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MULTI-FUZZY SEQUENCES IN METRIC SPACES

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Abstract. This paper introduces the concept of multi-fuzzy sequences and studies convergence within a metric space. It presents key definitions and illustrative examples, particularly focusing on the convergence of multi-fuzzy sequences, multi-fuzzy bounded sequences and multi-fuzzy Cauchy sequences. Theorems are provided to establish properties related to the uniqueness of limits and the relationships between boundedness and convergence. Furthermore, the theorems and results demonstrate connections among crisp sequences, multi-fuzzy sequences and multi-fuzzy Cauchy sequences. This article lays the groundwork for understanding the behaviour and properties of multi-fuzzy sequences.

1. Introduction

Fuzzy set theory was introduced by Zadeh in 1965 [23] for handling uncertainty and imprecision in various mathematical and computational contexts. Similarly, multiset theory was explored by Knuth in 1969, allows elements to repeat, unlike traditional set theory. For instance, the collection of elements $\{3, 3, 5, 5, 2, 3, 4, 4\}$ is not considered a set according to Cantor's set theory, as each element in this theory can appear in the set only once. Therefore, the membership (characteristic) function of a multiset takes on non-negative integer values. Subsequently W.D. Blizard [4] gave major contribution to multiset theory. Yager [22] introduced the notion of fuzzy bag in 1986, Miyamoto([7], [8]) renamed this concept as fuzzy multiset. Formally, a fuzzy multiset in some universal set X is a multiset in X $\times$ [0, 1].

Sabu Sebastian and T V Ramakrishnan ([12], [17], [19], [20], [21]) developed multi-fuzzy sets, an extension of fuzzy set, using multi dimensional membership functions. We introduced some similarity measures on multi-fuzzy set [11]. Many authors defined and studied the concept of fyzzy metric space and related concepts in different ways([2], [3], [5], [6], [13]). A multi-fuzzy extension of crisp functions is also developed by Sabu Sebastian and T. V. Ramakrishnan [18]. Specifically, we study multi-fuzzy extensions by utilizing fuzzy matrices as bridge functions [10]. Various types of fuzzy convergence and fuzzy Cauchy sequences in a fuzzy metric space have been studied in [1] , [2] and [14]. Fuzzy boundedness in a fuzzy metric spaces is introduced in [24].

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The concept of fuzzy sequence in a metric space is defined by M. Muthukumari, A. Nagarajan and M. Murugalingam in [9].

In this paper, we explore the concept of multi-fuzzy sets and sequences, investigating their definitions, properties and applications in the context of metric spaces. We build upon foundational concepts, such as fuzzy sequences and convergence, extending them to the multidimensional setting of multi-fuzzy sequences in a metric space. The foundation is laid in Section 2, where we establish essential definitions and terminologies. Section 3 introduces the core concept of our paper, multi-fuzzy sequences in metric spaces. This section bridges the gap between fuzzy sequences and multi-fuzzy sets.

2. Preliminary

In this paper, we use the following definitions and terminologies.

DEFINITION 2.1. ([15], [16]) Let l be a positive integer. A multi-fuzzy set μ in a universal set X is a set of ordered $(l+1)$-tuples

$$\mu = \{\langle x,\, \mu_1(x),\, \mu_2(x),\, \dots, \mu_l(x)\rangle : \, x \in X\}$$

where each μ_i is a function from a universal set X in to $[0,1]$. The positive integer l is called the dimension of μ. The collection of all multi-fuzzy sets of dimension l is denoted by $M^lFS(X)$.

DEFINITION 2.2. [9] Let X be a non-empty set. A fuzzy set on $\mathbb{N} \times X$ is called a fuzzy sequence in X, that is, $\mu : \mathbb{N} \times X \to [0,1]$ is called a fuzzy sequence in X.

Here after the ordinary sequence, which is a mapping from $\mathbb{N}$ to Xwill be named the crisp sequence. In [9] a crisp sequence f in X is identified with a fuzzy-sequence $\mu_f : \mathbb{N} \times X \to [0,1]$, given by

$$\mu_f(n,x) = \begin{cases} 1, & if\, f(n) = x \\ 0, & otherwise. \end{cases}$$

DEFINITION 2.3. [9] Let (X,d) be a metric space and let μ be a fuzzy sequence on X. Let $\alpha \in (0,1]$ and $a \in X$. μ is said to converge to a at a level α, if
1) for each $n \in \mathbb{N}$, there exists an element $x \in X$ such that $\mu(n,x) \geq \alpha$;
2) given $\epsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that $d(x,a) < \epsilon$, for all $n \geq n_0$ and for all x with $\mu(n,x) \geq \alpha$.

In this paper, for $\alpha, \beta \in [0,1]^r$, with $\alpha = [\alpha_1, \alpha_2, ..., \alpha_r]$ and $\beta = [\beta_1, \beta_2, ..., \beta_r]$, by $\beta \geq \alpha$ we mean $\beta_i \geq \alpha_i, \forall i = 1, 2, ..., r$.

3. Multi-fuzzy Sequences in a Metric Space

DEFINITION 3.1. Let X be a non-empty set, and let r be a positive integer. A multi-fuzzy set μ of dimension r on $\mathbb{N} \times X$ is called a multi-fuzzy sequence of dimension r in X. That is, $\mu : \mathbb{N} \times X \to [0,1]^r$ given by
$\mu(n,x) = (\mu_1(n,x),\, \mu_2(n,x),\, ...,\, \mu_r(n,x))\,,$ for $n \in \mathbb{N},\ x \in X,$where $\mu_i : \mathbb{N} \times X \to [0,1],$ for all $i = 1, 2, ..., r.$

Example 3.2. Let $X = \{0,1\}$. Define $\mu : \mathbb{N} \times X \to [0,1]^2$ by

$$\mu_1(n,0) = 0.3,\ \mu_1(n,1) = 0;$$
$$\mu_2(n,0) = 0.5,\ \mu_2(n,1) = 0.2.$$

Then μ is a multi-fuzzy sequence of dimension 2 in X.

Example 3.3. Let $X = \mathbb{N}$. Define $\mu : \mathbb{N} \times X \to [0,1]^r$ by

$$\mu_i(n,x) = \left(\frac{1}{n+x}\right)^i, \text{ for } n \in \mathbb{N},\ x \in X \text{ and for all } i = 1,2,...,r.$$

Clearly μ is a multi-fuzzy sequence of dimension r in $\mathbb{N}$.

Example 3.4. Let $X = \mathbb{R}$, then a multi-fuzzy sequence μ of dimension 2 in $\mathbb{R}$ is given by

$$\mu_1(n,x) = \frac{1}{e^{|nx|}},\ \ \mu_2(n,x) = \frac{1}{e^{2|nx|}},\ \text{ for } n \in \mathbb{N}, \text{ and } x \in X.$$

Definition 3.5. Let (X,d) be a metric space and let μ be a multi-fuzzy sequence of dimension r in X. Let $\alpha \in [0,1]^r \backslash \{(0,0,...,0)\}$ and $a \in X$. Then μ is said to converge to a at level α, if

1) for each $n \in \mathbb{N}$, the set $\{x \in X : \mu(n,x) \geq \alpha\}$ is non empty. That is, $\exists\, x \in X$ with $\mu(n,x) \geq \alpha$;

2) given $\varepsilon > 0$, $\exists k_\varepsilon \in \mathbb{N}$ such that $d(x,a) < \varepsilon$, $\forall n \geq k_\varepsilon$ and $\forall x \in X$ with $\mu(n,x) \geq \alpha$.

Example 3.6. Let $X = \{0,1\}$ as a subspace of $\mathbb{R}$ with the usual metric. Define $\mu : \mathbb{N} \times X \to [0,1]^2$ by

$$\mu_1(n,0) = 0.3,\ \mu_1(n,1) = 0;$$
$$\mu_2(n,0) = 0.5,\ \mu_2(n,1) = 0.2.$$

We can prove that the given multi-fuzzy sequence converges to 0 at level $\alpha \in [0,1]^2 \backslash \{(0,0)\}$, where $\alpha = [\alpha_1,\ \alpha_2]$, *with* $\alpha_1 \leq\ 0.3$ *and* $\alpha_2 \leq\ 0.5$. Because, for each $n \in \mathbb{N}$, *corresponding to* $0 \in X$, $\mu(n,0) = (0.3, 0.5) \geq \alpha$ and $\varepsilon > 0$, *and* $\forall n \in \mathbb{N}$, $x \in X$ *with* $\mu(n,x) \geq \alpha$ *implies* $x = 0$, *then* $d(x,0) = d(0,0) = |0-0| = 0$.

Example 3.7. Let $X = \mathbb{R}$ with the usual metric. Define $\mu : \mathbb{N} \times \mathbb{R} \to [0,1]^2$ by

$$\mu_i(n,x) = \left\{ \begin{array}{cc} 1 - \frac{1}{n+i}, & if\, x = \frac{1}{n} \\ 0, & otherwise \end{array} \right., \text{ for } i = 1,2.$$

We can prove that μ converges to 0 at any level α, where $\alpha = [\alpha_1,\ \alpha_2]$, *with* $\alpha_1 \leq 0.5$ *and* $\alpha_2 \leq \frac{2}{3}$. For

i) each $n \in \mathbb{N}$, $x = 1/n \in \mathbb{R}$ *and* $\mu(n,x) = \left(1 - \frac{1}{n+1},\, 1 - \frac{1}{n+2}\right) \geq \alpha$, *since* $1 - \frac{1}{n+1} \geq \frac{1}{2},\, 1 - \frac{1}{n+2} \geq \frac{2}{3}$;

ii) $\forall \varepsilon > 0$, $\exists n_0 \in \mathbb{N}$, such that $\frac{1}{\varepsilon} < n_0$. *Now* $\forall n \geq n_0$, $n > \varepsilon$ and $\mu(n,x) \geq \alpha$ implies $x = \frac{1}{n}$, otherwise, $\mu(n,x) = (0,0)$. *Hence* $d(x,0) = d(\frac{1}{n},0) = |\frac{1}{n} - 0| < \varepsilon$.

Remark 3.8. We can identify a crisp sequence f on a set X by the multi-fuzzy sequence μ^f of dimension r on X given by

$$\mu_i^f(n,x) = \left\{ \begin{array}{cc} 1\,, & if\, f(n) = x \\ 0\,, & otherwise \end{array} \right., \ i = 1,2,...,r.$$

Remark 3.9. A multi-fuzzy sequence of dimension one in a set X is a fuzzy sequence in X.

Remark 3.10. We say that a multi-fuzzy sequence μ of dimension r in a metric space (X, d) does not converge to $a \in X$ at level α if,
either $\exists n_0 \in \mathbb{N}$ such that the set $\{x \in X : \mu(n_0, x) \geq \alpha\} = \phi$. That is, $\forall x \in X$, $\mu(n_0, x) < \alpha$.
or
$\exists \varepsilon_0 > 0$, $\forall k \in \mathbb{N}$, $\exists n_k \geq k$ and $x_k \in X$ with $\mu(n_k, x_k) \geq \alpha$, but $d(x_k, a) \geq \varepsilon$.

Example 3.11. If we define $\mu : \mathbb{N} \times X \to [0,1]^2$, where $X = \{0, 1\}$ as a subspace of $\mathbb{R}$ with the usual metric, by

$$\mu_1(n, 0) = 0, \ \mu_1(n, 1) = 0;$$
$$\mu_2(n, 0) = 0.5, \ \mu_2(n, 1) = 0.2 \ .$$

Then μ does not converge to 0 or 1 at level α, where $\alpha = [\alpha_1, \alpha_2], with\ \alpha_1 > 0\ and\ \alpha_2 > \frac{1}{2}$. Because, for any $n \in \mathbb{N}$, we have, $\mu(n, 0) = (0, 0.5) < \alpha$ and $\mu(n, 1) = (0, 0.2) < \alpha$. That is, the set $\{x \in X : \mu(n, x) \geq \alpha\} = \phi$.

Theorem 3.12. *Uniqueness of limit*
If a multi-fuzzy sequence of dimension r in a metric space (X, d) converge to two elements $a, b \in X$ at the same level α, then $a = b$.

Proof. Let $\epsilon > 0$. Since μ converges to a at level α, for each $n \in \mathbb{N}$, $\exists x \in X$ *such that* $\mu(n, x) \geq \alpha$ and $\exists n_1 \in \mathbb{N}$, $\forall n \geq n_1$, $\mu(n, x) \geq \alpha$ implies $d(x, a) < \frac{\varepsilon}{2}$. Similarly, μ converges to b at level α, for each $n \in \mathbb{N}$, $\exists x \in X$ *such that* $\mu(n, x) \geq \alpha$ and $\exists n_2 \in \mathbb{N}$, $\forall n \geq n_2$, $\mu(n, x) \geq \alpha$ implies $d(x, b) < \frac{\varepsilon}{2}$. Now we choose $n \in \mathbb{N}$ such that $n \geq \max\{n_1, n_2\}$ and corresponding to this n, $\exists\, x \in X$ *such that* $\mu(n, x) \geq \alpha$, then $d(x, a) < \frac{\varepsilon}{2}$ and $d(x, b) < \frac{\varepsilon}{2}$. Therefore, $d(a, b) \leq d(a, x) + d(x, b) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$. That is, $0 \leq d(a, b) < \varepsilon$, $\forall \varepsilon > 0$, hence $d(a, b) = 0$, implies $a = b$.
□

Remark 3.13. A multi-fuzzy sequence may converge to different limits at different levels. For consider the following example.
Let $X = \{0, 1\}$. Define $\mu : \mathbb{N} \times X \to [0,1]^2$ by

$$\mu_1(n, 0) = 0.1, \ \mu_1(n, 1) = 0.5$$
$$\mu_2(n, 0) = 0.5, \ \mu_2(n, 1) = 0.1$$

Then μ converges to 0 at level α, where $\alpha = [\alpha_1, \alpha_2], with\ 0 < \alpha_1 \leq 0.1 < \alpha_2 \leq 0.5$ and μ converges to 1 at level α, where $\alpha' = [\alpha'_1, \alpha'_2]\ with\ 0.1 < \alpha'_1 \leq 0.5\, and\, 0 < \alpha'_2 \leq 0.1$.

3.1. Multi-fuzzy Bounded Sequence in $\mathbb{R}$.

Definition 3.14. Let μ be a multi-fuzzy sequence of dimension r in $\mathbb{R}$ with the usual metric. We say that μ is bounded at level $\alpha \in [0,1]^r \backslash \{(0, 0, ..., 0)\}$, if there exist $M > 0$ such that for every $n \in \mathbb{N}$ and for every $x \in \mathbb{R}$, whenever $\mu(n, x) \geq \alpha$, we have, $|x| \leq M$.

EXAMPLE 3.15. Consider the multi-fuzzy sequence μ of dimension r in $\mathbb{N}$ defined by

$$\mu_i(n,x) = \begin{cases} 1-\frac{1}{n+i}, & if\, x = \frac{1}{n} \\ 0, & otherwise \end{cases}, \forall i = 1,2,...,r.$$

We can prove that μ is a multi-fuzzy bounded sequence at level $\alpha = [\alpha_1, \alpha_2, ..., \ \alpha_r]$ $with\ 0 < \alpha_i \leq \frac{1}{2}$, $\forall i = 1,2,...,r$. For we have, if any $n \in \mathbb{N},\ x \in X\ with\ \mu(n,x)$ $\geq \alpha$ implies, $\mu_i\ (n,x) \neq 0\ for\, every\, i.\ Therefore,\ x = \frac{1}{n}.\ Hence\ |x| = \frac{1}{n} \leq 1.$

THEOREM 3.16. *If f is a bounded crisp sequence in $\mathbb{R}$. Then corresponding multi-fuzzy sequence μ^f is also bounded at any level α.*

Proof. Assume that f is a bounded crisp sequence in $\mathbb{R}$. That is, there exist $M > 0$ such that $|f(n)| \leq M,\ \forall n \in \mathbb{N}$. The multi-fuzzy sequence μ^f is given by,

$$\mu_i^f(n,x) = \begin{cases} 1, & if\ x = f(n) \\ 0, & otherwise \end{cases}, \ i = 1,2,...,r.$$

Therefore, for $\alpha \in (0,1]^r,\ \mu^f(n,x) \geq \alpha$ implies $\mu_i^f(n,x) = 1,\ \forall i = 1,2,...,r,$ hence $x = f(n),$ so $|x| = |f(n)| \leq M$. That is, μ^f is bounded at any level α.

□

THEOREM 3.17. *If f is a crisp sequence in $\mathbb{R}$ and μ^f is the corresponding multi-fuzzy sequence of dimension r. If μ^f is bounded at any level α, then f is also bounded.*

Proof. Assume that μ^f is a multi-fuzzy bounded sequence in X at any level $\alpha \in [0,1]^r \backslash \{(0,0,...,0)\}$. That is, there exist $M > 0$ such that for every $n \in \mathbb{N}$ and for every $x \in \mathbb{R}$, whenever $\mu^f(n,x) \geq \alpha$, we have, $|x| \leq M$.
Let $n \in \mathbb{N}$. Then corresponding to $x = f(n)$, we have $\mu^f(n,x) = (1,1,...1) \geq \alpha.\ Therefore, |x| \leq M$ implies $|f(n)| \leq M,$ that is, $|f(n)| \leq M, \forall n \in \mathbb{N}$.
Implies f is bounded. □

REMARK 3.18. The above results show that the concept of bounded multi-fuzzy sequence is an extension of bounded crisp sequence.

REMARK 3.19. We say that a multi-fuzzy sequence μ of dimension r in the metric space $\mathbb{R}$ is unbounded at a level $\alpha \in [0,1]^r \backslash \{(0,0,...,0)\}$, if for every $M > 0$, there exist $n_M \in \mathbb{N}\ and\, x_M \in \mathbb{R}$ such that $\mu(n_M, x_M\,) \geq \alpha,\ but\ |x_M| > M.$

EXAMPLE 3.20. Consider the multi-fuzzy sequence μ of dimension r in $\mathbb{N}$ defined by

$$\mu_i(n,x) = 1 - \frac{1}{nx+i},\ i = 1,2,...,r,\ \forall n, x \in \mathbb{N}.$$

Let $\alpha \in [0,1]^r \backslash \{(0,0,...,0)\}\ with\ \alpha_i \leq \frac{1}{2}, \forall i$. We can prove that μ is unbounded at this level α. Let $M > 0$. By Archimedean property there exists $n_M \in \mathbb{N}$ such that $M < n_M$. Then let $x_M = n_M$ and $\mu_i(n_M, x_M) = \mu_i(n_M, n_M) = 1 - \frac{1}{n_M^2+i} \geq \frac{1}{2},\ \forall i = 1,2,...,r.$ Therefore, $\mu(n_M, x_M\,) \geq \alpha,\ but\ |x_M| = n_M > M.$

THEOREM 3.21. *If X is a bounded subset of $\mathbb{R}$. Then any multi-fuzzy sequence of dimension r in X is also bounded at any level α.*

Proof. Assume that X is a bounded subset of $\mathbf{R}$ bounded by M. Let μ be a multi-fuzzy sequence of dimension r in X. Let $\alpha \in [0,1]^r \backslash \{(0,0,...,0)\}$. Since X is bounded by M, $|x| \leq M$, $\forall x \in X$. Therefore, in particular $|x| \leq M$, for any $n \in \mathbb{N}$ *and* $x \in X$, with $\mu(n,x) \geq \alpha$. □

THEOREM 3.22. *A convergent multi-fuzzy sequence of dimension r in a subspace X of $\mathbb{R}$ at a level α need not be bounded at that level.*

Proof. Consider the multi-fuzzy sequence of dimension r in $\mathbb{R}$ defined by

$$\mu_i(n,x) = \begin{cases} 1, & if\, n = 1 \; and \; \forall x \in \mathbb{R} \\ 1 - \frac{1}{n+i}, & if\, n > 1 \; and \; x = \frac{1}{n} \\ 0, & if\, n > 1 \; and \; x \neq \frac{1}{n} \end{cases}, \; \forall i = 1,2,...,r.$$

μ converges to 0 at a level $\alpha = [\alpha_1, \alpha_2, ... \, \alpha_r]$ *with* $0 < \alpha_i \leq \frac{1}{2}$, $\forall i$. Because, for any $n \in \mathbb{N}$, if $n = 1$, $\mu(n,x) = (1,1,...,1) \geq \alpha$, $\forall x \in \mathbb{R}$ *and if* $n > 1$, *let* $x = \frac{1}{n} \in \mathbb{R}$ *and* $\mu(n,x) = \left(1 - \frac{1}{n+1}, 1 - \frac{1}{n+2}, ..., \; 1 - \frac{1}{n+r}\right) \geq \alpha$. Now $\forall \varepsilon > 0$, *we choose* $n_0 > 1$ *such that* $\frac{1}{\varepsilon} < n_0$. *Then,* $\forall n \geq n_0$, *we have* $n > 1$, $\frac{1}{n} \leq \frac{1}{n_0} < \varepsilon$ *and* $\mu(n,x) \geq \alpha$ *implies* $x = \frac{1}{n}$, *otherwise,* $\mu(n,x) = (0,0,...,0)$, *for such* x, *we have* $d(x,0) = |x| = \frac{1}{n} < \varepsilon$. But this multi-fuzzy sequence is not bounded at this level α. For any $M > 0$, we choose $n = 1$ *and* $x = M+1$, then $\mu(n,x) = \mu(1, M+1) = (1,1,...,1) \geq \alpha$, *but* $|x| = |M+1| = M+1 > M$. □

THEOREM 3.23. *If a multi-fuzzy sequence μ of dimension r in a subspace X of $\mathbb{R}$ is convergent at a level α and if for each $n \in \mathbb{N}$, the set $\mu^n_{[\alpha]} = \{x \in X : \mu(n,x) \geq \alpha\}$ is a bounded subset of X, then μ is bounded at level α.*

Proof. Let μ be a multi-fuzzy sequence of dimension r in a subspace X of $\mathbb{R}$. Assume that μ converges to $a \in X$ at level α and the set $\mu^n_{[\alpha]} = \{x \in X : \mu(n,x) \geq \alpha\}$ is bounded for each $n \in \mathbb{N}$, that is, for $n \in \mathbb{N}$, there exists $M_n > 0$ such that $|x| \leq M_n$, $\forall x \in \mu^n_{[\alpha]}$. Since μ converges to a, corresponding to $1 > 0$, there exists $n_0 \in \mathbb{N}$ *such that* $\forall n \geq n_0$ *and for* $x \in X$, $\mu(n,x) \geq \alpha$ implies $|x - a| < 1$. Therefore, $\forall n \geq n_0$ *and for* $x \in X$, $\mu(n,x) \geq \alpha$, we have $|x| \leq |x-a| + |a| \; < 1 + |a|$.

Also we have the set $\mu^n_{[\alpha]}$ is bounded for each $n \in \mathbb{N}$, therefore, $\mu(n,x) \geq \alpha$ implies $|x| \leq M_n$, $\forall n = 1,2,..., n_{0-1}$. Let $M = \sup\{M_1, \, M_2, \, ..., \, M_{n_0 - 1}, \, 1 + |a| \, \}$. Then for any $n \in \mathbb{N}$ *and for* $x \in X$, *with* $\mu(n,x) \geq \alpha$, we have $|x| \leq M$. That is, μ is bounded at level α. □

REMARK 3.24. In the above theorem the convergence of the multi-fuzzy sequence is needed.
Consider the multi-fuzzy sequence μ of dimension r in $\mathbb{N}$ defined by

$$\mu_i(n,x) = \begin{cases} 1 - \frac{1}{nx+i}, & if \; x \leq n \\ 0, & otherwise \end{cases}, \; i = 1,2,...,r.$$

Let $\alpha = [\alpha_1, \alpha_2, ..., \, \alpha_r]$ *with* $0 < \alpha_i \leq \frac{1}{2}$, $\forall i$. For each $n \in \mathbb{N}$, the set $\mu^n_{[\alpha]} = \{1,2,...,n\}$ is a bounded subset of $\mathbb{N}$. But the sequence μ is unbounded.
For any $M > 0$, by the Archimedean property there exists $n_0 \in \mathbb{N}$ such that $M < n_0$. Now let $n_M = n_0 + 1$ *and* $x_M = n_0$, *then* $\mu(n_M, x_M) = \mu(n_0 + 1\,, n_0) \geq \alpha$, *but* $|x_M| =$

$n_0 > M$.

This is because this sequence does not converge to any $a \in \mathbb{N}$. For let $a \in \mathbb{N}$. We prove that μ does not converge to a. Let $\varepsilon_0 = 1$, $\forall k \in \mathbb{N}$, we can choose a natural number $n_k > k$ *such that* $n_k - 1 \neq a$. Then $x_k = n_k - 1 \in \mathbb{N}$ and $\mu(n_k, x_k) = \mu(n_k, n_k - 1) \geq \alpha$, *but* $d(x_k, a) = |n_k - 1 - a| \geq 1 = \varepsilon_0$, *since* $n_k - 1 \neq a$.

3.2. Multi-fuzzy Cauchy Sequences.

Definition 3.25. We say that a multi-fuzzy sequence μ of dimension r in a subspace X of $\mathbb{R}$ with the usual metric is a multi-fuzzy Cauchy sequence at a level $\alpha \in [0,1]^r \backslash \{(0,0,...,0)\}$, if
i) for each $n \in \mathbb{N}$, there exist $x \in X$ such that $\mu(n,x) \geq \alpha$;
ii) for every $\varepsilon > 0$, there exist $H \in \mathbb{N}$ such that for all $n, m > H$ and for any $x, y \in X$ with $\mu(n,x) \geq \alpha$ and $\mu(m,y) \geq \alpha$ implies $d(x,y) < \varepsilon$.

Theorem 3.26. *If f is a Cauchy crisp sequence in a subspace X of $\mathbb{R}$ with the usual metric, then the corresponding multi-fuzzy sequence μ^f is a multi-fuzzy Cauchy sequence at any level $\alpha \in (0,1]^r$.*

Proof. Assume that f is a Cauchy crisp sequence in X, the corresponding multi-fuzzy sequence is given by

$$\mu_i^f(n,x) = \begin{cases} 1, & if\ x = f(n) \\ 0, & otherwise \end{cases}, \ i = 1,2,...,r.$$

Let $\alpha \in (0,1]^r$.
i) For each $n \in \mathbb{N}$, let $x = f(n)$, then $\mu_i^f(n,x) = 1$, for every $i = 1,2,...,r$, hence $\mu^f(n,x) = (1,1,...,1) \geq \alpha$;
ii) for every $\varepsilon > 0$, there exist $H \in \mathbb{N}$ such that for all $n, m \geq H$, $d(f(n), f(m)) < \varepsilon$. Therefore, for all $n, m \geq H$, $\mu^f(n,x) \geq \alpha$ *and* $\mu^f(m,y) \geq \alpha$ implies $\mu_i^f(n,x) = 1$ *and* $\mu_i^f(m,y) = 1$ for every i, since each $\alpha_i > 0$. Therefore, $x = f(n)$ *and* $y = f(m)$, hence $d(x,y) = d(f(n), f(m)) < \varepsilon$. □

Theorem 3.27. *Let f be a crisp sequence in the subspace X of $\mathbb{R}$ with the usual metric. If the corresponding multi-fuzzy sequence μ^f is a multi-fuzzy Cauchy sequence at a level α, then f is a Cauchy crisp sequence in X.*

Proof. Let $\varepsilon > 0$. Since μ^f is a multi-fuzzy Cauchy sequence at a level α, for each $n \in \mathbb{N}$, there exist $x \in X$ such that $\mu(n,x) \geq \alpha$ and corresponding to the given $\varepsilon > 0$, there exist $H \in \mathbb{N}$ such that for all $n, m > H$ and for any $x, y \in X$ with $\mu(n,x) \geq \alpha$ and $\mu(m,y) \geq \alpha$ implies $d(x,y) < \varepsilon$. Therefore, for all $n, m \geq H$, we have $\mu^f(n, f(n)) = (1,1,...,1) \geq \alpha\, and\, \mu^f(m, f(m)) = (1,1,...,1) \geq \alpha$, hence $d(f(n), f(m)) < \varepsilon$. □

Example 3.28. Consider the multi-fuzzy sequence μ of dimension r in $\mathbb{N}$ as a subspace of $\mathbb{R}$ with the usual metric defined by

$$\mu_i(n,x) = \begin{cases} 1 - \frac{1}{n+i}, & if\, x = \frac{1}{n} \\ 0, & otherwise \end{cases}, \ \forall i = 1,2,...,r.$$

We can prove that μ is a multi-fuzzy Cauchy sequence at level α with $0 < \alpha_i < 0.5$, for every i.

For each $n \in \mathbb{N}$, let $x = \frac{1}{n}$, then $\mu(n,x) = (1-\frac{1}{n+1}, 1-\frac{1}{n+2}, ..., 1-\frac{1}{n+r}) \geq \alpha$. For any $\varepsilon > 0$, by the Archimedean Property we can choose a natural number n_0 such that $\frac{1}{n_0} < \frac{\varepsilon}{2}$. Now for all $n,m \geq n_0$, we have $\frac{1}{m} \geq \frac{1}{n_0}$ *and* $\frac{1}{n} \geq \frac{1}{n_0}$ and if $\mu(n,x) \geq \alpha$ *and* $\mu(m,x) \geq \alpha$, then since $\alpha_i > 0$ for every i, $x = \frac{1}{n}$ *and* $y = \frac{1}{m}$, $d(x,y) = |\frac{1}{n} - \frac{1}{m}| \leq \frac{1}{n} + \frac{1}{m} < \varepsilon$.

Remark 3.29. We say that a multi-fuzzy sequence μ of dimension r in a subspace X of $\mathbb{R}$ is not a multi-fuzzy Cauchy sequence at a level α, if either one of the following conditions hold:
i) there exist $n_0 \in \mathbb{N}$ such that $\mu(n_0,x) < \alpha$, for every $x \in X$;
ii) there exist $\varepsilon_0 > 0$ such that for every $k \in \mathbb{N}$, there exist $n_k, m_k \in \mathbb{N}$ and $x_k, y_k \in X$ such that $\mu(n_k,x_k) \geq \alpha \, and \, \mu(m_k,y_k) \geq \alpha$, but $d(x_k,y_k) \geq \varepsilon_0$.

Example 3.30. Let $\mu : \mathbb{N} \times \{0,1\} \to [0,1]^2$ defined by,

$$\mu_1(n,0) = 0.1, \;\; \mu_1(n,1) = 0.2;$$

$$\mu_2(n,0) = 0.2, \;\; \mu_1(n,1) = 0.1.$$

μ is not a multi-fuzzy Cauchy sequence at any level $\alpha = [\alpha_1, \, \alpha_2] \, with \, \alpha_1, \alpha_2 > 0.2$, because,

$$\mu(n,0) = (0.1, 0.2) < \alpha, \; \mu(n,1) = (0.2, 0.1) < \alpha.$$

Theorem 3.31. *If a multi-fuzzy sequence μ of dimension r in a subspace X of $\mathbb{R}$ is convergent at level α, then it is a multi-fuzzy Cauchy sequence at the same level α.*

Proof. Assume that μ converges to an element $a \in X$ at a level α. To show that μ is a multi-fuzzy Cauchy sequence at the same level α.
i) For each $n \in \mathbb{N}$, there exist $x \in X$ such that $\mu(n,x) \geq \alpha$, since μ is convergent.
ii)For any $\varepsilon > 0$, since μ is convergent to $a \in X$ at a level α, $\exists K \in \mathbb{N} \, such \, that$ $d(x,a) < \frac{\varepsilon}{2}$, $\forall n \geq K \, and \, \forall x \in X \, with \; \mu(n,x) \geq \alpha$.

Therefore, for all $n,m \geq K$, $\mu(n,x) \geq \alpha$ *and* $\mu(m,y) \geq \alpha$, then $d(x,a) < \frac{\varepsilon}{2}$ *and* $d(y,a) < \frac{\varepsilon}{2}$. Hence $d(x,y) \leq d(x,a) + d(y,a) < \varepsilon$. That is, μ is a multi-fuzzy Cauchy sequence at the same level α. □

Theorem 3.32. *A multi-fuzzy Cauchy sequence need not be a bounded multi-fuzzy sequence.*

Proof. Consider the multi-fuzzy sequence μ of dimension r in $\mathbb{R}$ defined by

$$\mu_i(n,x) = \begin{cases} 1, & if\, n = n_0 \; and \; x \in \mathbb{R} \\ 1 - \frac{1}{n+i}, & if\, n \neq n_0 \; and \; x = \frac{1}{n} \\ 0, & if\, n \neq n_0 \; and \; x \neq \frac{1}{n} \end{cases}, \; for \; i = 1, 2, ..., r.$$

where n_0 is a fixed natural number.

Then μ is a multi-fuzzy Cauchy sequence at level α with $0 < \alpha_i < 0.5$, for every i,
i) for each $n \in \mathbb{N}$, if $n = n_0$, then for every $x \in \mathbb{R}$, $\mu(n_0,x) = (1,1,...,1) \geq \alpha$ and if $n \neq n_0$, let $x = \frac{1}{n}$, for this x

$$\mu(n,x) = (1 - \frac{1}{n+1}, 1 - \frac{1}{n+2}, ..., 1 - \frac{1}{n+r}) \geq \alpha;$$

ii) for any $\varepsilon > 0$, we choose a natural number n_1 such that $n_1 > n_0$ *and* $n_1 > \frac{2}{\varepsilon}$. Then for all $n,m \geq \varepsilon$, we have $\frac{1}{n} \leq \frac{1}{n_1}$ *and* $\frac{1}{m} \leq \frac{1}{n_1}$ *and* $\mu(n,x) \geq \alpha$ *and* $\mu(m,y) \geq$

α *implies* $x = \frac{1}{n}$ *and* $y = \frac{1}{m}$. Therefore, $|x - y| = |\frac{1}{n} - \frac{1}{m}| \leq \frac{1}{n} + \frac{1}{m} < \varepsilon$.

But this multi-fuzzy sequence is not bounded, for any $M > 0$, let $n_M = n_0$ *and* $x_M = M + 1$. Then $\mu(n_M, x_M) = \mu(n_0, M+1) = (1, 1, .., 1) \geq \alpha$, but $|x_M| = |M+1| = M + 1 > M$. □

THEOREM 3.33. *A bounded multi-fuzzy sequence need not be a multi-fuzzy Cauchy sequence.*

Proof. There are bounded multi-fuzzy sequences which are not multi-fuzzy Cauchy sequences. For consider the following example.
Let $X = \{1, 2, ..., M\}$ as a subspace of $\mathbb{R}$ with the usual metric. Define $\mu : \mathbb{N} \times X \to [0, 1]^r$ by

$$\mu_i(n, x) = \begin{cases} 1, & if\, x\, is\, even \\ \frac{1}{nx}, & if\, x\, is\, odd \end{cases} \quad for\ i = 1, 2, ..., r.$$

Since X is bounded, μ is a bounded multi-fuzzy sequence for any α, but it is not a multi-fuzzy Cauchy sequence. For let $\varepsilon_0 = 1$. For any natural number k, let $n_k\, and\, m_k$ be any natural numbers and let $x_k = 2\, and\, y_k = 4$. Then $\mu(n_k, x_k) = (1, 1, ..., 1) \geq \alpha\, and\, \mu(m_k, y_k) = (1, 1, ..., 1) \geq \alpha$, *but* $d(x_k, y_k) = |2 - 4| = 2 > \varepsilon_0$. □

THEOREM 3.34. *If a multi-fuzzy sequence* μ *of dimension* r *in a subspace* X *of* $\mathbb{R}$ *is a multi-fuzzy Cauchy sequence at a level* α *and if for each* $n \in \mathbb{N}$*, the set* $\mu^n_{[\alpha]} = \{x \in X : \mu(n, x) \geq \alpha\}$ *is a bounded subset of* X*, then* μ *is bounded at level* α*.*

Proof. Let μ be a multi-fuzzy sequence of dimension r in a subspace X of $\mathbb{R}$. Assume that μ is a multi-fuzzy Cauchy sequence at level α and the set $\mu^n_{[\alpha]} = \{x \in X : \mu(n, x) \geq \alpha\}$ is bounded for each $n \in \mathbb{N}$, that is for $n \in \mathbb{N}$, there exist $M_n > 0$ such that $|x| \leq M_n,\ \forall x \in \mu^n_{[\alpha]}$.
Since μ is a multi-fuzzy Cauchy sequence at level α, corresponding to $1 > 0$, there exist $H \in \mathbb{N}$ such that for all $n, m \geq H$ and for any $x, y \in X$ with $\mu(n, x) \geq \alpha\ and\ \mu(m, y) \geq \alpha$ implies $|x - y| < 1$. In particular, let $m = H$, there exist $y \in X$ such that $\mu(H, y) \geq \alpha$. Now, if $n \geq H$, $\mu(n, x) \geq \alpha$ implies $|x - y| < 1$, *since* $\mu(H, y) \geq \alpha$. Therefore, if $n \geq H\, and\, for\, any\, x \in X\, with\, \mu(n, x) \geq \alpha$, we have $|x| \leq |x - y| + |y| < 1 + |y|$. Also we have the set $\mu^n_{[\alpha]}$ is bounded for each $n = 1, 2, ..., H - 1$, therefore, there exist $M_n > 0$ such that $|x| \leq M_n,\ \forall x \in X,\ with\ \mu(n, x) \geq \alpha$, for every $n = 1, 2, ..H - 1$. Let $M = max\{M_1, M_2, ...M_{H-1}, 1 + |y|\}$, then for any $n \in \mathbb{N}$ and for $x \in X$ with $\mu(n, x) \geq \alpha$, we have $|x| \leq M$, that is, μ is bounded at level α. □

4. Conclusion

This article contributes a theoretical foundation to multi-fuzzy mathematics, providing insights into the behaviour of multi-fuzzy sequences in metric spaces. This study establishes a basis for researchers in the theory of multi-fuzzy sets. Various concepts and results presented here lead to further research and applications in areas with inherent uncertainty and imprecision. The multidimensional perspective offered promises to enhance the applicability and relevance of multi-fuzzy set theory in tackling the complexities of real-world problems.

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