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APPLICATIONS OF THE GAUSSIAN HYPERGEOMETRIC FUNCTION TO SOME SUBCLASSES OF ANALYTIC FUNCTIONS

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Abstract. In this paper, we derive the necessary and sufficient conditions for the Gaussian hypergeometric function to be in some subclasses of analytic functions.

1. Introduction

Let $\mathcal{A}$ denote the family of functions that are analytic in the open unit disc $U = \{\zeta \in \mathbb{C} : |\zeta| < 1\}$ with normalized form

$$g(\zeta) = \zeta + \sum_{t=2}^{\infty} a_t \zeta^t \ . \tag{1.1}$$

We denote by $\mathcal{S}$ the subclass of $\mathcal{A}$ consisting of univalent functions. Further, let $\mathcal{T}$ be a subclass of $\mathcal{S}$ consisting of functions g of the form

$$g(\zeta) = \zeta - \sum_{t=2}^{\infty} a_t \zeta^t, \quad a_t \geq 0. \tag{1.2}$$

Definition 1. [5]A function $g \in \mathcal{T}$ is in the class $\overline{H}\left(\nu, \tau\right)$ if it satisfies the following condition

$$\Re \left\{ \frac{\zeta g'(\zeta) + \nu \zeta^2 g''(\zeta)}{g(\zeta)} \right\} > \tau, \quad 0 \leq \nu, \tau < 1.$$

Definition 2. [3]A function $g \in \mathcal{T}$ is in the class $\mathcal{J}^*\left(\nu, \tau\right)$ if it satisfies the following condition

$$\Re \left\{ \frac{(1-\nu)\, g(\zeta)}{\zeta} + \nu g'(\zeta) \right\} > \tau, \quad 0 \leq \nu, \tau < 1.$$

Definition 3. [8]A function $g \in \mathcal{T}$ is in the class $\mathcal{R}^1\left(\nu, \tau\right)$ if it satisfies the following condition

$$\Re \left\{ g'(\zeta) + \nu \zeta g''(\zeta) \right\} > \tau, \quad 0 \leq \nu, \tau < 1.$$

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Definition 4. A function $g \in \mathcal{T}$ is in the class $\mathcal{X}^*(\nu,\tau)$ if it satisfies the following condition

$$\Re\left\{(1-\nu)\frac{g(\zeta)}{\zeta g'(\zeta)}+\nu(1+\frac{\zeta g''(\zeta)}{g'(\zeta)})\right\}>\tau, \qquad 0\leq \nu,\tau<1.$$

It is known that the Gaussian hypergeometric function ${}_2F_1(\gamma,\mu,\eta,\zeta)$ is one of the special functions and it is a solution of a second order linear ordinary differential equation

$$\zeta(1-\zeta)y''+[\eta-(\gamma+\mu+1)\zeta]y'-\gamma\mu y=0,$$

where γ,μ and η are constants. It is defined by

$${}_2F_1(\gamma,\mu,\eta,\zeta)=\sum_{t=0}^{\infty}\frac{(\gamma)_t(\mu)_t}{(\eta)_t\, t!}\zeta^t, \tag{1.3}$$

such that $\eta\neq 0,-1,-2,...$ and the Pochhammer symbol $(\alpha)_t$ defined by

$$(\alpha)_t=\begin{cases}\alpha(\alpha+1)...(\alpha+t-1) & \text{if} \quad t=1,2,...\\ 1 & \text{if} \quad t=0.\end{cases} \tag{1.4}$$

If $\Re(\eta-\gamma-\mu)>0$ and $\eta\neq 0,-1,-2,...$, then

$${}_2F_1(\gamma,\mu,\eta,1)=\frac{\Gamma(\eta)\Gamma(\eta-\gamma-\mu)}{\Gamma(\eta-\gamma)\Gamma(\eta-\mu)}. \tag{1.5}$$

We note that

$$(\alpha)_t=\alpha(\alpha+1)_{t-1}, \tag{1.6}$$

$$\begin{aligned}\zeta{}_2F_1(\gamma,\mu,\eta,\zeta) &= \zeta+\sum_{t=2}^{\infty}\frac{(\gamma)_{t-1}(\mu)_{t-1}}{(\eta)_{t-1}(t-1)!}\zeta^t\\ &= \zeta+\frac{\gamma\mu}{\eta}\sum_{t=2}^{\infty}\frac{(\gamma+1)_{t-2}(\mu+1)_{t-2}}{(\eta+1)_{t-2}(t-1)!}\zeta^t\\ &= \zeta-\left|\frac{\gamma\mu}{\eta}\right|\sum_{t=2}^{\infty}\frac{(\gamma+1)_{t-2}(\mu+1)_{t-2}}{(\eta+1)_{t-2}(t-1)!}\zeta^t.\end{aligned} \tag{1.7}$$

Now, let

$$\begin{aligned}F_1(\gamma,\mu,\eta,\zeta) &= \zeta(2-{}_2F_1(\gamma,\mu,\eta,\zeta))\\ &= \zeta-\sum_{t=2}^{\infty}\frac{(\gamma)_{t-1}(\mu)_{t-1}}{(\eta)_{t-1}(t-1)!}\zeta^t.\end{aligned} \tag{1.8}$$

The hypergeometric function has many applications in mathematics, physics and engineering. In particular, in mathematics it has been used in different areas such as fractional calculus, geometric function theory, mathematical modeling and etc. The study of the hypergeometric function in the field of geometric function theory began to be in the researchers interest after the de Branges proof for the Bieberbach's conjecture in 1985 as he used the Gaussian hypergeometric function in his proof.

Necessary and sufficient conditions for $\zeta{}_2F_1(\gamma,\mu,\eta,\zeta)$ to be in various subclasses of analytic functions were investigated extensively by Silverman [12], Know and Cho [4], Mostafa [6] and Ramachandran et al. [10], and various geometric properties of this function were discussed in [1, 7, 9]. This paper will determine the necessary and sufficient conditions for the Gaussian hypergeometric function to be in the classes $\overline{H}(\nu,\tau), \mathcal{J}^*(\nu,\tau), \mathcal{X}^*(\nu,\tau)$ and $\mathcal{R}^1(\nu,\tau)$.

2. Main Results

We need the following lemmas to prove our results.

Lemma 1. [5] *A function $g \in \mathcal{T}$ is in the class $\overline{H}(\nu,\tau)$ if and only if*

$$\sum_{t=2}^{\infty}\{(t-1)(\nu t+1)+(1-\tau)\}a_t \leq 1-\tau, \quad 0 \leq \tau < 1.$$

Lemma 2. [8] *A function $g \in \mathcal{T}$ is in the class $\mathcal{R}^1(\nu,\tau)$ if and only if*

$$\sum_{t=2}^{\infty} t[\nu(t-1)+1]a_t \leq 1-\tau, \quad 0 \leq \tau < 1.$$

Since $g \in \mathcal{R}^1(\nu,\tau) \Leftrightarrow \zeta\, g' \in \mathcal{J}^*(\nu,\tau)$, we have the following:

Lemma 3. *A function $g \in \mathcal{T}$ is in the class $\mathcal{J}^*(\nu,\tau)$ if and only if*

$$\sum_{t=2}^{\infty}[\nu(t-1)+1]a_t \leq 1-\tau, \quad 0 \leq \tau < 1.$$

Using the same method and technique given by Silverman [11], we easily establish the following theorem, and so we omit the details.

Theorem 1. *Let $\frac{1}{3} \leq \nu < 1$. A function $g \in \mathcal{T}$ is in the class $\mathcal{X}^*(\nu,\tau)$ if and only if*

$$\sum_{t=2}^{\infty}\{(t-1)[\nu(t+1)-1]+(1-\tau)t\}a_t \leq 1-\tau, \quad 0 \leq \tau < 1.$$

In this paper, we derive the necessary and sufficient conditions for the Gaussian hypergeometric function to be in $\overline{H}(\nu,\tau), \mathcal{J}^*(\nu,\tau), \mathcal{X}^*(\nu,\tau)$ and $\mathcal{R}^1(\nu,\tau)$.

Theorem 2. *If $\gamma,\mu > -1, \gamma\mu < 0$ and $\eta > \gamma+\mu+2$, then $\zeta_2F_1(\gamma,\mu,\eta,\zeta) \in \overline{H}(\nu,\tau)$ if and only if*

$$(2\nu+1)\gamma\mu\eta - \gamma\mu(2\nu+1)(\gamma+\mu+2)+\nu(\gamma)_2(\mu)_2+(1-\tau)(\eta-\gamma-\mu-2)_2 \geq 0. \tag{2.1}$$

Proof. According to Lemma 1 and using (1.7), we need to show that

$$\sum_{t=2}^{\infty}\{(t-1)(\nu t+1)+(1-\tau)\}\frac{(\gamma+1)_{t-2}(\mu+1)_{t-2}}{(\eta+1)_{t-2}(t-1)!} \leq \left|\frac{\eta}{\gamma\mu}\right|(1-\tau).$$

Applying (1.5) and (1.6) , we get

$$
\begin{aligned}
&\sum_{t=2}^{\infty}\{(t-1)(\nu t+1)+(1-\tau)\}\frac{(\gamma+1)_{t-2}(\mu+1)_{t-2}}{(\eta+1)_{t-2}(t-1)!} \\
=\ & \frac{\nu(\gamma+1)(\mu+1)}{(\eta+1)}\sum_{t=0}^{\infty}\frac{(\gamma+2)_t(\mu+2)_t}{(\eta+2)_t\,t!}+(2\nu+1)\sum_{t=0}^{\infty}\frac{(\gamma+1)_t(\mu+1)_t}{(\eta+1)_t\,t!} \\
&+\frac{\eta(1-\tau)}{\gamma\mu}\sum_{t=1}^{\infty}\frac{(\gamma)_t(\mu)_t}{(\eta)_t\,t!} \\
=\ & \frac{\nu(\gamma+1)(\mu+1)}{(\eta+1)}\frac{\Gamma(\eta+2)\Gamma(\eta-\gamma-\mu-2)}{\Gamma(\eta-\gamma)\Gamma(\eta-\mu)}+(2\nu+1)\frac{\Gamma(\eta+1)\Gamma(\eta-\gamma-\mu-1)}{\Gamma(\eta-\gamma)\Gamma(\eta-\mu)} \\
&+\frac{\eta(1-\tau)}{\gamma\mu}\left[\frac{\Gamma(\eta)\Gamma(\eta-\gamma-\mu)}{\Gamma(\eta-\gamma)\Gamma(\eta-\mu)}-1\right] \\
\leq\ & \left|\frac{\eta}{\gamma\mu}\right|(1-\tau),
\end{aligned}
$$

which is equivalent to

$$
\begin{aligned}
&\frac{\Gamma(\eta+1)\Gamma(\eta-\gamma-\mu-2)}{\Gamma(\eta-\gamma)\Gamma(\eta-\mu)}\Big[\nu(\gamma+1)(\mu+1)+(2\nu+1)(\eta-\gamma-\mu-2) \\
&\quad+\frac{(1-\tau)(\eta-\gamma-\mu-2)_2}{\gamma\mu}\Big] \\
\leq\ & (1-\tau)\left(\frac{\eta}{\gamma\mu}+\frac{\eta}{|\gamma\mu|}\right)=0,
\end{aligned}
$$

which is valid if (2.1) holds. This completes the proof. □

THEOREM 3. *If $\gamma,\mu>0$ and $\eta>\gamma+\mu+2$, then $F_1(\gamma,\mu,\eta,\zeta)\in\overline{H}(\nu,\tau)$ if and only if*

$$
\frac{\Gamma(\eta)\Gamma(\eta-\gamma-\mu-2)}{\Gamma(\eta-\gamma)\Gamma(\eta-\mu)}\left[\frac{\nu(\gamma)_2(\mu)_2}{1-\tau}+\frac{\gamma\mu(2\nu+1)(\eta-\gamma-\mu-2)}{1-\tau}+(\eta-\gamma-\mu-2)_2\right]\leq 2. \tag{2.2}
$$

Proof. According to Lemma 1 and using (1.8), we need to show that

$$
\sum_{t=2}^{\infty}\{(t-1)(\nu t+1)+(1-\tau)\}\frac{(\gamma)_{t-1}(\mu)_{t-1}}{(\eta)_{t-1}(t-1)!}\leq 1-\tau. \tag{2.3}
$$

Using (1.5) and (1.6), (2.3) we get

$$\begin{aligned}
&\sum_{t=2}^{\infty}\{(t-1)(\nu t+1)+(1-\tau)\}\frac{(\gamma)_{t-1}(\mu)_{t-1}}{(\eta)_{t-1}(t-1)!}\\
=\ &\frac{\nu\gamma\mu(\gamma+1)(\mu+1)}{\eta(\eta+1)}\sum_{t=0}^{\infty}\frac{(\gamma+2)_t(\mu+2)_t}{(\eta+2)_t\,t!}+\frac{(2\nu+1)\gamma\mu}{\eta}\sum_{t=0}^{\infty}\frac{(\gamma+1)_t(\mu+1)_t}{(\eta+1)_t\,t!}\\
&+(1-\tau)\sum_{t=1}^{\infty}\frac{(\gamma)_t(\mu)_t}{(\eta)_t\,t!}\\
=\ &\frac{\nu\gamma\mu(\gamma+1)(\mu+1)}{\eta(\eta+1)}\frac{\Gamma(\eta+2)\Gamma(\eta-\gamma-\mu-2)}{\Gamma(\eta-\gamma)\Gamma(\eta-\mu)}+\frac{(2\nu+1)\gamma\mu}{\eta}\frac{\Gamma(\eta+1)\Gamma(\eta-\gamma-\mu-1)}{\Gamma(\eta-\gamma)\Gamma(\eta-\mu)}\\
&+(1-\tau)\left[\frac{\Gamma(\eta)\Gamma(\eta-\gamma-\mu)}{\Gamma(\eta-\gamma)\Gamma(\eta-\mu)}-1\right]\\
\leq\ &1-\tau,
\end{aligned}$$

which is equivalent to

$$\frac{\Gamma(\eta)\Gamma(\eta-\gamma-\mu-2)}{\Gamma(\eta-\gamma)\Gamma(\eta-\mu)}\left[\nu(\gamma)_2(\mu)_2+(2\nu+1)\gamma\mu(\eta-\gamma-\mu-2)+(1-\tau)(\eta-\gamma-\mu-2)_2\right]$$
$$\leq 2(1-\tau).$$

This completes the proof of Theorem 3. □

THEOREM 4. *If $\gamma,\mu>-1,\gamma\mu<0$ and $\eta>\gamma+\mu+1$, then $\zeta{}_2F_1(\gamma,\mu,\eta,\zeta)\in\mathcal{J}^*(\nu,\tau)$ if and only if*

$$\Gamma(\eta+1)\Gamma(\eta-\gamma-\mu-1)[\eta-\gamma-\mu-1+\nu\gamma\mu]-\tau\eta\Gamma(\eta-\gamma)\Gamma(\eta-\mu)\geq 0. \quad (2.4)$$

Proof. By using (1.7) and applying Lemma 3, it is adequate to show that

$$\sum_{t=2}^{\infty}[\nu(t-1)+1]\frac{(\gamma+1)_{t-2}(\mu+1)_{t-2}}{(\eta+1)_{t-2}(t-1)!}\leq\left|\frac{\eta}{\gamma\mu}\right|(1-\tau). \quad (2.5)$$

It follows from (1.5), (1.6) and (2.5) that

$$\begin{aligned}
&\sum_{t=2}^{\infty}[\nu(t-1)+1]\frac{(\gamma+1)_{t-2}(\mu+1)_{t-2}}{(\eta+1)_{t-2}(t-1)!}\\
=\ &\nu\sum_{t=0}^{\infty}\frac{(\gamma+1)_t(\mu+1)_t}{(\eta+1)_t\,t!}+\frac{\eta}{\gamma\mu}\sum_{t=1}^{\infty}\frac{(\gamma)_t(\mu)_t}{(\eta)_t\,t!}\\
=\ &\nu\frac{\Gamma(\eta+1)\Gamma(\eta-\gamma-\mu-1)}{\Gamma(\eta-\gamma)\Gamma(\eta-\mu)}+\frac{\eta}{\gamma\mu}\left[\frac{\Gamma(\eta)\Gamma(\eta-\gamma-\mu)}{\Gamma(\eta-\gamma)\Gamma(\eta-\mu)}-1\right]\\
\leq\ &\left|\frac{\eta}{\gamma\mu}\right|(1-\tau).
\end{aligned}$$

Hence

$$\frac{\Gamma(\eta+1)\Gamma(\eta-\gamma-\mu-1)}{\Gamma(\eta-\gamma)\Gamma(\eta-\mu)}[\eta-\gamma-\mu-1+\nu\gamma\mu]\geq\tau\eta,$$

which is equivalent to (2.4). This completes the proof of Theorem 4. □

THEOREM 5. *If $\gamma, \mu > 0$ and $\eta > \gamma + \mu + 1$, then $F_1(\gamma, \mu, \eta, \zeta) \in \mathcal{J}^*(\nu, \tau)$ if and only if*

$$\frac{\Gamma(\eta)\Gamma(\eta-\gamma-\mu-1)}{\Gamma(\eta-\gamma)\Gamma(\eta-\mu)}[\nu\gamma\mu+\eta-\gamma-\mu-1]+\tau \leq 2. \tag{2.6}$$

Proof. According to Lemma 3 and using (1.8), we need to show that

$$\sum_{t=2}^{\infty}[\nu(t-1)+1]\frac{(\gamma)_{t-1}(\mu)_{t-1}}{(\eta)_{t-1}(t-1)!} \leq 1-\tau.$$

Thus,

$$\begin{aligned}
&\sum_{t=2}^{\infty}[\nu(t-1)+1]\frac{(\gamma)_{t-1}(\mu)_{t-1}}{(\eta)_{t-1}(t-1)!}\\
=\ & \frac{\nu\gamma\mu}{\eta}\sum_{t=0}^{\infty}\frac{(\gamma+1)_t(\mu+1)_t}{(\eta+1)_t\, t!}+\sum_{t=1}^{\infty}\frac{(\gamma)_t(\mu)_t}{(\eta)_t\, t!}\\
=\ & \frac{\nu\gamma\mu}{\eta}\frac{\Gamma(\eta+1)\Gamma(\eta-\gamma-\mu-1)}{\Gamma(\eta-\gamma)\Gamma(\eta-\mu)}+\frac{\Gamma(\eta)\Gamma(\eta-\gamma-\mu)}{\Gamma(\eta-\gamma)\Gamma(\eta-\mu)}-1\\
\leq\ & 1-\tau,
\end{aligned}$$

if

$$\frac{\Gamma(\eta)\Gamma(\eta-\gamma-\mu-1)}{\Gamma(c-\gamma)\Gamma(\eta-\mu)}[\nu\gamma\mu+\eta-\gamma-\mu-1] \leq 2-\tau,$$

which is equivalent to (2.6). This completes the proof of Theorem 5. □

THEOREM 6. *Let $\frac{1}{3} \leq \nu < 1$. If $\gamma, \mu > -1, \gamma\mu < 0$ and $\eta > \gamma + \mu + 2$, then $\zeta_2 F_1(\gamma, \mu, \eta, \zeta) \in \mathcal{X}^*(\nu, \tau)$ if and only if*

$$\nu\gamma\mu(\gamma+1)(\mu+1)+\gamma\mu(3\nu-\tau)(\eta-\gamma-\mu-2)+(1-\tau)(\eta-\gamma-\mu-2)_2 \geq 0. \tag{2.7}$$

Proof. According to Theorem 1 and using (1.7), we need to show that

$$\sum_{t=2}^{\infty}\{(t-1)[\nu(t+1)-1]+(1-\tau)t\}\frac{(\gamma+1)_{t-2}(\mu+1)_{t-2}}{(\eta+1)_{t-2}(t-1)!} \leq \left|\frac{\eta}{\gamma\mu}\right|(1-\tau).$$

Applying (1.5) and (1.6), we have

$$\begin{aligned}
&\sum_{t=2}^{\infty}\{(t-1)\left[\nu(t+1)-1\right]+(1-\tau)\,t\}\frac{(\gamma+1)_{t-2}\,(\mu+1)_{t-2}}{(\eta+1)_{t-2}\,(t-1)!}\\
=\;&\frac{\nu\,(\gamma+1)\,(\mu+1)}{(\eta+1)}\sum_{t=0}^{\infty}\frac{(\gamma+2)_t\,(\mu+2)_t}{(\eta+2)_t\,t!}+(3\nu-\tau)\sum_{t=0}^{\infty}\frac{(\gamma+1)_t\,(\mu+1)_t}{(\eta+1)_t\,t!}\\
&+\frac{\eta\,(1-\tau)}{\gamma\mu}\sum_{t=1}^{\infty}\frac{(\gamma)_t\,(\mu)_t}{(\eta)_t\,t!}\\
=\;&\frac{\nu\,(\gamma+1)\,(\mu+1)}{(\eta+1)}\frac{\Gamma\,(\eta+2)\,\Gamma\,(\eta-\gamma-\mu-2)}{\Gamma\,(\eta-\gamma)\,\Gamma\,(\eta-\mu)}+(3\nu-\tau)\,\frac{\Gamma\,(\eta+1)\,\Gamma\,(\eta-\gamma-\mu-1)}{\Gamma\,(\eta-\gamma)\,\Gamma\,(\eta-\mu)}\\
&+\frac{\eta\,(1-\tau)}{\gamma\mu}\left[\frac{\Gamma\,(\eta)\,\Gamma\,(\eta-\gamma-\mu)}{\Gamma\,(\eta-\gamma)\,\Gamma\,(\eta-\mu)}-1\right]\\
\le\;&\left|\frac{\eta}{\gamma\mu}\right|(1-\tau)\,,
\end{aligned}$$

which is equivalent to

$$\begin{aligned}
\frac{\Gamma\,(\eta+1)\,\Gamma\,(\eta-\gamma-\mu-2)}{\Gamma\,(\eta-\gamma)\,\Gamma\,(\eta-\mu)}\Big[&\nu\,(\gamma+1)\,(\mu+1)+(3\nu-\tau)\,(\eta-\gamma-\mu-2)\\
&+\frac{1-\tau}{\gamma\mu}\,(\eta-\gamma-\mu-2)_2\Big]\le 0.
\end{aligned}$$

This completes the proof of Theorem 6. □

Theorem 7. *Let* $\frac{1}{3}\le\nu<1$. *If* $\gamma,\mu>0$ *and* $\eta>\gamma+\mu+2$, *then* $F_1\,(\gamma,\mu,\eta,\zeta)\in\mathcal{X}^*\,(\nu,\tau)$ *if and only if*

$$\begin{aligned}
\frac{\Gamma\,(\eta)\,\Gamma\,(\eta-\gamma-\mu-2)}{\Gamma\,(\eta-\gamma)\,\Gamma\,(\eta-\mu)}\Big[&\frac{\nu\,(\gamma)_2\,(\mu)_2}{1-\tau}+\frac{\gamma\mu\,(3\nu-\tau)\,(\eta-\gamma-\mu-2)}{1-\tau}\\
&+(\eta-\gamma-\mu-2)_2\Big]\le 2.
\end{aligned}\tag{2.8}$$

Proof. By using (1.8) and applying Theorem 1 , we need to show that

$$\sum_{t=2}^{\infty}\{(t-1)\left[\nu(t+1)-1\right]+(1-\tau)\,t\}\frac{(\gamma)_{t-1}\,(\mu)_{t-1}}{(\eta)_{t-1}\,(t-1)!}\le 1-\tau.$$

It follows from (1.5) and (1.6) that

$$\begin{aligned}
&\sum_{t=2}^{\infty}\{(t-1)\,[\nu(t+1)-1]+(1-\tau)\,t\}\frac{(\gamma)_{t-1}\,(\mu)_{t-1}}{(\eta)_{t-1}\,(t-1)!}\\
=\;&\frac{\nu\gamma\mu\,(\gamma+1)\,(\mu+1)}{\eta\,(\eta+1)}\sum_{t=0}^{\infty}\frac{(\gamma+2)_t\,(\mu+2)_t}{(\eta+2)_t\,t!}+\frac{(3\nu-\tau)\,\gamma\mu}{\eta}\sum_{t=0}^{\infty}\frac{(\gamma+1)_t\,(\mu+1)_t}{(\eta+1)_t\,t!}\\
&+(1-\tau)\sum_{t=1}^{\infty}\frac{(\gamma)_t\,(\mu)_t}{(\eta)_t\,t!}\\
=\;&\frac{\nu\,(\gamma)_2\,(\mu)_2}{(\eta)_2}\frac{\Gamma(\eta+2)\,\Gamma(\eta-\gamma-\mu-2)}{\Gamma(\eta-\gamma)\,\Gamma(\eta-\mu)}+\frac{(3\nu-\tau)\,\gamma\mu}{\eta}\frac{\Gamma(\eta+1)\,\Gamma(\eta-\gamma-\mu-1)}{\Gamma(\eta-\gamma)\,\Gamma(\eta-\mu)}\\
&+(1-\tau)\left[\frac{\Gamma(\eta)\,\Gamma(\eta-\gamma-\mu)}{\Gamma(\eta-\gamma)\,\Gamma(\eta-\mu)}-1\right]\\
\leq\;&1-\tau,
\end{aligned}$$

which is equivalent to

$$\begin{aligned}
&\frac{\Gamma(\eta)\,\Gamma(\eta-\gamma-\mu-2)}{\Gamma(\eta-\gamma)\,\Gamma(\eta-\mu)}\left[\nu\,(\gamma)_2\,(\mu)_2+(3\nu-\tau)\,\gamma\mu\,(\eta-\gamma-\mu-2)+(1-\tau)\,(\eta-\gamma-\mu-2)_2\right]\\
&\leq 2\,(1-\tau)\,.
\end{aligned}$$

This completes the proof of Theorem 7. □

THEOREM 8. *If $\gamma,\mu>-1,\gamma\mu<0$ and $\eta>\gamma+\mu+2$, then $\zeta_2F_1(\gamma,\mu,\eta,\zeta)\in\mathcal{R}^1(\nu,\tau)$ if and only if*

$$\begin{aligned}
&\Gamma(\eta+1)\,\Gamma(\eta-\gamma-\mu-2)\left[\nu\,(\gamma)_2\,(\mu)_2+(1+2\nu)\,(\eta-\gamma-\mu-2)\,\gamma\mu+(\eta-\gamma-\mu-2)_2\right]\\
&-\eta\tau\Gamma(\eta+1)\,\Gamma(\eta-\gamma-\mu-2)\geq 0. \qquad (2.9)
\end{aligned}$$

Proof. According to Lemma 2 and using (1.7), we need to show that

$$\sum_{t=2}^{\infty}t[\nu(t-1)+1]\frac{(\gamma+1)_{t-2}\,(\mu+1)_{t-2}}{(\eta+1)_{t-2}\,(t-1)!}\leq\left|\frac{\eta}{\gamma\mu}\right|(1-\tau)\,.$$

From (1.5) and (1.6) it follows that

$$\begin{aligned}
&\sum_{t=2}^{\infty}t[\nu(t-1)+1]\frac{(\gamma+1)_{t-2}\,(\mu+1)_{t-2}}{(\eta+1)_{t-2}\,(t-1)!}\\
=\;&\frac{\nu\,(\gamma+1)\,(\mu+1)}{(\eta+1)}\sum_{t=0}^{\infty}\frac{(\gamma+2)_t\,(\mu+2)_t}{(\eta+2)_t\,t!}+(1+2\nu)\sum_{t=0}^{\infty}\frac{(\gamma+1)_t\,(\mu+1)_t}{(\eta+1)_t\,t!}\\
&+\frac{\eta}{\gamma\mu}\sum_{t=1}^{\infty}\frac{(\gamma)_t\,(\mu)_t}{(\eta)_t\,t!}\\
=\;&\frac{\nu\,(\gamma+1)\,(\mu+1)}{(\eta+1)}\frac{\Gamma(\eta+2)\,\Gamma(\eta-\gamma-\mu-2)}{\Gamma(\eta-\gamma)\,\Gamma(\eta-\mu)}+(1+2\nu)\,\frac{\Gamma(\eta+1)\,\Gamma(\eta-\gamma-\mu-1)}{\Gamma(\eta-\gamma)\,\Gamma(\eta-\mu)}\\
&+\frac{\eta}{\gamma\mu}\left[\frac{\Gamma(\eta)\,\Gamma(\eta-\gamma-\mu)}{\Gamma(\eta-\gamma)\,\Gamma(\eta-\mu)}-1\right]\\
\leq\;&\left|\frac{\eta}{\gamma\mu}\right|(1-\tau)\,,
\end{aligned}$$

which is equivalent to

$$\frac{\Gamma(\eta+1)\Gamma(\eta-\gamma-\mu-2)}{\Gamma(\eta-\gamma)\Gamma(\eta-\mu)}\left[\nu(\gamma)_2(\mu)_2+(1+2\nu)(\eta-\gamma-\mu-2)\gamma\mu+(\eta-\gamma-\mu-2)_2\right]\geq\eta\tau.$$

This completes the proof of Theorem 8. □

THEOREM 9. *If $\gamma,\mu>0$ and $\eta>\gamma+\mu+2$, then $F_1(\gamma,\mu,\eta,\zeta)\in\mathcal{R}^1(\nu,\tau)$ if and only if*

$$\frac{\Gamma(\eta)\Gamma(\eta-\gamma-\mu-2)}{\Gamma(\eta-\gamma)\Gamma(\eta-\mu)}\left[(1+2\nu)\gamma\mu(\eta-\gamma-\mu-2)+\nu(\gamma)_2(\mu)_2+(\eta-\gamma-\mu-2)_2\right]+\tau\leq 2. \tag{2.10}$$

Proof. According to Lemma 2 and using (1.6) and (1.8), we need to show that

$$\sum_{t=2}^{\infty}t[\nu(t-1)+1]\frac{(\gamma)_{t-1}(\mu)_{t-1}}{(\eta)_{t-1}(t-1)!}\leq 1-\tau.$$

Thus,

$$\begin{aligned}
&\sum_{t=2}^{\infty}t[\nu(t-1)+1]\frac{(\gamma)_{t-1}(\mu)_{t-1}}{(\eta)_{t-1}(t-1)!}\\
=\;&\frac{\nu\gamma\mu(\gamma+1)(\mu+1)}{\eta(\eta+1)}\sum_{t=0}^{\infty}\frac{(\gamma+2)_t(\mu+2)_t}{(\eta+2)_t\,t!}+\frac{(1+2\nu)\gamma\mu}{\eta}\sum_{t=0}^{\infty}\frac{(\gamma+1)_t(\mu+1)_t}{(\eta+1)_t\,t!}\\
&+\sum_{t=1}^{\infty}\frac{(\gamma)_t(\mu)_t}{(\eta)_t\,t!}\\
=\;&\frac{\nu(\gamma)_2(\mu)_2}{(\eta)_2}\frac{\Gamma(\eta+2)\Gamma(\eta-\gamma-\mu-2)}{\Gamma(\eta-\gamma)\Gamma(\eta-\mu)}+\frac{(1+2\nu)\gamma\mu}{\eta}\frac{\Gamma(\eta+1)\Gamma(\eta-\gamma-\mu-1)}{\Gamma(\eta-\gamma)\Gamma(\eta-\mu)}\\
&+\frac{\Gamma(\eta)\Gamma(\eta-\gamma-\mu)}{\Gamma(\eta-\gamma)\Gamma(\eta-\mu)}-1\\
\leq\;&1-\tau,
\end{aligned}$$

if

$$\frac{\Gamma(\eta)\Gamma(\eta-\gamma-\mu-2)}{\Gamma(\eta-\gamma)\Gamma(\eta-\mu)}\left[(1+2\nu)\gamma\mu(\eta-\gamma-\mu-2)+\nu(\gamma)_2(\mu)_2+(\eta-\gamma-\mu-2)_2\right]\leq 2-\tau,$$

which is equivalent to (2.10). This completes the proof of Theorem 9. □

3. Conclusion

Seeking for the necessary and sufficient conditions for the hypergeometric function to belong to different subclasses of analytic functions was investigated by many researchers intensively. In our paper, we investigated the necessary and sufficient conditions for the Gaussian hypergeometric function $\zeta_2F_1(\gamma,\mu,\eta,\zeta)$ and $F_1(\gamma,\mu,\eta,\zeta)$ to belong to the classes $\overline{H}(\nu,\tau),\mathcal{J}^*(\nu,\tau),\mathcal{X}^*(\nu,\tau)$ and $\mathcal{R}^1(\nu,\tau)$. For future work one can find the conditions for the Gaussian hypergeometric function to belong to different classes of analytic functions.

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