$$

Case II: for $x = (3,4)$, we have

$$\begin{aligned} \frac{1}{2}D((3,4),T(3,4)) &= \frac{1}{2}\inf\{d((3,4),y) : y \in T(3,4)\} \\ &= \frac{1}{2}d((3,4),(5,5)) \\ &= 2. \end{aligned}$$

Case III: for $x = (5,5)$, we have

$$\begin{aligned} \frac{1}{2}D((5,5),T(5,5)) &= \frac{1}{2}\inf\{d((5,5),y) : y \in T(5,5)\} \\ &= \frac{1}{2}d((5,5),(5,5)) \\ &= 0. \end{aligned}$$

Now, for $x \in X$ with $D(x, Tx) > 0$, that is, $x \in \{(0,2),(3,4)\}$, we have
$H(T(0,2), T(3,4)) = H((0,2),(3,4)) = 2$,

$$\begin{aligned}
\psi[J_T^s((0,2),(3,4))] &= \psi[\delta_1(\frac{D((0,2),T(0,2))D((3,4),T(3,4))}{1+d((0,2),(3,4))})^s + \delta_2(d((0,2),(3,4)))^s]^{\frac{1}{s}}] \\
&= \psi[\delta_1(\frac{d((0,2),(0,2),(3,4))d((3,4),(5,5))}{1+d((0,2),(3,4))})^s + \delta_2(d((0,2),(3,4)))^s]^{\frac{1}{s}}] \\
&= \psi[\delta_1(\frac{d((0,2),(3,4))d((3,4),(5,5))}{1+d((0,2),(3,4))})^s + \delta_2(d((0,2),(3,4)))^s]^{\frac{1}{s}}] \\
&\leq \psi[\delta_1(\frac{d((0,2),(3,4))d((3,4),(5,5))}{d((0,2),(3,4))})^s + \delta_2(d((0,2),(3,4)))^s]^{\frac{1}{s}}] \\
&= \psi[\delta_1(d((3,4),(5,5)))^s + \delta_2(d((0,2),(3,4)))^s]^{\frac{1}{s}}] \\
&\leq \psi[\delta_1(d((0,2),(3,4)))^s + \delta_2(d((0,2),(3,4)))^s]^{\frac{1}{s}}] \\
&= \psi[(\delta_1+\delta_2)(d((0,2),(3,4)))^s]^{\frac{1}{s}}] \\
&= \psi[(d((0,2),(3,4)))^s]^{\frac{1}{s}}] \\
&= \psi[d((0,2),(3,4))] \\
&= \psi[3] = 2.25.
\end{aligned}$$

Now,
$H(T(0,2), T(3,4)) = 2 \leq 2.25 = \psi[J_T^s((0,2),(3,4))]$.
Thus, for all $x, y \in X$ with $x \neq y$, $\frac{1}{2}D(x, Tx) > 0$ and $\frac{1}{2}D(y, Ty) > 0$
imply $H(Tx, Ty) \leq \psi(J_T^s(x,y))$, where

$$\psi(J_T^s(x,y)) = \begin{cases} [\delta_1\left(\frac{D(x,Tx)D(y,Ty)}{1+d(x,y)}\right)^s + \delta_2(d(x,y))^s]^{\frac{1}{s}} & \text{if } s > 0, x, y \in X, x \neq y, \\ \\ (D(x,Tx))^{\delta_1}(D(y,Ty))^{\delta_2} & \text{if } s = 0, x, y \in X \setminus F_T(X). \end{cases}$$

It follows that all the hypotheses of Theorem 2.8 are satisfied. We see that T has a fixed point.

Moreover, a nondiscrete example is constructed to support the hypotheses of Theorem 2.8.

Example 2.11. Let $X = [0, \infty)$ and $d(x, y) = |x - y|$ for all $x, y \in X$. Then, (X, d) is a complete metric space. Consider a multivalued mapping $T : X \longrightarrow K(X)$ defined as foloows;

$$Tx = \begin{cases} [0,1], & \text{if } x \in \{0,1,2\} \\ \\ \{\frac{x}{3}\} & \textit{Otherwise.} \end{cases}$$

Define the mapping $\psi : [0,1] \longrightarrow [0,\infty)$ as $\psi(t) = \frac{t}{2}$ for all $t \in [0,1]$. Clearly, $\psi \in \Psi$. We now verify that under the above constructions, the inequality in (1) is satisfied. In (1), take $s = 1$, $\delta_1 = 0$ and $\delta_2 = 1$. Then, note that for all $x, y \in \{0,1,2\}$, there is nothing to show. So, we examine the following two cases:
Case I: For $x, y \in X \setminus \{0,1,2\}$, if $x = y$, the computation is again direct.

Case II: For $x \neq y$, without loss of generality, let $x < y$. Then,

$$\begin{aligned} H(Tx,Ty) &= H(\left\{\frac{x}{3}\right\},\left\{\frac{y}{3}\right\}) \\ &= \frac{1}{3}|x-y| \\ &= \frac{1}{3}d(x,y) \\ &\leq \psi(J_T^s(x,y)). \end{aligned}$$

Hence, all the conditions of Theorem 2.8 are satisfied. We see therefore that $F_T = \{0,1\}$.

In what follows, a few consequences of the main result obtained herein are pointed out.

Corollary 2.12. *Suppose that (X,d) is a complete metric space and $T : X \longrightarrow K(X)$ a multivalued mapping. If there exists $\psi \in \Psi$ such that $\frac{1}{2}D(x,Tx) \leq d(x,y)$ implies*

$$H(Tx,Ty) \leq (D(x,Tx))^{\delta_1}(D(y,Ty))^{\delta_2}, \tag{13}$$

for all $x,y \in X$, where $\delta_i \geq 0$, $i = 1,2,...$, such that $\delta_1 + \delta_2 = 1$. Then, T has a fixed point in X.

Proof. Setting $s = 0$ in Theorem 2.8, the result follows. □

Corollary 2.13. *[21, Theorem 8] Let (X,d) be a complete metric space and $E : X \longrightarrow X$ be a Jaggi-Suzuki-type hybrid contraction. Assume also that E is w-orbital admissible mapping and $w(u_0, Eu_0) \geq 1$, for some $u_0 \in X$. Then, E has a fixed point in X.*

Proof. Consider a multivalued mapping $T : X \longrightarrow K(X)$ defined by $Tx = \{Ex\}$, for all $x \in X$, where $E : X \longrightarrow X$ is a single-valued mapping. We see that

$$d(Ex,Ey) \leq \psi(J_E^s(x,y)) = \begin{cases} [\delta_1\left(\frac{d(x,Ex)d(y,Ey)}{d(x,y)}\right)^s + \delta_2(d(x,y))^s]^{\frac{1}{s}} & \text{if } s > 0, x,y \in X, x \neq y, \\ (d(x,Ex))^{\delta_1}(d(y,Ey))^{\delta_2} & \text{if } s = 0, x,y \in X \setminus F_E(X), \end{cases}$$

where $F_E(X) = \{x \in X : x = Ex\}$. It follows that the assumption of Theorem 2.8 coincides with that of Corollary 2.13. Hence, there exists $u \in X$ such that $u \in Tu = \{Eu\}$; that is $u = Eu$. □

Corollary 2.14. *[21, Corollary 11] Let (X,d) be a complete metric space and $E : X \longrightarrow X$ be a continuous mapping. If there exists $\varphi \in \Psi$ such that $\frac{1}{2}d(x,Ex) \leq d(x,y)$ implies*

$$d(Ex,Ey) \leq J_E^s(x,y), \tag{14}$$

for all each $x,y \in X$, where $s \geq 0$ and $\delta_i \geq 0$, $i = 1,2,...$, such that $\delta_1 + \delta_2 = 1$ and $\alpha \in (0,1)$

$$J_E^s(x,y = \begin{cases} [\delta_1\left(\frac{d(x,Ex)d(y,Ey)}{1+d(x,y)}\right)^s + \delta_2(d(x,y))^s]^{\frac{1}{s}} & \text{if } s > 0, x,y \in X, x \neq y, \\ (d(x,Ex))^{\alpha}(d(y,Ey))^{1-\alpha} & \text{if } s = 0, x,y \in X \setminus F_E(X). \end{cases}$$

Then, E has a fixed point in X.

Corollary 2.15. *[21, Corollary 12] Let (X,d) be a complete metric space and $E : X \longrightarrow X$ be a continuous mapping. If there exists $\lambda \in (0,1)$ such that, $\frac{1}{2}d(x,Ex) \leq d(x,y)$ implies*

$$d(Ex,Ey) \leq J^s_E(x,y), \tag{15}$$

for all each $x,y \in X$, where $s \geq 0$ and $\delta_i \geq 0$, $i = 1,2,...$, such that $\delta_1 + \delta_2 = 1$ and $\alpha \in (0,1)$

$$J^s_E(x,y) = \begin{cases} [\delta_1 \left(\frac{d(x,Ex)d(y,Ey)}{1+d(x,y)}\right)^s + \delta_2(d(x,y))^s]^{\frac{1}{s}} & \text{if } s > 0, x,y \in X, x \neq y, \\ (d(x,Ex))^\alpha (d(y,Ey))^{1-\alpha} & \text{if } s = 0, x,y \in X \setminus F_E(X). \end{cases}$$

Then, E has a fixed point in X.

3. Conclusion

In this paper, a new concept of multivalued contraction was defined from a combination of Jaggi-type hybrid contraction and Suzuki-type hybrid contraction in the framework of a complete metric space. Conditions for the existence of fixed points for such contractions equipped with some suitable hypotheses were established.

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