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THE n-GENERALIZED COMPOSITION OPERATORS FROM ZYGMUND SPACES TO $Q_K(p,q)$ SPACES

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Abstract. The boundedness and compactness of the so-called n-generalized composition operator $c_{\varphi}^{\mathfrak{g},n}$ from the class of Zygmund-type spaces into $Q_K(p,q)$ spaces are characterized in this paper.

1. Introduction

Let $\Lambda = \{\zeta \in \mathbb{C} : |\zeta| < 1\}$ be the open unit disc in the complex plane $\mathbb{C}$, $H(\Lambda)$ denote the class of all analytic functions in Λ.

In [4], for each $\sigma \in \Lambda$, $\varphi_\sigma : \Lambda \to \Lambda$ denotes the Möbius transformation defined by

$$\varphi_\sigma(\zeta) := \frac{\sigma - \zeta}{1 - \bar{\sigma}\zeta}, \quad \text{for } \zeta \in \Lambda.$$

Green's function of Λ with a logarithmic singularity at σ, is defined as follows,

$$g(\zeta, \sigma) := \log \left| \frac{1 - \bar{\sigma}\zeta}{\zeta - \sigma} \right| = \log \frac{1}{|\varphi_\sigma(\zeta)|} \ .$$

The known composition operator $C_\varphi \mathfrak{f}(\zeta) = \mathfrak{f}(\varphi(\zeta))$, $\mathfrak{f} \in \Lambda$ has been studied for many years (see [5–7, 17]). From the recent research on the operator theory of complex-type function spaces, we can introduce the n-generalized composition operators $c_{\varphi}^{\mathfrak{g},n}$ used in the current paper as

$$\left(c_{\varphi}^{\mathfrak{g},n}\ \mathfrak{f} \right)(\zeta) \ = \ \int_0^{\zeta} \mathfrak{f}'(\varphi(\xi))\mathfrak{g}^{(n-1)}(\xi) d\xi,$$

where, $\mathfrak{g} \in H(\Lambda)$ and $\mathfrak{g}^{(n-1)(\zeta)} = \frac{d^{n-1}\mathfrak{g}(\zeta)}{d\zeta^{n-1}}$, with $''n-1''$ order derivatives, $n \in N$.

Definition 1.1. (see [14]) Let $K : [0,\infty) \to [0,\infty)$ be a right continuous and nondecreasing function. For $0 < p < \infty$ and $-2 < q < \infty$, the space $Q_K(p,q)$ is defined by

$$Q_K(p,q) := \sup_{\sigma \in \Lambda} \int_{\Lambda} |(\mathfrak{f})'(\zeta)|^p (1 - |\zeta|^2)^q K(g(\zeta,\sigma)) dA(\zeta) < \infty.$$

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If

$$\lim_{|\sigma|\to 1^-} \sup_{\sigma\in\Lambda} \int\limits_{\Lambda} |\mathfrak{f}'(\zeta)|^p \big(1-|\zeta|^2\big)^q K(g(\zeta,\sigma))dA(\zeta) = 0,$$

then $\mathfrak{f} \in Q_{K,0}(p,q)$.

Wulan and Zhou [13] mentioned the following properties of these spaces:
(1) For $p=2,\ q=0$, we obtain $Q_K(p,q) = Q_K$ (see [4]).
(2) For $p=2,\ q=0$, and $K(t)=t^p$, we obtain $Q_K(p,q) = Q_p$ (see [1]).
(3) For $K(t)=t^s$, $Q_K(p,q) = F(p,q,s)$ [3, 16].

Definition 1.2. For $0<\beta<\infty$, a function $\mathfrak{f} \in H(\Lambda)$ belongs to the Zygmund space $\mathcal{Z}_\beta$ if

$$\sup_{\sigma\in\Lambda} |(\mathfrak{f})''(\zeta)|(1-|\zeta|^2)^\beta < \infty,$$

$$\|\mathfrak{f}\|_{\mathcal{Z}_\beta} = |\mathfrak{f}(0)| + |\mathfrak{f}'(0)| + +\sup_{\sigma\in\Lambda} |(\mathfrak{f})''(\zeta)|(1-|\zeta|^2)^\beta < \infty.$$

Zygmund-type spaces and some operators on them were studied in [2, 9].

Definition 1.3. (see [11]) The analytic function $\mathfrak{f} \in \Lambda$ has the Hadamard gap (also called as lacunary series) if $\mathfrak{f}(\zeta) = \sum_{k=1}^{\infty} a_k \zeta^{n_k}$, (with $n_k \in \mathbb{N}$; for all $k \in \mathbb{N}$) and there exists a constant $\lambda > 1$ such that $\frac{n_{k+1}}{n_k} \geq \lambda$ for all $k \in \mathbb{N}$.

2. Preliminaries

We need the following lemmas to derive our results.

Lemma 2.1. *[15] Let $\mathfrak{f}$ be a holomorphic function in Λ with the gap series expansion*

$$\mathfrak{f}(\zeta) = \sum_{k=1}^{\infty} a_k \zeta^{n_k}, \quad \zeta \in \Lambda,$$

where, for a constant $\lambda > 1$, the natural numbers $n_k,\ k \geq 1$ satisfy $\frac{n_{k+1}}{n_k} \geq \lambda$. Then $\mathfrak{f} \in \mathfrak{B}_\alpha$ if and only if $\limsup_{k\to\infty} |a_k| n_k^{1-\alpha} < \infty$.

Lemma 2.2. *[18] Assume that $\{n_k\}$ is an increasing sequence of positive integers satisfying $\frac{n_{k+1}}{n_k} \geq \lambda > 1$ for all $k \in N$. Let $0 < p < \infty$. Then there are two positive constants, C_1 and C_2, depending only on p and λ such that*

$$C_1\Big(\sum_{k=0}^{\infty} |a_k|^2\Big)^{\frac{1}{2}} \leq \Big(\frac{1}{2\pi}\int_0^{2\pi} \Big|\sum_{k=0}^{\infty} a_k e^{in_k\vartheta}\Big| d\vartheta\Big)^{\frac{1}{p}} \leq C_2\Big(\sum_{k=0}^{\infty} |a_k|^2\Big)^{\frac{1}{2}}.$$

Lemma 2.3. *[8] Let $0<\beta<\infty$, for $\mathfrak{f} \in \mathcal{Z}_\beta$.*

1. *For $0<\beta<1$. $|\mathfrak{f}\,'(\zeta)| \leq \frac{2}{1-\beta}\|\mathfrak{f}\|_{\mathcal{Z}_\beta}$ and $|\mathfrak{f}(\zeta)| \leq \frac{2}{1-\beta}\|\mathfrak{f}\|_{\mathcal{Z}_\beta}$;*

2. *For $\beta = 1$. $|\mathfrak{f}\,'(\zeta)| \leq 2\ \ln \frac{2}{1-|\zeta|^2}\|\mathfrak{f}\|_{\mathcal{Z}_\beta}$ and $|\mathfrak{f}(\zeta)| \leq \|\mathfrak{f}\|_{\mathcal{Z}_\beta}$;*

3. *For $\beta > 1$. $|\mathfrak{f}\,'(\zeta)| \leq \frac{2\|\mathfrak{f}\|_{\mathcal{Z}_\beta}}{(\beta-1)\ (1-|\zeta|^2)^{\beta-1}}$;*

4. *For* $1 < \beta < 2$. $|\mathfrak{f}\,(\zeta)| \leq \frac{2}{(\beta-1)\,(\beta-2)}\|\mathfrak{f}\|_{\mathcal{Z}_\beta}$;

5. *For* $\beta = 2$. $|\mathfrak{f}\,(\zeta)| \leq 2\|\mathfrak{f}\|_{\mathcal{Z}_\beta} \ln \frac{2}{1-|\zeta|^2}$;

6. *For* $\beta > 2$. $|\mathfrak{f}\,(\zeta)| \leq \frac{2\|\mathfrak{f}\|_{\mathcal{Z}_\beta}}{(\beta-1)\,(\beta-2)(1-|\zeta|^2)^{\beta-2}}$.

The following lemma follows by standard arguments of the corresponding results in [12].

LEMMA 2.4. *Let* $0 < \beta < \infty$ *and* K *be a nonnegative non decreasing continuous function on* $[0, \infty)$. *Assume* φ *is an analytic mapping from* Λ *into itself. Then* $C_{\varphi}^{\mathfrak{g},n} : \mathcal{Z}_\beta \to Q_K(p,q)$ *is compact if and only if for any bounded sequence* $\{n_\imath\} \in \mathcal{Z}_\beta$ *which converges to zero uniformly on compact subsets of* Λ, $\lim_{\imath\to\infty} ||C_{\varphi}^{\mathfrak{g},n} n_\imath||_{Q_K(p,q)} = 0$.

The proof of following lemma is similar to the proof of Lemma 1 in [10].

LEMMA 2.5. *Let* $0 < \beta < \infty$ *and* K *be a nonnegative non decreasing continuous function on* $[0, \infty)$. *Assume* φ *is an analytic mapping from* Λ *into itself. If* $C_{\varphi}^{\mathfrak{g},n} : \mathcal{Z}_\beta \to Q_K(p,q)$ *is compact, then for any* ε *there exists a* δ, $0 < \delta < 1$ *such that for all* $\mathfrak{f} \in \mathcal{Z}_\beta$,

$$\sup_{\sigma\in\Lambda} \int_{|\varphi(\zeta)>r|} |\mathfrak{f}'(\varphi(\zeta))|^p |\mathfrak{g}(\zeta)|^{p(n-1)} (1-|\zeta|^2)^q K(g(\zeta,\sigma))dA(\zeta) < \varepsilon$$

holds whenever $\delta < r < 1$.

3. The boundedness of $C_{\varphi}^{\mathfrak{g},n} : \mathcal{Z}_\beta \to Q_K(p,q)$

3.1. The case $0 < \beta < 1$.

THEOREM 3.1. *Let* $0 < \beta < 1$, *and* $\mathfrak{g} \in H(\Lambda)$. *Let* $\varphi \in \Lambda$ *be an analytic self-mapping. Then,* $C_{\varphi}^{\mathfrak{g},n} : \mathcal{Z}_\beta \to Q_K(p,q)$ *is bounded if and only if*

$$(1) \qquad l_1 : \;=\; \sup_{\sigma\in\Lambda} \int_\Lambda |\mathfrak{g}(\zeta)|^{p(n-1)} (1-|\zeta|^2)^q K(g(\zeta,\sigma))dA(\zeta) < \infty.$$

Proof. Assume that (1) holds and let $\mathfrak{f} \in \mathcal{Z}_\beta$. By Lemma 2.3 (1) we have

$$\mathfrak{f}'(\zeta) \leq \frac{2}{1-\beta}\|\mathfrak{f}\|_{\mathcal{Z}_\beta}.$$

Hence

$$\begin{aligned}
||C_{\varphi}^{\mathfrak{g},n}\mathfrak{f}||^p_{Q_K(p,q)} &= \sup_{\sigma\in\Lambda}\int_{\Lambda}|(C_{\varphi}^{\mathfrak{g},n}\mathfrak{f})'(\zeta)|^p(1-|\zeta|^2)^qK(g(\zeta,\sigma))dA(\zeta)\\
&= \sup_{\sigma\in\Lambda}\int_{\Lambda}|\mathfrak{f}(\varphi)'(\zeta)|^p|\mathfrak{g}(\zeta)|^{p(n-1)}(1-|\zeta|^2)^qK(g(\zeta,\sigma))dA(\zeta)\\
&\le \left(\frac{2}{1-\beta}\right)^p\|\mathfrak{f}\|^p_{\mathcal{Z}_\beta}\sup_{\sigma\in\Lambda}\int_{\Lambda}|\mathfrak{g}(\zeta)|^{p(n-1)}(1-|\zeta|^2)^qK(g(\zeta,\sigma))dA(\zeta)\\
&= C\|\mathfrak{f}\|^p_{\mathcal{Z}_\beta}\sup_{\sigma\in\Lambda}\int_{\Lambda}|\mathfrak{g}(\zeta)|^{p(n-1)}(1-|\zeta|^2)^qK(g(\zeta,\sigma))dA(\zeta)\\
&= C\|\mathfrak{f}\|^p_{\mathcal{Z}_\beta}\cdot l_1\\
&< \infty.
\end{aligned}$$

It follows that $C_{\varphi}^{\mathfrak{g},n}:\mathcal{Z}_\beta\to Q_K(p,q)$ is bounded.

Conversely, we assume that $C_{\varphi}^{\mathfrak{g},n}:\mathcal{Z}_\beta\to Q_K(p,q)$ is bounded. Let $\mathfrak{h}(\zeta)=\zeta\in\mathcal{Z}_\beta$, then

$$\begin{aligned}
\infty &> ||C_{\varphi}^{\mathfrak{g},n}\mathfrak{h}||^p_{Q_K(p,q)}\\
&= \sup_{\sigma\in\Lambda}\int_{\Lambda}|(C_{\varphi}^{\mathfrak{g},n}\mathfrak{h})'(\zeta)|^p(1-|\zeta|^2)^qK(g(\zeta,\sigma))dA(\zeta)\\
&= \sup_{\sigma\in\Lambda}\int_{\Lambda}|\mathfrak{h}(\varphi)'(\zeta)|^p|\mathfrak{g}(\zeta)|^{p(n-1)}(1-|\zeta|^2)^qK(g(\zeta,\sigma))dA(\zeta)\\
&= \sup_{\sigma\in\Lambda}\int_{\Lambda}|\mathfrak{g}(\zeta)|^{p(n-1)}(1-|\zeta|^2)^qK(g(\zeta,\sigma))dA(\zeta)\\
&= l_1.
\end{aligned}$$

Then (1) holds. The proof of this theorem is completed.

3.2. The case $\beta = 1$.

THEOREM 3.2. *Let $\beta=1$ and $\mathfrak{g}\in H(\Lambda)$. Let $\varphi\in\Lambda$ be an analytic self-mapping. Then*

(a) $C_{\varphi}^{\mathfrak{g},n}:\mathcal{Z}_\beta\to Q_K(p,q)$ *is bounded if*

$$l_2 := \sup_{\sigma\in\Lambda}\int_{\Lambda}|\mathfrak{g}(\zeta)|^{p(n-1)}\left(\ln\frac{2}{1-|\varphi(\zeta)|^2}\right)^p(1-|\zeta|^2)^qK(g(\zeta,\sigma))dA(\zeta)<\infty.$$

(b) *If* $C_{\varphi}^{\mathfrak{g},n}:\mathcal{Z}_\beta\to Q_K(p,q)$ *is bounded, then*

$$l_3 := \sup_{\sigma\in\Lambda}\int_{\Lambda}|\mathfrak{g}(\zeta)|^P(n-1)\ln\frac{2}{1-|\varphi(\zeta)|^2}(1-|\zeta|^2)^qK(g(\zeta,\sigma))dA(\zeta)<\infty.$$

Proof. (a) Let $\mathfrak{f}\in\mathcal{Z}_\beta$, by Lemma 2.3 (2), we have

$$|\mathfrak{f}'(\zeta)|\le 2\ \ln\frac{2}{1-|\zeta|^2}\ \ ||\mathfrak{f}||_{\mathcal{Z}_\beta}.$$

Then we have

$$\begin{aligned}
||C_\varphi^{\mathfrak{g},n}\mathfrak{f}||^p_{Q_K(p,q)} &= \sup_{\sigma\in\Lambda}\int_\Lambda |(C_\varphi^{\mathfrak{g},n}\mathfrak{f})'(\zeta)|^p(1-|\zeta|^2)^q K(g(\zeta,\sigma))dA(\zeta)\\
&= \sup_{\sigma\in\Lambda}\int_\Lambda |\mathfrak{f}(\varphi)'(\zeta)|^p|\mathfrak{g}(\zeta)|^{p(n-1)}(1-|\zeta|^2)^q K(g(\zeta,\sigma))dA(\zeta)\\
&\leq 2^p\|\mathfrak{f}\|^p_{\mathcal{Z}_\beta}\sup_{\sigma\in\Lambda}\int_\Lambda |\mathfrak{g}(\zeta)|^{p(n-1)}\left(\ln\frac{2}{1-|\zeta|^2}\right)^p(1-|\zeta|^2)^q K(g(\zeta,\sigma))dA(\zeta)\\
&= C\|\mathfrak{f}\|^p_{\mathcal{Z}_\beta}\sup_{\sigma\in\Lambda}\int_\Lambda |\mathfrak{g}(\zeta)|^{p(n-1)}(1-|\zeta|^2)^q K(g(\zeta,\sigma))dA(\zeta)\\
&= C\|\mathfrak{f}\|^p_{\mathcal{Z}_\beta}\cdot l_2\\
&< \infty.
\end{aligned}$$

Hence $C_\varphi^{\mathfrak{g},n}:\mathcal{Z}_\beta\to Q_K(p,q)$ is bounded.

(b) Assume that $C_\varphi^{\mathfrak{g},n}:\mathcal{Z}_\beta\to Q_K(p,q)$ is bounded. Let $h(\zeta)=\zeta\in\mathcal{Z}_\beta$. Then

$$\sup_{\sigma\in\Lambda}\int_\Lambda |\mathfrak{g}(\zeta)|^{p(n-1)}(1-|\zeta|^2)^q K(g(\zeta,\sigma))dA(\zeta)<\infty.$$

Hence

$$\begin{aligned}
&\sup_{\sigma\in\Lambda}\int_{|\varphi(\zeta)|\leq\frac{1}{\sqrt{2}}} |\mathfrak{g}(\zeta)|^{p(n-1)}\ln\frac{1}{1-|\varphi(\zeta)|^2}(1-|\zeta|^2)^q K(g(\zeta,\sigma))dA(\zeta)\\
&\leq \ln 2\sup_{\sigma\in\Lambda}\int_{|\varphi(\zeta)|\leq\frac{1}{\sqrt{2}}} |\mathfrak{g}(\zeta)|^{p(n-1)}(1-|\zeta|^2)^q K(g(\zeta,\sigma))dA(\zeta)\\
&\leq \ln 2\sup_{\sigma\in\Lambda}\int_\Lambda |\mathfrak{g}(\zeta)|^{p(n-1)}(1-|\zeta|^2)^q K(g(\zeta,\sigma))dA(\zeta)<\infty. \qquad (2)
\end{aligned}$$

Now, let

$$\mathfrak{t}(\zeta)=\sum_{k=0}^{k}\frac{1}{2^k+1}\cdot\zeta^{2^k+1}$$

where $\zeta\in\lambda$, such that $|\zeta|=r\leq\frac{1}{\sqrt{2}}$. Then

$$\mathfrak{t}'(\zeta)=\sum_{k=0}^{k}\zeta^{2^k}\in\mathfrak{B}_\beta.$$

By Lemma 2.1, from the relationship of Bloch space and Zygmund space, $\mathfrak{t}\in\mathcal{Z}$. Let

$$\mathfrak{t}_\vartheta(\zeta)=t(e^{j\vartheta}\zeta)\sum_{k=0}^{k=\infty}\frac{1}{2^k+1}\cdot(e^{j\vartheta}\zeta)^{2^k+1}.$$

Then we have $\mathfrak{t}_\vartheta \in \mathcal{Z}_\beta$. Thus

$$\begin{aligned} \infty &> ||c_\varphi^{g,n}||^p||\mathfrak{t}_\vartheta||^p_{\mathcal{Z}_\beta} \\ &\geq \sup_{\sigma\in\Lambda}\int_\Lambda |(c_\varphi^{g,n}\mathfrak{t}_\vartheta)'|^p(1-|\zeta|^2)^qK(g(\zeta,\sigma))dA(\zeta) \\ &\geq \sup_{\sigma\in\Lambda}\int\limits_{|\varphi(\zeta)|>\frac{1}{\sqrt{2}}}\left|\sum_{k=0}^{k=\infty}e^{j(2^k+1)\vartheta}\varphi^{2^k}(\zeta)\right|^p|\mathfrak{g}(\zeta)|^{p(n-1)}(1-|\zeta|^2)^qK(g(\zeta,\sigma))dA(\zeta). \end{aligned} \tag{3}$$

So

$$\begin{aligned} ||c_\varphi^{g,n}||^p||\mathfrak{t}_\vartheta||^p_{\mathcal{Z}_\beta} &= \frac{1}{2\pi}\int_0^{2\pi}||c_\varphi^{g,n}||^p||t||^pd\vartheta \\ &= \frac{1}{2\pi}\int_0^{2\pi}||c_\varphi^{g,n}||^p||t_\vartheta||^pd\vartheta. \end{aligned}$$

Using Fubini's theorem, Lemma 2.2 and (3), we have

$$\begin{aligned} \infty &> \frac{1}{2\pi}\int_0^{2\pi}||c_\varphi^{g,n}||^p||t_\vartheta||^pd\vartheta \\ &\geq \frac{1}{2\pi}\int_0^{2\pi}\sup_{\sigma\in\Lambda}\int_{|\varphi(\zeta)|>\frac{1}{\sqrt{2}}}\left|\sum_{k=0}^{k=\infty}e^{j(2^k+1)\vartheta}\varphi^{2^k}(\zeta)\right|^p|\mathfrak{g}(\zeta)|^{p(n-1)}(1-|\zeta|^2)^qK(g(\zeta,\sigma))dA(\zeta) \\ &\geq \sup_{\sigma\in\Lambda}\int_{|\varphi(\zeta)|>\frac{1}{\sqrt{2}}}\left\{\frac{1}{2\pi}\int_0^{2\pi}\left|\sum_{k=0}^{k=\infty}e^{j(2^k+1)\vartheta}\varphi^{2^k}(\zeta)\right|^pdA(\zeta)\right\}\mathfrak{g}(\zeta)|^{p(n-1)}(1-|\zeta|^2)^qK(g(\zeta,\sigma)) \\ &\geq C\sup_{\sigma\in\Lambda}\int_{|\varphi(\zeta)|>\frac{1}{\sqrt{2}}}\left(\sum_{k=0}^{k=\infty}|\varphi(\zeta)|^{2^k+1}\right)|\mathfrak{g}(\zeta)|^{p(n-1)}(1-|\zeta|^2)^qK(g(\zeta,\sigma)). \end{aligned}$$

For $0<r<1$,

$$\ln\frac{1}{1-r^2}=\sum_{k=1}^{\infty}\frac{r^{2k}}{k}=\sum_{k=0}^{\infty}\sum_{\mathfrak{j}=2^k}^{2^{k+1}-1}\frac{r^{2\mathfrak{j}}}{\mathfrak{j}}\leq\sum_{k=0}^{\infty}r^{2^{k+1}}.$$

Then, we have

$$\sum_{k=0}^{k=\infty}|\varphi(\zeta)|^{2^{k+1}}\leq\ln\frac{1}{1-|\varphi(\zeta)|^2}.$$

Thus

$$\begin{aligned} \infty &> \frac{1}{2\pi}\int_0^{2\pi}||c_\varphi^{g,n}||^p||t_\vartheta||^pd\vartheta \\ &\geq \sup_{\sigma\in\Lambda}\int_{|\varphi(\zeta)|>\frac{1}{\sqrt{2}}}|\mathfrak{g}(\zeta)|^{p(n-1)}\ln\frac{1}{1-|\varphi(\zeta)|^2}(1-|\zeta|^2)^qK(g(\zeta,\sigma))dA(\zeta). \end{aligned} \tag{4}$$

By using (2) and (4), l_3 holds.

3.3. The case $\beta > 1$.

THEOREM 3.3. *Let $\beta > 1$, and $\mathfrak{g} \in H(\Lambda)$. Let $\varphi \in \Lambda$ be an analytic self-mapping. Then $C_\varphi^{\mathfrak{g},n} : \mathcal{Z}_\beta \to Q_K(p,q)$ is bounded if*

$$\sup_{\sigma\in\Lambda}\int_\Lambda \frac{|\mathfrak{g}(\zeta)|^{p(n-1)}}{(1-|\varphi(\zeta)|^2)^{p(\beta-1)}}(1-|\zeta|^2)^q K(g(\zeta,\sigma))dA(\zeta) < \infty.$$

Proof. Let $\mathfrak{f} \in \mathcal{Z}_\beta$. Then by Lemma 2.3 (3), we have

$$|\mathfrak{f}'(\zeta)| \le \frac{2}{1-\beta}\,\frac{||\mathfrak{f}||_{\mathcal{Z}_\beta}}{(1-|\zeta|^2)^{\beta-1}}.$$

Then we have

$$\begin{aligned}
||C_\varphi^{\mathfrak{g},n}\mathfrak{f}||^p_{Q_K(p,q)} &= \sup_{\sigma\in\Lambda}\int_\Lambda |(C_\varphi^{\mathfrak{g},n}\mathfrak{f})'(\zeta)|^p(1-|\zeta|^2)^q K(g(\zeta,\sigma))dA(\zeta)\\
&= \sup_{\sigma\in\Lambda}\int_\Lambda |\mathfrak{f}(\varphi)'(\zeta)|^p|\mathfrak{g}(\zeta)|^{p(n-1)}(1-|\zeta|^2)^q K(g(\zeta,\sigma))dA(\zeta)\\
&\le \frac{2^p||\mathfrak{f}||^p_{\mathcal{Z}_\beta}}{(1-\beta)^p}\sup_{\sigma\in\Lambda}\int_\Lambda \frac{|\mathfrak{g}(\zeta)|^{p(n-1)}}{(1-|\varphi(\zeta)|^2)^{p(\beta-1)}}(1-|\zeta|^2)^q K(g(\zeta,\sigma))dA(\zeta)\\
&= C\|\mathfrak{f}\|^p_{\mathcal{Z}_\beta}\sup_{\sigma\in\Lambda}\int_\Lambda \frac{|\mathfrak{g}(\zeta)|^{p(n-1)}}{(1-|\varphi(\zeta)|^2)^{p(\beta-1)}}(1-|\zeta|^2)^q K(g(\zeta,\sigma))dA(\zeta)\\
&< \infty.
\end{aligned}$$

Hence $C_\varphi^{\mathfrak{g},n} : \mathcal{Z}_\beta \to Q_K(p,q)$ is bounded.

4. The compactness of $C_\varphi^{\mathfrak{g},n} : \mathcal{Z}_\beta \to Q_K(p,q)$

THEOREM 4.1. *Let $0 < \beta < 1$, and $\mathfrak{g} \in H(\Lambda)$. Let $\varphi \in \Lambda$ be an analytic self-mapping. Then $C_\varphi^{\mathfrak{g},n} : \mathcal{Z}_\beta \to Q_K(p,q)$ is compact if and only if*

$$\sup_{\sigma\in\Lambda}\int_\Lambda |\mathfrak{g}(\zeta)|^{p(n-1)}(1-|\zeta|^2)^q K(g(\zeta,\sigma))dA(\zeta) < \infty \tag{5}$$

and

$$\lim_{r\to 1}\sup_{\sigma\in\Lambda}\int_{|\varphi(\zeta)|>r} |\mathfrak{g}(\zeta)|^{p(n-1)}(1-|\zeta|^2)^q K(g(\zeta,\sigma))dA(\zeta) = 0. \tag{6}$$

Proof. Let $\{n_\imath\}$ be a bounded sequence in $\mathcal{Z}_\beta$, which converges to 0 uniformly on compact subsets of Λ. We need to prove that $||C_\varphi^{\mathfrak{g},n}||_{Q_K(p,q)} \to 0$, $\imath \to \infty$. From (6), we have that, for any $\varepsilon > 0$, there exists an r, $0 < r < 1$ such that

$$\sup_{\sigma\in\Lambda}\int_{|\varphi(\zeta)|>r} |\mathfrak{g}(\zeta)|^{p(n-1)}(1-|\zeta|^2)^q K(g(\zeta,\sigma))dA(\zeta) < \varepsilon. \tag{7}$$

Using Lemma 2.3 (1), we have

$$
\begin{aligned}
&\sup_{\sigma\in\Lambda}\int_{\Lambda}|(C_{\varphi}^{\mathfrak{g},n}n_\imath)'(\zeta)|^p(1-|\zeta|^2)^qK(g(\zeta,\sigma))dA(\zeta)\\
\le\ &\sup_{\sigma\in\Lambda}\int_{|\varphi(\zeta)|>r}|n_\imath'(\varphi(\zeta))|^p|\mathfrak{g}(\zeta)|^{p(n-1)}(1-|\zeta|^2)^qK(g(\zeta,\sigma))dA(\zeta)\\
+\ &\sup_{\sigma\in\Lambda}\int_{|\varphi(\zeta)|\le r}|n_\imath'(\varphi(\zeta))|^p|\mathfrak{g}(\zeta)|^{p(n-1)}(1-|\zeta|^2)^qK(g(\zeta,\sigma))dA(\zeta)\\
\le\ &C||n_\imath||_{\mathcal{Z}_\beta}^p\sup_{\sigma\in\Lambda}\int_{|\varphi(\zeta)|>r}\mathfrak{g}(\zeta)|^{p(n-1)}(1-|\zeta|^2)^qK(g(\zeta,\sigma))dA(\zeta)\\
+\ &\sup_{|\varpi|\le r}|n_\imath'(\varpi)|^p\sup_{\sigma\in\Lambda}\int_{\sigma\in\Lambda}|\mathfrak{g}(\zeta)|^{p(n-1)}(1-|\zeta|^2)^qK(g(\zeta,\sigma))dA(\zeta).
\end{aligned}
$$

By Cauchy's estimate, $\{n_\imath'\}$ also converges to 0 uniformly on compact subsets of Λ, then

$$\sup_{|\varpi|\le r}|n_\imath'(\varpi)|^p\to 0,\imath\to\infty.$$

Hence by (5) and (7), $||C_{\varphi}^{\mathfrak{g},n}n_\imath||_{Q_K(p,q)}^p\to 0$, $\imath\to\infty$. By Lemma 2.4, $C_{\varphi}^{\mathfrak{g},n}:\mathcal{Z}_\beta\to Q_K(p,q)$ is compact.

Conversely, assume that $C_{\varphi}^{\mathfrak{g},n}:\mathcal{Z}_\beta\to Q_K(p,q)$ is compact. Let $\hbar(\zeta)=\zeta\in\mathcal{Z}_\beta$, then (5) holds. Using Lemma 2.5 we have

$$\sup_{\sigma\in\Lambda}\int_{|\varphi(\zeta)|>r}|\mathfrak{f}'(\varphi(\zeta))|^p|\mathfrak{g}(\zeta)|^{p(n-1)}(1-|\zeta|^2)^qK(g(\zeta,\sigma))dA(\zeta)<\varepsilon. \tag{8}$$

Let $\mathfrak{f}(\varphi(\zeta))=\zeta\in\mathcal{Z}_\beta$ in (8), then

$$\sup_{\sigma\in\Lambda}\int_{|\varphi(\zeta)|>r}|\mathfrak{g}(\zeta)|^{p(n-1)}(1-|\zeta|^2)^qK(g(\zeta,\sigma))dA(\zeta)<\varepsilon.$$

THEOREM 4.2. *Let $\beta=1$, and $\mathfrak{g}\in H(\Lambda)$. Let $\varphi\in\Lambda$ be an analytic-self mapping. If*

$$\sup_{\sigma\in\Lambda}\int_{\Lambda}|\mathfrak{g}(\zeta)|^{p(n-1)}\ (1-|\zeta|^2)^qK(g(\zeta,\sigma))dA(\zeta)<\infty \tag{9}$$

and

$$\lim_{r\to1}\sup_{\sigma\in\Lambda}\int_{|\varphi(\zeta)|>r}|\mathfrak{g}(\zeta)|^{p(n-1)}\big(\ln\frac{2}{1-|\varphi(\zeta)|^2}\big)^p(1-|\zeta|^2)^qK(g(\zeta,\sigma))dA(\zeta)=0, \tag{10}$$

then $C_{\varphi}^{\mathfrak{g},n}:\mathcal{Z}_\beta\to Q_K(p,q)$ is compact.

Proof. Let $\{n_\imath\}$ be a bounded sequence in $\mathcal{Z}_\beta$ which converges to 0 uniformly on compact subsets of Λ. Using Lemma 2.4, we need to prove that $||C_{\varphi}^{\mathfrak{g},n}||_{Q_K(p,q)}\to 0$, $\imath\to\infty$. From (10), for any $\varepsilon>0$, there exists an r, $0<r<1$ such that,

$$\sup_{\sigma\in\Lambda}\int_{|\varphi(\zeta)|>r}|\mathfrak{g}(\zeta)|^{p(n-1)}\left(\ln\frac{2}{1-|\varphi(\zeta)|^2}\right)^p(1-|\zeta|^2)^qK(g(\zeta,\sigma))dA(\zeta)<\varepsilon. \tag{11}$$

Using Lemma 2.3(2), we have

$$\begin{aligned}
&\sup_{\sigma\in\Lambda}\int_{\Lambda}|(C_{\varphi}^{\mathfrak{g},n}n_{\imath})'(\zeta)|^{p}(1-|\zeta|^{2})^{q}K(g(\zeta,\sigma))dA(\zeta)\\
\leq\ &\sup_{\sigma\in\Lambda}\int_{|\varphi(\zeta)|>r}|n_{\imath}'(\varphi(\zeta))|^{p}|\mathfrak{g}(\zeta)|^{p(n-1)}(1-|\zeta|^{2})^{q}K(g(\zeta,\sigma))dA(\zeta)\\
+\ &\sup_{\sigma\in\Lambda}\int_{|\varphi(\zeta)|\leq r}|n_{\imath}'(\varphi(\zeta))|^{p}|\mathfrak{g}(\zeta)|^{p(n-1)}(1-|\zeta|^{2})^{q}K(g(\zeta,\sigma))dA(\zeta)\\
\leq\ &C||n_{\imath}||_{\mathcal{Z}_{\beta}}^{p}\sup_{\sigma\in\Lambda}\int_{|\varphi(\zeta)|>r}|\mathfrak{g}(\zeta)|^{p(n-1)}\left(\ln\frac{2}{1-|\varphi(\zeta)|^{2}}\right)^{p}(1-|\zeta|^{2})^{q}K(g(\zeta,\sigma))dA(\zeta)\\
+\ &\sup_{|\varpi|\leq r}|n_{\imath}'(\varpi)|^{p}\sup_{\sigma\in\Lambda}\int_{\sigma\in\Lambda}|\mathfrak{g}(\zeta)|^{p(n-1)}(1-|\zeta|^{2})^{q}K(g(\zeta,\sigma))dA(\zeta).
\end{aligned}$$

By Cauchy's estimate, $\{n_{\imath}'\}$ also converges to 0 uniformly on compact subsets of Λ. Then

$$\sup_{|\varpi|\leq r}|n_{\imath}'(\varpi)|^{p}\to 0,\imath\to\infty. \tag{12}$$

Hence, by (9), (11) and (12), $||C_{\varphi}^{\mathfrak{g},n}n_{\imath}||_{Q_K(p,q)}^{p}\to 0,\ \imath\to\infty$. By Lemma 2.4, $C_{\varphi}^{\mathfrak{g},n}:\mathcal{Z}_{\beta}\to Q_K(p,q)$ is compact.

THEOREM 4.3. *Let $\beta>1$, and $\mathfrak{g}\in H(\Lambda)$. Let $\varphi\in\Lambda$ be an analytic self-mapping. If*

$$\sup_{\sigma\in\Lambda}\int_{\Lambda}|\mathfrak{g}(\zeta)|^{p(n-1)}(1-|\zeta|^{2})^{q}K(g(\zeta,\sigma))dA(\zeta)<\infty \tag{13}$$

and

$$\sup_{\sigma\in\Lambda}\int_{|\varphi(\zeta)|>r}\frac{|\mathfrak{g}(\zeta)|^{p(n-1)}}{(1-|\varphi(\zeta)|^{2})^{p(\beta-1)}}(1-|\zeta|^{2})^{q}K(g(\zeta,\sigma))dA(\zeta)=0, \tag{14}$$

then $C_{\varphi}^{\mathfrak{g},n}:\mathcal{Z}_{\beta}\to Q_K(p,q)$ is compact.

Proof. Let $\{n_{\imath}\}$ be a bounded sequence in $\mathcal{Z}_{\beta}$ which converges to 0 uniformly on compact subsets of Λ. Using Lemma 2.4, we need to prove that $||C_{\varphi}^{\mathfrak{g},n}||_{Q_K(p,q)}\to 0$, $\imath\to\infty$. From (14), for any $\varepsilon>0$, there exists an r, $0<r<1$ such that,

$$\sup_{\sigma\in\Lambda}\int_{|\varphi(\zeta)|>r}\frac{|\mathfrak{g}(\zeta)|^{p(n-1)}}{(1-|\varphi(\zeta)|^{2})^{p(\beta-1)}}(1-|\zeta|^{2})^{q}K(g(\zeta,\sigma))dA(\zeta)<\varepsilon. \tag{15}$$

By Lemma 2.3 (3),

$$\begin{aligned}
&\sup_{\sigma\in\Lambda}\int_{\Lambda}|(C_{\varphi}^{\mathfrak{g},n}n_{\imath})'(\zeta)|^p(1-|\zeta|^2)^qK(g(\zeta,\sigma))dA(\zeta)\\
\leq\ &\sup_{\sigma\in\Lambda}\int_{|\varphi(\zeta)|>r}|n_{\imath}'(\varphi(\zeta))|^p|\mathfrak{g}(\zeta)|^{p(n-1)}(1-|\zeta|^2)^qK(g(\zeta,\sigma))dA(\zeta)\\
+\ &\sup_{\sigma\in\Lambda}\int_{|\varphi(\zeta)|\leq r}|n_{\imath}'(\varphi(\zeta))|^p|\mathfrak{g}(\zeta)|^{p(n-1)}(1-|\zeta|^2)^qK(g(\zeta,\sigma))dA(\zeta)\\
\leq\ &C||n_{\imath}||_{\mathcal{Z}_\beta}^p\sup_{\sigma\in\Lambda}\int_{|\varphi(\zeta)|>r}\frac{|\mathfrak{g}(\zeta)|^{p(n-1)}}{(1-|\varphi(\zeta)|^2)^{p(\beta-1)}}(1-|\zeta|^2)^qK(g(\zeta,\sigma))dA(\zeta)\\
+\ &\sup_{|\varpi|\leq r}|n_{\imath}'(\varpi)|^p\sup_{\sigma\in\Lambda}\int_{\sigma\in\Lambda}|\mathfrak{g}(\zeta)|^{p(n-1)}(1-|\zeta|^2)^qK(g(\zeta,\sigma))dA(\zeta).
\end{aligned}$$

By Cauchy's estimate, $\{n_{\imath}'\}$ also converges to 0 uniformly on compact subsets of Λ. Then

$$\sup_{|\varpi|\leq r}|n_{\imath}'(\varpi)|^p\to 0,\imath\to\infty. \tag{16}$$

Hence, by (13), (15) and (16), $||C_{\varphi}^{\mathfrak{g},n}n_{\imath}||_{Q_K(p,q)}^p\to 0,\ \imath\to\infty$. By Lemma 2.4, $C_{\varphi}^{\mathfrak{g},n}:\mathcal{Z}_\beta\to Q_K(p,q)$ is compact.

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