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INTEGRAL MEAN ESTIMATES FOR SOME OPERATOR PRESERVING INEQUALITIES

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Abstract. In this paper, some integral mean estimates for the polar derivative of a polynomial with complex coefficients are proved. We will see that these type of estimates are new in this direction and discuss their importance with respect to existing results comparatively. In addition, the obtained results provide valuable insights into the behavior of integrals involving operator preserving inequalities.

1. Statements of preliminary results

Let $\mathcal{P}$ be the class of complex polynomials $P(z) = \sum\limits_{v=0}^{n} a_v z^v$ of degree n and $P'(z)$ be the derivative of $P(z)$. For $P \in \mathcal{P}$, we define

$$\|P\|_0 = \exp\left(\frac{1}{2\pi}\int_0^{2\pi} \log|P(e^{i\theta})|d\theta\right),$$

$$\|P\|_p = \left(\frac{1}{2\pi}\int_0^{2\pi} |P(e^{i\theta})|^p d\theta\right)^{\frac{1}{p}} \quad \text{for } 0 < p < \infty$$

and

$$\|P\|_\infty = \max_{|z|=1} |P(z)|.$$

Notice that $\|P\|_0 = \lim_{p\to 0^+} \|P\|_p$ and $\|P\|_\infty = \lim_{p\to\infty} \|P\|_p$. For the sake of simplicity, we denote $\|P\|_\infty$ simply by $\|P\|$. We begin with the Turán's classical inequality [11], which asserts that

$$\|P'\| \geq \frac{n}{2}\|P\| \tag{1.1}$$

holds for all polynomials $P \in \mathcal{P}$ having all zeros in $|z| \leq 1$. This result is best possible and the extremal polynomial for (1.1) is $P(z) = z^n + 1$.

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An exceptional inequality for polynomials of a complex variable which consists (1.1) as one of the special cases is due to Govil [6], and which states that the inequality

$$\|P'\| \geq \frac{n}{1+k^n}\|P\| \tag{1.2}$$

holds for all polynomials $P \in \mathcal{P}$ having all zeros in $|z| \leq k$, where $k \geq 1$. It is easy to see that (1.2) becomes equality when $P(z) = z^n + k^n$. Inequality (1.2) is one of the most known polynomial inequality in the theory of polynomial inequalities and will be beneficial as well for our results. Again, to improve the bound in (1.2), Govil [5] under the same hypothesis as of (1.2) showed that the following inequality holds

$$\|P'\| \geq \frac{n}{1+k^n}\{\|P\| + \min_{|z|=k}|P(z)|\}. \tag{1.3}$$

Polar Derivative: Let $P \in \mathcal{P}$, and α be any complex number, then

$$\begin{aligned} D_\alpha P(z) &= -\left[\frac{P(z)}{(z-\alpha)^n}\right]'(z-\alpha)^{n+1} \\ &= nP(z) + (\alpha - z)P'(z), \end{aligned}$$

is called the *polar derivative* of $P(z)$. Note that $D_\alpha P(z)$ is a polynomial of degree at most $n-1$ and it generalizes the concept of "ordinary derivative" which is evident and convincing from the fact that

$$\lim_{\alpha\to\infty}\frac{D_\alpha P(z)}{\alpha} = P'(z) \tag{1.4}$$

uniformly with respect to z for $|z| \leq R$, $R > 0$.

In the polar derivative contexture, all the above inequalities have been widely investigated, even there are variety of inequalities related to the polar derivative which include all the above mentioned inequalities as special cases (see [7], [9], [12], [8]). The main aim of this paper is to focus on the following result, which states that the inequality

$$\max_{|z|=1}|D_\alpha P(z)| \geq \frac{n(|\alpha|-k)}{1+k^n}\|P\| \tag{1.5}$$

holds for all polynomials $P \in \mathcal{P}$ which have all its zeros in $|z| \leq k$, $k \geq 1$ and for every $\alpha \in \mathbb{C}$ with $|\alpha| \geq k$. This result is ascribed to Aziz and Rather [3].

Look at the following result concerning integral norm estimates due to Aziz [1] of which the inequality (1.2) is a special case and it states that the inequality

$$n\left\{\int_0^{2\pi}|P(e^{i\theta})|^p d\theta\right\}^{\frac{1}{p}} \leq \left\{\int_0^{2\pi}|1+k^n e^{i\theta}|^p d\theta\right\}^{\frac{1}{p}} \max_{|z|=1}|P'(z)| \tag{1.6}$$

holds for all polynomials $P \in \mathcal{P}$ which have all its zeros in $|z| \leq k$, $k \geq 1$ and for each $p \geq 1$. The result is best possible and equality in (1.6) holds for the polynomial $P(z) = \alpha z^n + \beta k^n$, where $|\alpha| = |\beta|$. The problem of estimating the integral norm of inequalities like (1.5) and associated results is still open and in this paper we make an endeavor to solve some inequalities of the same type.

2. Main results

We first prove the following result concerning integral norm to inequality (1.5) in a slightly different style and we will see that both the inequalities have equivalent consequences.

Theorem 2.1. *If $P \in \mathcal{P}$ and $P(z)$ has all its zeros in $|z| \le k$, $k \ge 1$, then for each $r \ge 1$ and for every $\alpha \in \mathbb{C}$ with $|\alpha| \ge k^n$*

$$n(|\alpha| - k^n)\left\{\int_0^{2\pi} |P(e^{i\theta})|^r d\theta\right\}^{\frac{1}{r}} \le \left\{\int_0^{2\pi} |1 + k^n e^{i\theta}|^r d\theta\right\}^{\frac{1}{r}} \max_{|z|=1} |D_\alpha P(z)|. \tag{2.1}$$

Remark 2.1. Dividing both sides of inequality (2.1) by $|\alpha|$ and letting $|\alpha| \to \infty$, and taking (1.4) into consideration, we obtain inequality (1.6) as a special case.

Remark 2.2. Letting $r \to \infty$ in (2.1), we obtain the following inequality under the same hypothesis

$$\max_{|z|=1} |D_\alpha P(z)| \ge \frac{n(|\alpha| - k^n)}{1 + k^n}\|P\|, \text{ for } |\alpha| \ge k^n. \tag{2.2}$$

It seems that inequalities (2.2) and (1.5) are same but they are equivalent up to the restrictions on α. As mentioned above, on dividing both sides of inequalities (2.2) and (1.5) by $|\alpha|$ and letting $|\alpha| \to \infty$, and taking (1.4) into consideration, they yield the same inequality (1.2) as a special case.

Theorem 2.2. *If $P \in \mathcal{P}$ and $P(z)$ has all its zeros in $|z| \le k$, $k \ge 1$, then for each $r \ge 1$, $p > 1$, $q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$ and for every $\alpha \in \mathbb{C}$ with $|\alpha| \ge k^n$*

(2.3)
$$n(|\alpha| - k^n)\left\{\int_0^{2\pi} |P(e^{i\theta})|^r d\theta\right\}^{\frac{1}{r}} \le C_n \left\{\int_0^{2\pi} |1 + e^{i\theta}|^{pr} d\theta\right\}^{\frac{1}{pr}} \left\{\int_0^{2\pi} |D_\alpha P(e^{i\theta})|^{qr} d\theta\right\}^{\frac{1}{qr}},$$

where $C_n = \dfrac{\left\{\int_0^{2\pi} |1+k^n e^{i\theta}|^r d\theta\right\}^{\frac{1}{r}}}{\left\{\int_0^{2\pi} |1+e^{i\theta}|^r d\theta\right\}^{\frac{1}{r}}}$.

Remark 2.3. The limit $q \to \infty$ implies $p \to 1$ in Theorem 2.2. Therefore, inequality (2.3) reduces to inequality (2.1). On dividing both sides of inequalities (2.3) by $|\alpha|$ and letting $|\alpha| \to \infty$, and taking (1.4) into consideration, it yields the following inequality due to Aziz and Ahemad [2] as a special case.

$$n\left\{\int_0^{2\pi} |P(e^{i\theta})|^r d\theta\right\}^{\frac{1}{r}} \le C_n \left\{\int_0^{2\pi} |1 + e^{i\theta}|^{pr} d\theta\right\}^{\frac{1}{pr}} \left\{\int_0^{2\pi} |P'(e^{i\theta})|^{qr} d\theta\right\}^{\frac{1}{qr}}. \tag{2.4}$$

THEOREM 2.3. *Let $P \in \mathcal{P}$ and $P(z)$ have all its zeros in $|z| \leq k$, $k \geq 1$, then for each $r \geq 1$, $p > 1$, $q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, $|\lambda| < 1$ and for every real or complex number α with $|\alpha| \geq 2k^n + 1$*

$$\begin{aligned} n\,(|\alpha| - k^n) \left\{ \int_0^{2\pi} |P(e^{i\theta}) + \lambda m|^r d\theta \right\}^{\frac{1}{r}} &\leq C_n \left\{ \int_0^{2\pi} |1 + e^{i\theta}|^{pr} d\theta \right\}^{\frac{1}{pr}} \\ &\quad \times \left\{ \int_0^{2\pi} |D_\alpha P(e^{i\theta}) + m\lambda n|^{qr} d\theta \right\}^{\frac{1}{qr}}, \end{aligned} \tag{2.5}$$

where $C_n = \dfrac{\left\{ \int_0^{2\pi} |1 + k^n e^{i\theta}|^r d\theta \right\}^{\frac{1}{r}}}{\left\{ \int_0^{2\pi} |1 + e^{i\theta}|^r d\theta \right\}^{\frac{1}{r}}}$ *and* $m = \min\limits_{|z|=k} |P(z)|$.

REMARK 2.4. If $q \to \infty$, then $p \to 1$, and so the inequality (2.5) reduces to the following inequality

$$n\,(|\alpha| - k^n) \left\{ \int_0^{2\pi} |P(e^{i\theta}) + \lambda m|^r d\theta \right\}^{\frac{1}{r}} \leq \left\{ \int_0^{2\pi} |1 + k^n e^{i\theta}|^r d\theta \right\}^{\frac{1}{r}} |D_\alpha P(z) + m\lambda n|, \tag{2.6}$$

where $m = \min\limits_{|z|=k} |P(z)|$. If $r \to \infty$ and choosing an argument of λ suitably in (2.6), we get the following result.

COROLLARY 2.4. *Let $P \in \mathcal{P}$ and $P(z)$ have all its zeros in $|z| \leq k$, $k \geq 1$, then for every real or complex number α with $|\alpha| \geq 2k^n + 1$*

$$\max_{|z|=1} |D_\alpha P(z)| \geq \frac{n}{1 + k^n} \left[(|\alpha| - k^n)\|P\| + \{|\alpha| - (2k^n + 1)\} \min_{|z|=k} |P(z)| \right]. \tag{2.7}$$

On dividing both sides of inequality (2.6) by $|\alpha|$ and letting $|\alpha| \to \infty$, and taking (1.4) into consideration, we obtain the following result.

COROLLARY 2.5. *Let $P \in \mathcal{P}$ and $P(z)$ have all its zeros in $|z| \leq k$, $k \geq 1$, then for each $r > 0$ and $|\lambda| < 1$*

$$n \left\{ \int_0^{2\pi} |P(e^{i\theta}) + \lambda m|^r d\theta \right\}^{\frac{1}{r}} \leq \left\{ \int_0^{2\pi} |1 + k^n e^{i\theta}|^r d\theta \right\}^{\frac{1}{r}} \max_{|z|=1} |P'(z)|, \tag{2.8}$$

where $m = \min\limits_{|z|=k} |P(z)|$.

REMARK 2.5. For $k = 1$, Corollary 2.5 reduces to result proved by Aziz [1, Theorem 3].

3. Auxiliary results

Lemma 3.1. *[6] If $P \in \mathcal{P}$ and $P(z)$ has all its zeros in $|z| \leq k$, $k \geq 1$, then*

$$\max_{|z|=1} |Q'(z)| \leq k^n \max_{|z|=1} |P'(z)|, \tag{3.1}$$

where $Q(z) = z^n \overline{P(\frac{1}{\bar{z}})}$.

Lemma 3.2. *[4] If $P \in \mathcal{P}$ and $P(z)$ does not vanish in $|z| < 1$, then for every $R \geq 1$ and $r \geq 1$*

$$\left\{ \int_0^{2\pi} |P(Re^{i\theta})|^r d\theta \right\} \leq B_r \left\{ \int_0^{2\pi} |P(e^{i\theta})|^r d\theta \right\}. \tag{3.2}$$

where

$$B_r = \frac{\left\{ \int_0^{2\pi} |1 + R^n e^{i\theta}|^r d\theta \right\}}{\left\{ \int_0^{2\pi} |1 + e^{i\theta}|^r d\theta \right\}}.$$

4. Proofs of the main results

Proof of Theorem 2.1. Let $Q(z) = z^n \overline{P(\frac{1}{\bar{z}})}$, then it can be easily verified for $|z| = 1$ that

$$|Q'(z)| = |nP(z) - zP'(z)|.$$

Now for any real or complex number α with $|\alpha| > k^n$, the polar derivative of $P(z)$ with respect to α is

$$D_\alpha P(z) = nP(z) + (\alpha - z)P'(z).$$

This implies by using Lemma 3.1 for $|z| = 1$

$$\begin{aligned} |D_\alpha P(z)| &\geq |\alpha||P'(z)| - |nP(z) - zP'(z)| \\ &\geq |\alpha||P'(z)| - |Q'(z)| \\ &\geq |\alpha||P'(z)| - k^n|P'(z)|. \end{aligned}$$

This gives

$$|D_\alpha P(z)| \geq (|\alpha| - k^n)|P'(z)|.$$

Consequently for $|z| = 1$, we have

$$\max_{|z|=1} |D_\alpha P(z)| \geq (|\alpha| - k^n) \max_{|z|=1} |P'(z)|. \tag{4.1}$$

From inequality (1.6), we have for $r \geq 1$

$$\max_{|z|=1} |P'(z)| \geq n \frac{\left\{ \int_0^{2\pi} |P(e^{i\theta})|^r d\theta \right\}^{\frac{1}{r}}}{\left\{ \int_0^{2\pi} |1 + k^n e^{i\theta}|^r d\theta \right\}^{\frac{1}{r}}}.$$

Using this in (4.1), we get for each $\theta,\ 0 \le \theta < 2\pi$

$$\max_{|z|=1} |D_\alpha P(z)| \ge n(|\alpha| - k^n) \frac{\left\{ \int\limits_0^{2\pi} |P(e^{i\theta})|^r d\theta \right\}^{\frac{1}{r}}}{\left\{ \int\limits_0^{2\pi} |1 + k^n e^{i\theta}|^r d\theta \right\}^{\frac{1}{r}}}.$$

Hence

$$n(|\alpha| - k^n) \left\{ \int\limits_0^{2\pi} |P(e^{i\theta})|^r d\theta \right\}^{\frac{1}{r}} \le \left\{ \int\limits_0^{2\pi} |1 + k^n e^{i\theta}|^r d\theta \right\}^{\frac{1}{r}} \max_{|z|=1} |D_\alpha P(z)|.$$

This completes the proof of Theorem 2.1.

Proof of Theorem 2.2. Assume that all the zeros of polynomial $P(z)$ lie in $|z| \le k,\ k \ge 1$, it follows that all the zeros of polynomial $S(z) = P(kz)$ lie in $|z| \le 1$. Hence the polynomial $T(z) = z^n \overline{S(\frac{1}{\bar{z}})}$ has all its zeros in $|z| \ge 1$. If $\{z_v : v = 1, 2, 3, ..., n\}$ is the set of zeros of $T(z)$, then it is clear that $|z_v| \ge 1,\ v = 1, 2, .., n$ and we have by Fundamental theorem of algebra

$$\frac{zT'(z)}{T(z)} = \sum_{v=1}^{n} \frac{z}{z - z_v}.$$

Therefore for the points $e^{i\theta},\ 0 \le \theta < 2\pi$, for which $T(e^{i\theta}) \ne 0$, we get

$$\begin{aligned} \Re\left(\frac{e^{i\theta}T'(e^{i\theta})}{T(e^{i\theta})}\right) &= \sum_{v=1}^{n} \Re\left(\frac{e^{i\theta}}{e^{i\theta} - z_v}\right) \\ &\le \sum_{v=1}^{n} \frac{1}{2} \\ &= \frac{n}{2}, \end{aligned}$$

which implies

$$\left|\frac{e^{i\theta}T'(e^{i\theta})}{nT(e^{i\theta})}\right| \le \left|1 - \frac{e^{i\theta}T'(e^{i\theta})}{nT(e^{i\theta})}\right|.$$

Equivalently

$$|T'(e^{i\theta})| \le |nT(e^{i\theta}) - e^{i\theta}T'(e^{i\theta})|,$$

which is clearly true for the points $e^{i\theta},\ 0 \le \theta < 2\pi$, for which $T(e^{i\theta}) = 0$. Hence it follows for $|z| = 1$ that

$$|T'(z)| \le |nT(z) - zT'(z)|. \tag{4.2}$$

Since $T(z) = z^n \overline{S(\frac{1}{\bar{z}})}$, the derivative of $T(z)$ is

$$T'(z) = nz^{n-1}\ \overline{S(1/\bar{z})} - z^{n-2}\ \overline{S'(1/\bar{z})}.$$

Since all the zeros of $S(z)$ lie in $|z| \le 1$, by Gauss-Lucas theorem all the zeros of $S'(z)$ also lie in $|z| \le 1$ and hence the polynomial

$$z^{n-1}\ \overline{S'(1/\bar{z})} = nT(z) - zT'(z)$$

has all its zeros in $|z| \geq 1$. Therefore it follows that the function

$$W(z) = \frac{zT'(z)}{nT(z) - zT'(z)}$$

is analytic for $|z| \leq 1$ with $|W(z)| \leq 1$ for $|z| \leq 1$ and $W(0) = 0$. Thus the function $1 + W(z)$ is subordinate to the function $1 + z$ for $|z| \leq 1$. Hence by subordination property for analytic functions, we have for each $r \geq 1$

$$\int_0^{2\pi} |1 + W(e^{i\theta})|^r d\theta \leq \int_0^{2\pi} |1 + e^{i\theta}|^r d\theta. \tag{4.3}$$

Now

$$1 + W(z) = \frac{nT(z)}{nT(z) - zT'(z)}$$

and for $|z| = 1$

$$|S'(z)| = |z^{n-1}\ \overline{S'(1/\overline{z})}| = |nT(z) - zT'(z)|.$$

Consequently for $|z| = 1$

$$\begin{aligned} n|T(z)| &= |1 + W(z)|\ |nT(z) - zT'(z)| \\ &= |1 + W(z)|\ |S'(z)|. \end{aligned} \tag{4.4}$$

Now from (4.3) and (4.4), we obtain for each $r \geq 1$

$$\begin{aligned} n^r \int_0^{2\pi} |T(e^{i\theta})|^r d\theta &\leq \int_0^{2\pi} |1 + e^{i\theta}|^r\ |S'(e^{i\theta})|^r d\theta \\ &\leq k^r \int_0^{2\pi} |1 + e^{i\theta}|^r\ |P'(ke^{i\theta})|^r d\theta. \end{aligned} \tag{4.5}$$

Since $T(z)$ does not vanish in $|z| < 1$, applying Lemma 3.2 with $R = k \geq 1$ to the polynomial $T(z)$, we get for each $r \geq 1$

$$\int_0^{2\pi} |T(ke^{i\theta})|^r d\theta \leq (C_n)^r \int_0^{2\pi} |T(e^{i\theta})|^r d\theta, \tag{4.6}$$

where $C_n = \dfrac{\left\{\int_0^{2\pi} |1+k^n e^{i\theta}|^r d\theta\right\}^{\frac{1}{r}}}{\left\{\int_0^{2\pi} |1+e^{i\theta}|^r d\theta\right\}^{\frac{1}{r}}}$. Since

$$T(z) = z^n \overline{S(1/\overline{z})} = z^n \overline{P(k/\overline{z})},$$

it follows for $0 \leq \theta < 2\pi$

$$|T(ke^{i\theta})| = k^n \overline{|e^{in\theta} P(e^{i\theta})|} = k^n |P(e^{i\theta})|. \tag{4.7}$$

By inequality (4.5) in conjunction with (4.6) and (4.7), we get for each $r \geq 1$

$$\begin{aligned} n^r k^{nr} \int_0^{2\pi} |P(e^{i\theta})|^r d\theta &\leq n^r (C_n)^r \int_0^{2\pi} |T(e^{i\theta})|^r d\theta \\ &\leq k^r (C_n)^r \int_0^{2\pi} |1+e^{i\theta}|^r \, |P'(ke^{i\theta})|^r d\theta. \end{aligned} \tag{4.8}$$

From (4.1), we have

$$|D_\alpha P(z)| \geq (|\alpha| - k^n)|P'(z)|.$$

Therefore for each $r \geq 1$, we get

$$|D_\alpha P(e^{i\theta})|^r \geq (|\alpha| - k^n)^r |P'(e^{i\theta})|^r,$$

which gives

$$|D_\alpha P(ke^{i\theta})|^r \geq (|\alpha| - k^n)^r |P'(ke^{i\theta})|^r. \tag{4.9}$$

From (4.8) and (4.9), it follows that

$$n^r k^{nr} (|\alpha| - k^n)^r \int_0^{2\pi} |P(e^{i\theta})|^r d\theta \leq k^r (C_n)^r \int_0^{2\pi} |1+e^{i\theta}|^r \, |D_\alpha P(ke^{i\theta})|^r d\theta,$$

which gives with the help of Holder's inequality for each $r \geq 1, \ p > 1, \ q > 1$ with $p^{-1} + q^{-1} = 1$

$$n^r k^{nr} (|\alpha| - k^n)^r \int_0^{2\pi} |P(e^{i\theta})|^r d\theta \leq k^r (C_n)^r \left\{ \int_0^{2\pi} |1+e^{i\theta}|^{pr} d\theta \right\}^{\frac{1}{p}} \left\{ \int_0^{2\pi} |D_\alpha P(ke^{i\theta})|^{qr} d\theta \right\}^{\frac{1}{q}},$$

i.e.,

$$n k^n (|\alpha| - k^n) \left\{ \int_0^{2\pi} |P(e^{i\theta})|^r d\theta \right\}^{\frac{1}{r}} \leq k C_n \left\{ \int_0^{2\pi} |1+e^{i\theta}|^{pr} d\theta \right\}^{\frac{1}{pr}} \left\{ \int_0^{2\pi} |D_\alpha P(ke^{i\theta})|^{qr} d\theta \right\}^{\frac{1}{qr}}. \tag{4.10}$$

Since the polar derivative $D_\alpha P(z)$ is a polynomial of degree at most $n-1$, so it is easy to verify that for each $t \geq 1$ and $R \geq 1$

$$\left\{ \int_0^{2\pi} |D_\alpha P(Re^{i\theta})|^t d\theta \right\}^{\frac{1}{t}} \leq R^{n-1} \left\{ \int_0^{2\pi} |D_\alpha P(e^{i\theta})|^t d\theta \right\}^{\frac{1}{t}}. \tag{4.11}$$

Finally, on applying (4.11) to (4.10) with $R = k$ and $t = qr$, we obtain for $r \geq 1$

$$n (|\alpha| - k^n) \left\{ \int_0^{2\pi} |P(e^{i\theta})|^r d\theta \right\}^{\frac{1}{r}} \leq C_n \left\{ \int_0^{2\pi} |1+e^{i\theta}|^{pr} d\theta \right\}^{\frac{1}{pr}} \left\{ \int_0^{2\pi} |D_\alpha P(e^{i\theta})|^{qr} d\theta \right\}^{\frac{1}{qr}}.$$

This proves Theorem 2.2 completely.

Proof of Theorem 2.3. Let

$$m = \min_{|z|=k} |P(z)|$$

then $|P(z)| \geq m$ on $|z| = k$. Therefore, for every λ with $|\lambda| < 1$

$$|P(z)| > |\lambda m| \text{ on } |z| = k.$$

If $P(z)$ has a zero on $|z| = k$, then $m = 0$ and the result follows from Theorem 2.2. Therefore from now onwards we will assume that $P(z)$ has all its zeros in $|z| < k$, where $k \geq 1$. By Rouche's theorem the polynomial

$$F(z) = P(z) + \lambda m$$

also has all its zeros in $|z| < k$. Thus, on applying Theorem 2.2 to $F(z)$, we obtain for each $r \geq 1$ and $|\alpha| \geq 2k^n + 1 \geq k^n$

$$n(|\alpha| - k^n) \left\{ \int_0^{2\pi} |F(e^{i\theta})|^r d\theta \right\}^{\frac{1}{r}} \leq C_n \left\{ \int_0^{2\pi} |1 + e^{i\theta}|^{pr} d\theta \right\}^{\frac{1}{pr}} \left\{ \int_0^{2\pi} |D_\alpha F(e^{i\theta})|^{qr} d\theta \right\}^{\frac{1}{qr}},$$

i.e.,

$$n(|\alpha| - k^n) \left\{ \int_0^{2\pi} |P(e^{i\theta}) + \lambda m|^r d\theta \right\}^{\frac{1}{r}} \leq C_n \left\{ \int_0^{2\pi} |1 + e^{i\theta}|^{pr} d\theta \right\}^{\frac{1}{pr}} \left\{ \int_0^{2\pi} |D_\alpha P(e^{i\theta}) + n\lambda m|^{qr} d\theta \right\}^{\frac{1}{qr}}.$$

This completes the proof of Theorem 2.3.

5. Conclusion

We mainly focused on estimating integral mean for operator preserving inequalities of the type (1.5) and associated results by proving some of new results through a thorough analysis and investigation. The main results provide generalizations and refinements, and a link with some of the classical results as seen in remarks. Furthermore, the findings shed light on the interplay between various operators, their impact on preserving inequalities and integral norm.

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