

ON THE WEAKENED HYPOTHESES-BASED GENERALIZATIONS OF THE ENESTRÖM-KAKEYA THEOREM

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ABSTRACT. According to the well-known Eneström-Kakeya Theorem, all the zeros of a polynomial $P(z) = \sum_{s=0}^n a_s z^s$ of degree n with real coefficients satisfying $a_n \geq a_{n-1} \geq \cdots \geq a_1 \geq a_0 > 0$ lie in the complex plane $|z| \leq 1$. We provide comparable results with hypotheses relating to the real and imaginary parts of the coefficients as well as the coefficients' moduli in response to recent findings about an Eneström-Kakeya “type” condition on real coefficients. Our findings so broadly extend the other previous findings.

1. Introduction

The classical Eneström-Kakeya Theorem addresses where the complex zeros of a real polynomial with non-negative monotone coefficients are located. Gustav Eneström [3] and Sōichi Kakeya [8] separately demonstrated it in 1893 and 1912, respectively.

THEOREM 1.1. (Eneström-Kakeya) *If $P(z) = \sum_{s=0}^n a_s z^s$ is a complex polynomial of degree n and its real coefficients meet the conditions $a_n \geq a_{n-1} \geq \cdots \geq a_1 \geq a_0 > 0$, then all the zeros of $P(z)$ are located in the circle $|z| \leq 1$.*

There are several different generalizations of the Eneström-Kakeya Theorem. The majority of these entail loosening the restriction on the coefficients. See [5] for a summary of these outcomes until 2014. Govil and Rahman [6], for instance, demonstrated the following in 1968.

THEOREM 1.2. *If $P(z) = \sum_{s=0}^n a_s z^s$ is a polynomial of degree n with complex coefficients such that $|\arg(a_s) - \mu| \leq \theta \leq \pi/2$ for $0 \leq s \leq n$ for some real μ , and $|a_n| \geq |a_{n-1}| \geq \cdots \geq |a_0|$, then all the zeros of P lie in*

$$|z| \leq \cos \theta + \sin \theta + \frac{2 \sin \theta}{|a_n|} \sum_{s=0}^{n-1} |a_s|.$$

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Theorem 1.2 reduces to the Eneström-Keakeya Theorem 1.1 when $\mu = \theta = 0$. The following result regarding a monotonicity constraint on the real and imaginary parts of the coefficients was stated by Gardner and Govil ([4], Corollary 1) as a corollary to a more comprehensive result.

THEOREM 1.3. *Let $P(z) = \sum_{s=0}^n a_s z^s$ be a polynomial of degree n with complex coefficients where $Re(a_s) = \alpha_s$ and $Im(a_s) = \beta_s$ for $s = 0, 1, \dots, n$. Suppose that $\alpha_n \geq \alpha_{n-1} \geq \dots \geq \alpha_0$ and $\beta_n \geq \beta_{n-1} \geq \dots \geq \beta_0$. Then all the zeros of P lie in*

$$\frac{|a_0|}{|a_n| - (\alpha_0 + \beta_0) + (\alpha_n + \beta_n)} \leq |z| \leq \frac{|a_0| - (\alpha_0 + \beta_0) + (\alpha_n + \beta_n)}{|a_n|}.$$

According to Theorem 1.3, a result of Joyal, Labelle, and Rahman ([7], Theorem 3) is implied when $\beta_s = 0$ for $s = 0, 1, \dots, n$. In 2012, Aziz and Zargar [1] published the following result, which involves a slight generalization of the Eneström-Keakeya monotonicity condition on the real coefficients of a polynomial.

THEOREM 1.4. *Let $P(z) = \sum_{s=0}^n a_s z^s$ be a polynomial of degree n with real coefficients such that, for some positive numbers k and ρ with $k \geq 1$ and $0 < \rho \leq 1$, the coefficients satisfy $ka_n \geq a_{n-1} \geq \dots \geq a_1 \geq \rho a_0 \geq 0$. Then all the zeros of P lie in the closed disk*

$$|z + k - 1| \leq k + 2a_0(1 - \rho)/a_n.$$

The Eneström-Keakeya Theorem 1.1 is implied by Theorem 1.4 with $k = \rho = 1$. Shah et al. [9] recently established the following result, which keeps the monotonicity criterion on the “central” coefficients but does not impose one on the “tail” coefficients.

THEOREM 1.5. *Let $P(z) = \sum_{s=0}^n a_s z^s$ be a polynomial of degree n with real coefficients such that, for some positive numbers p and q with $0 \leq q \leq p \leq n$, the coefficients satisfy $a_p \geq a_{p-1} \geq \dots \geq a_{q+1} \geq a_q$. Then all the zeros of P lie in the closed annulus*

$$\min \left\{ 1, \frac{|a_0|}{N_q - a_q + a_p + M_p + |a_n|} \right\} \leq |z| \leq \frac{|a_0| + N_q - a_q + a_p + M_p}{|a_n|},$$

where $N_q = \sum_{s=1}^q |a_s - a_{s-1}|$ and $M_p = \sum_{s=p+1}^n |a_s - a_{s-1}|$.

A result from Joyal, Labelle, and Rahman [8] is implied by Theorem 1.5 for $p = n$ and $q = 0$.

Furthermore, when $a_0 \geq 0$, it entails the Eneström-Keakeya Theorem 1.1.

This study combines Theorems 1.4 and 1.5’s hypotheses and applies them to polynomials with complex coefficients while allowing the possibility of $\alpha_s \leq \alpha_{s-1}$ and $\beta_s \leq \beta_{s-1}$ for $1 \leq s \leq n$. We apply the hypotheses to the real and imaginary parts of the coefficients, and to the moduli of the coefficients.

2. Statement of Results

THEOREM 2.1. *Let $P(z) = \sum_{s=0}^n a_s z^s$ be a polynomial of degree n with complex coefficients. Let $\alpha_s = Re(a_s)$ and $\beta_s = Im(a_s)$ for $0 \leq s \leq n$. Suppose that, for some*

positive numbers $k_s, k'_s, \rho, \rho', p$ and q with $k_s \geq 1, k'_s \geq 1, 0 < \rho \leq 1, 0 < \rho' \leq 1, 1 \leq s \leq p - q$ and $0 \leq q \leq p \leq n$, the coefficients satisfy

$$\rho\alpha_q \leq k_1\alpha_{q+1} \leq k_2\alpha_{q+2} \leq \dots \leq k_{p-q-1}\alpha_{p-1} \leq k_{p-q}\alpha_p,$$

and

$$\rho'\beta_q \leq k'_1\beta_{q+1} \leq k'_2\beta_{q+2} \leq \dots \leq k'_{p-q-1}\beta_{p-1} \leq k'_{p-q}\beta_p.$$

Then all the zeros of P lie in the closed annulus

$$\begin{aligned} \min \left\{ 1, \frac{|a_0|}{\left(N_q + (k_{p-q}\alpha_p - \rho\alpha_q) + (1 - \rho)|\alpha_q| + 2 \sum_{s=1}^{p-q} (k_s - 1)|\alpha_{q+s}| \right.} \right. \\ \left. \left. + (k'_{p-q}\beta_p - \rho'\beta_q) + (1 - \rho')|\beta_q| + 2 \sum_{s=1}^{p-q} (k'_s - 1)|\beta_{q+s}| + M_p + |a_n| \right) \right\} \leq |z| \\ \leq \left(|a_0| + N_q + (k_{p-q}\alpha_p - \rho\alpha_q) + (1 - \rho)|\alpha_q| + 2 \sum_{s=1}^{p-q} (k_s - 1)|\alpha_{q+s}| \right. \\ \left. + (k'_{p-q}\beta_p - \rho'\beta_q) + (1 - \rho')|\beta_q| + 2 \sum_{s=1}^{p-q} (k'_s - 1)|\beta_{q+s}| + M_p \right) / |a_n|, \end{aligned}$$

where $N_q = \sum_{s=1}^q |a_s - a_{s-1}|$ and $M_p = \sum_{s=p+1}^n |a_s - a_{s-1}|$.

It is important to note that Theorem 2.1 becomes Theorem 1.5 when $\beta_s = 0$ for $0 \leq s \leq n$ and $k_s = \rho = 1$ for $1 \leq s \leq p - q$ and also Theorem 2.1 reduces to a result of Joyal, Labelle, and Rahman [8] when $\beta_s = 0$ for $0 \leq s \leq n, k_s = \rho = 1$ for $1 \leq s \leq p - q, q = 0$, and $p = n$. It further simplifies to the Enestrom-Kakeya Theorem 1.1 if, moreover, $a_0 \geq 0$.

The following example illustrates Theorem 2.1.

Example: Consider the polynomial

$$100z^6 + (-1 - i)z^5 + (5 + 4i)z^4 + (1 + i)z^3 + (2 + i)z^2 + (2 - 2i)z + 5.$$

Take, $q = 1, p = 4$, so that $N_q = \sqrt{13}$ and $M_p = \sqrt{61} + \sqrt{10202}$.

Choose,

$$\rho = 1/3, k_1 = 1, k_2 = 2, k_3 = 1,$$

$$\rho' = 1/2, k'_1 = 1, k'_2 = 1, k'_3 = 1.$$

Clearly, the coefficients satisfy the hypotheses

$$\rho\alpha_q \leq k_1\alpha_{q+1} \leq k_2\alpha_{q+2} \leq k_3\alpha_{q+3},$$

$$\rho'\beta_q \leq k'_1\beta_{q+1} \leq k'_2\beta_{q+2} \leq k'_3\beta_{q+3}.$$

Applying Theorem 2.1, it is easy to verify that all the zeros of p lie in $\alpha \leq |z| \leq \beta$, where $\alpha \approx 0.02246$ and $\beta \approx 1.3108$.

Next, we prove the following result by imposing a similar hypothesis as that of Theorem 2.1 on the moduli of the coefficients of a polynomial.

THEOREM 2.2. Let $P(z) = \sum_{s=0}^n a_s z^s$ be a polynomial of degree n with complex coefficients. Suppose that, for some positive numbers k_s, ρ, p and q with $k_s \geq 1$,

$0 < \rho \leq 1, 1 \leq s \leq p - q$ and $0 \leq q \leq p \leq n$, the coefficients satisfy $|\arg(k_s a_{q+s}) - \mu| \leq \theta \leq \pi/2$ for some real μ , and

$$\rho|a_q| \leq k_1|a_{q+1}| \leq k_2|a_{q+2}| \leq \dots \leq k_{p-q-1}|a_{p-1}| \leq k_{p-q}|a_p|.$$

Then all the zeros of P lie in the closed annulus

$$\min \left\{ 1, \frac{|a_0|}{\left(N_q + |a_q| + 2 \sum_{s=1}^{p-q-1} (k_s - 1)|a_{q+s}| + \rho|a_q|(\sin \theta - \cos \theta - 1) \right)} \right. \\ \left. + 2 \sin \theta \sum_{s=1}^{p-q-1} k_s|a_{q+s}| + k_{p-q}|a_p|(\sin \theta + \cos \theta + 1) - |a_p| + M_p \right. \\ \left. + |a_n| \right\} \leq |z| \leq \frac{1}{|a_n|} \left(|a_0| + N_q + |a_q| + 2 \sum_{s=1}^{p-q-1} (k_s - 1)|a_{q+s}| + \rho|a_q|(\sin \theta - \cos \theta - 1) \right. \\ \left. + 2 \sin \theta \sum_{s=1}^{p-q-1} k_s|a_{q+s}| + k_{p-q}|a_p|(\sin \theta + \cos \theta + 1) - |a_p| + M_p \right),$$

where N_q and M_p are same as defined in Theorem 2.1.

REMARK 2.3. For $q = 0, p = n$ and $\rho = k_s = 1, 1 \leq s \leq n$, Theorem 2.2 implies that all the zeros of P lie in

$$|z| \leq \frac{1}{|a_n|} \left((\cos \theta + \sin \theta)|a_n| + |a_0|(1 - \cos \theta - \sin \theta) + 2 \sin \theta \sum_{s=0}^{n-1} |a_s| \right).$$

This is an improvement of Theorem 1.2. More over choosing $\theta = \mu = 0$, it results in the Eneström-Kakeya Theorem 1.1.

The class of polynomials of the form $P(z) = a_0 + \sum_{s=m}^n a_s z^s$ was taken into consideration in relation to Bernstein type inequalities by Chan and Malik [2]. By enforcing the inequality hypotheses of Theorems 2.1 and 2.2 to this class of polynomials, we get the following.

COROLLARY 2.4. Let $P(z) = a_0 + \sum_{s=m}^n a_s z^s$ be a polynomial of degree n with complex coefficients. Let $\alpha_s = \text{Re}(a_s)$ and $\beta_s = \text{Im}(a_s)$ for $0 \leq s \leq n$. Suppose that, for some positive numbers $k_s, k'_s, \rho, \rho', p$ and q with $k_s \geq 1, k'_s \geq 1, 0 < \rho \leq 1, 0 < \rho' \leq 1, 1 \leq s \leq p - q$ and $0 \leq q \leq p \leq n$, the coefficients satisfy

$$\rho\alpha_q \leq k_1\alpha_{q+1} \leq k_2\alpha_{q+2} \leq \dots \leq k_{p-q-1}\alpha_{p-1} \leq k_{p-q}\alpha_p,$$

and

$$\rho'\alpha_q \leq k'_1\alpha_{q+1} \leq k'_2\alpha_{q+2} \leq \dots \leq k'_{p-q-1}\alpha_{p-1} \leq k'_{p-q}\alpha_p.$$

Then all the zeros of P lie in the closed annulus defined in Theorem 2.1 where N_q is to be replaced as $N_q = \sum_{s=m}^q |a_s - a_{s-1}|$.

COROLLARY 2.5. Let $P(z) = a_0 + \sum_{s=m}^n a_s z^s$ be a polynomial of degree n with complex coefficients. Suppose that, for some positive numbers k_s, ρ, p and q with $k_s \geq 1, 0 < \rho \leq 1, 1 \leq s \leq p - q$ and $0 \leq q \leq p \leq n$, the coefficients satisfy

$|\arg(k_s a_{q+s}) - \mu| \leq \theta \leq \pi/2$ for some real μ , and

$$\rho|a_q| \leq k_1|a_{q+1}| \leq k_2|a_{q+2}| \leq \dots \leq k_{p-q-1}|a_{p-1}| \leq k_{p-q}|a_p|.$$

Then all the zeros of P lie in the closed annulus defined in Theorem 2.1 where N_q is to be replaced as $N_q = \sum_{s=m}^q |a_s - a_{s-1}|$.

3. Lemma

In proving Theorem 2.2, we require the following lemma by Govil and Rahman ([6], equation (6)).

LEMMA 3.1. Let $\{a_s\}_{s=1}^n$ be a set of complex numbers which satisfy $|\arg(a_s) - \mu| \leq \theta \leq \pi/2$ for $0 \leq s \leq n$ and for some real μ . Suppose that $|a_0| \leq |a_1| \leq \dots \leq |a_n|$, then

$$|a_k - a_{k-1}| \leq (|a_k| - |a_{k-1}|) \cos \theta + (|a_k| + |a_{k-1}|) \sin \theta, \quad 1 \leq k \leq n.$$

4. Proof of the Results

We first give a proof of Theorem 2.1.

Proof. Let $P(z) = \sum_{s=0}^n a_s z^s$, $a_n \neq 0$ be the given polynomial satisfying the stated hypotheses. Define f by the equation

$$P(z)(1 - z) = a_0 + \sum_{s=1}^n (a_s - a_{s-1})z^s = f(z) - a_n z^{n+1}.$$

For $|z| = 1$, we have

$$\begin{aligned} |f(z)| &= \left| a_0 + \sum_{s=1}^n (a_s - a_{s-1})z^s \right| \\ &\leq |a_0| + \sum_{s=1}^n |a_s - a_{s-1}| \\ &= |a_0| + \sum_{s=1}^q |a_s - a_{s-1}| + |\alpha_{q+1} + i\beta_{q+1} - \alpha_q - i\beta_q| + |\alpha_{q+2} + i\beta_{q+2} - \alpha_{q+1} - i\beta_{q+1}| \\ &\quad + \dots + |\alpha_{p-1} + i\beta_{p-1} - \alpha_{p-2} - i\beta_{p-2}| + |\alpha_p + i\beta_p - \alpha_{p-1} - i\beta_{p-1}| + \sum_{s=p+1}^n |a_s - a_{s-1}|. \end{aligned}$$

Let $N_q = \sum_{s=1}^q |a_s - a_{s-1}|$ and $M_p = \sum_{s=p+1}^n |a_s - a_{s-1}|$. Therefore for $|z| = 1$, we get

$$\begin{aligned} |f(z)| &\leq |a_0| + N_q + (|\alpha_{q+1} - \alpha_q| + |\alpha_{q+2} - \alpha_{q+1}| + \dots + |\alpha_{p-1} - \alpha_{p-2}| + |\alpha_p - \alpha_{p-1}|) \\ &\quad (|\beta_{q+1} - \beta_q| + |\beta_{q+2} - \beta_{q+1}| + \dots + |\beta_{p-1} - \beta_{p-2}| + |\beta_p - \beta_{p-1}|) + M_p. \end{aligned}$$

That is

$$\begin{aligned}
|f(z)| &\leq |a_0| + N_q + |k_1\alpha_{q+1} - \rho\alpha_q + \alpha_q(\rho - 1) + \alpha_{q+1}(1 - k_1)| + |k_2\alpha_{q+2} - k_1\alpha_{q+1} \\
&\quad + \alpha_{q+1}(k_1 - 1) + \alpha_{q+2}(1 - k_2)| + \cdots + |k_{p-q}\alpha_p - k_{p-q-1}\alpha_{p-1} + \alpha_{p-1}(k_{p-q-1} - 1) \\
&\quad + \alpha_p(1 - k_{p-q})| + |k'_1\beta_{q+1} - \rho'\beta_q + \beta_q(\rho' - 1) + \beta_{q+1}(1 - k'_1)| + |k'_2\beta_{q+2} - k'_1\beta_{q+1} \\
&\quad + \beta_{q+1}(k'_1 - 1) + \beta_{q+2}(1 - k'_2)| + \cdots + |k'_{p-q}\beta_p - k'_{p-q-1}\beta_{p-1} + \beta_{p-1}(k'_{p-q-1} - 1) \\
&\quad + \beta_p(1 - k'_{p-q})| + M_p \\
&\leq |a_0| + N_q + (k_{p-q}\alpha_p - \rho\alpha_q) + (1 - \rho)|\alpha_q| + 2 \sum_{s=1}^{p-q} (k_s - 1)|\alpha_{q+s}| \\
&\quad + (k'_{p-q}\beta_p - \rho'\beta_q) + (1 - \rho')|\beta_q| + 2 \sum_{s=1}^{p-q} (k'_s - 1)|\beta_{q+s}| + M_p.
\end{aligned}$$

Notice that, we have

$$\max_{|z|=1} \left| z^n f\left(\frac{1}{z}\right) \right| = \max_{|z|=1} \left| f\left(\frac{1}{z}\right) \right| = \max_{|z|=1} |f(z)|,$$

it is clear that $z^n f\left(\frac{1}{z}\right)$ has the same bound on $|z| = 1$ as f , that is

$$\begin{aligned}
\left| z^n f\left(\frac{1}{z}\right) \right| &\leq |a_0| + N_q + (k_{p-q}\alpha_p - \rho\alpha_q) + (1 - \rho)|\alpha_q| + 2 \sum_{s=1}^{p-q} (k_s - 1)|\alpha_{q+s}| \\
&\quad + (k'_{p-q}\beta_p - \rho'\beta_q) + (1 - \rho')|\beta_q| + 2 \sum_{s=1}^{p-q} (k'_s - 1)|\beta_{q+s}| + M_p \quad \text{for } |z| = 1.
\end{aligned}$$

Since $z^n f\left(\frac{1}{z}\right)$ is a polynomial and hence is analytic in $|z| \leq 1$, it follows by Maximum Modulus Theorem, that

$$\begin{aligned}
\left| z^n f\left(\frac{1}{z}\right) \right| &\leq |a_0| + N_q + (k_{p-q}\alpha_p - \rho\alpha_q) + (1 - \rho)|\alpha_q| + 2 \sum_{s=1}^{p-q} (k_s - 1)|\alpha_{q+s}| \\
&\quad + (k'_{p-q}\beta_p - \rho'\beta_q) + (1 - \rho')|\beta_q| + 2 \sum_{s=1}^{p-q} (k'_s - 1)|\beta_{q+s}| + M_p \quad \text{for } |z| \leq 1.
\end{aligned}$$

Hence

$$\begin{aligned}
\left| f\left(\frac{1}{z}\right) \right| &\leq \frac{1}{|z|^n} \left(|a_0| + N_q + (k_{p-q}\alpha_p - \rho\alpha_q) + (1 - \rho)|\alpha_q| + 2 \sum_{s=1}^{p-q} (k_s - 1)|\alpha_{q+s}| \right. \\
&\quad \left. + (k'_{p-q}\beta_p - \rho'\beta_q) + (1 - \rho')|\beta_q| + 2 \sum_{s=1}^{p-q} (k'_s - 1)|\beta_{q+s}| + M_p \right) \quad \text{for } |z| \leq 1.
\end{aligned}$$

Replacing z by $1/z$, we see that

$$\begin{aligned}
|f(z)| &\leq \left(|a_0| + N_q + (k_{p-q}\alpha_p - \rho\alpha_q) + (1 - \rho)|\alpha_q| + 2 \sum_{s=1}^{p-q} (k_s - 1)|\alpha_{q+s}| \right. \\
&\quad \left. + (k'_{p-q}\beta_p - \rho'\beta_q) + (1 - \rho')|\beta_q| + 2 \sum_{s=1}^{p-q} (k'_s - 1)|\beta_{q+s}| + M_p \right) |z|^n \quad \text{for } |z| \geq 1.
\end{aligned}$$

Therefore, for $|z| \geq 1$, we have

$$\begin{aligned} |(1-z)P(z)| &= |f(z) - a_n z^{n+1}| \geq |a_n||z|^{n+1} - |f(z)| \\ &\geq |z|^n \left[|a_n||z| - \left(|a_0| + N_q + (k_{p-q}\alpha_p - \rho\alpha_q) + (1-\rho)|\alpha_q| \right. \right. \\ &\quad \left. \left. + 2 \sum_{s=1}^{p-q} (k_s - 1)|\alpha_{q+s}| + (k'_{p-q}\beta_p - \rho'\beta_q) + (1-\rho')|\beta_q| \right. \right. \\ &\quad \left. \left. + 2 \sum_{s=1}^{p-q} (k'_s - 1)|\beta_{q+s}| + M_p \right) \right]. \end{aligned}$$

Hence, if

$$\begin{aligned} |z| > \frac{1}{|a_n|} &\left(|a_0| + N_q + (k_{p-q}\alpha_p - \rho\alpha_q) + (1-\rho)|\alpha_q| + 2 \sum_{s=1}^{p-q} (k_s - 1)|\alpha_{q+s}| \right. \\ &\left. + (k'_{p-q}\beta_p - \rho'\beta_q) + (1-\rho')|\beta_q| + 2 \sum_{s=1}^{p-q} (k'_s - 1)|\beta_{q+s}| + M_p \right), \end{aligned}$$

then $|(1-z)P(z)| > 0$, that is $(1-z)P(z) \neq 0$ and so $P(z) \neq 0$. Therefore all the zeros of P lie in

$$\begin{aligned} |z| \leq \frac{1}{|a_n|} &\left(|a_0| + N_q + (k_{p-q}\alpha_p - \rho\alpha_q) + (1-\rho)|\alpha_q| + 2 \sum_{s=1}^{p-q} (k_s - 1)|\alpha_{q+s}| \right. \\ &\left. + (k'_{p-q}\beta_p - \rho'\beta_q) + (1-\rho')|\beta_q| + 2 \sum_{s=1}^{p-q} (k'_s - 1)|\beta_{q+s}| + M_p \right). \end{aligned}$$

Next, consider the polynomial

$$T(z) = z^n P\left(\frac{1}{z}\right) = \sum_{s=0}^n a_s z^{n-s}.$$

Let

$$\begin{aligned} R(z) &= (1-z)T(z) \\ &= -a_0 z^{n+1} + \sum_{s=1}^q (a_{s-1} - a_s) z^{n+1-s} + \sum_{s=q+1}^p (a_{s-1} - a_s) z^{n+1-s} + \sum_{s=p+1}^n (a_{s-1} - a_s) z^{n+1-s} + a_n. \end{aligned}$$

With $\alpha_s = Re(a_s)$ and $\beta_s = Im(a_s)$ for $q \leq s \leq p$, we have

$$\begin{aligned}
|R(z)| &\geq |a_0||z|^{n+1} - \left(|a_0 - a_1||z|^n + |a_1 - a_2||z|^{n-1} + \cdots + |a_q - a_{q+1}||z|^{n-q} + \cdots \right. \\
&\quad \left. + |a_p - a_{p+1}||z|^{n-p} + \cdots + |a_{n-1} - a_n||z| + |a_n| \right) \\
&= |a_0||z|^{n+1} - \left(|a_0 - a_1||z|^n + |a_1 - a_2||z|^{n-1} + \cdots + |a_{q-1} - a_q||z|^{n-q+1} \right. \\
&\quad + |\alpha_q + i\beta_q - \alpha_{q+1} - i\beta_{q+1}| + |a_{q-1} - a_q||z|^{n-q} + |\alpha_{q+1} + i\beta_{q+1} - \alpha_{q+2} \\
&\quad - i\beta_{q+2}||z|^{n-q-1} + \cdots + |\alpha_{p-1} + i\beta_{p-1} - \alpha_p - i\beta_p||z|^{n-p+1} + |a_p - a_{p+1}||z|^{n-p} + \\
&\quad \left. \cdots + |a_{n-1} - a_n||z| + |a_n| \right) \\
&\geq |a_0||z|^{n+1} - \left(|a_0 - a_1||z|^n + |a_1 - a_2||z|^{n-1} + \cdots + |a_{q-1} - a_q||z|^{n-q+1} \right. \\
&\quad + |\rho\alpha_q - k_1\alpha_{q+1} + \alpha_q(1 - \rho) + \alpha_{q+1}(k_1 - 1)||z|^{n-q} + |\rho'\beta_q - k'_1\beta_{q+1} \\
&\quad + \beta_q(1 - \rho') + \beta_{q+1}(k'_1 - 1)||z|^{n-q} + |k_1\alpha_{q+1} - k_2\alpha_{q+2} + \alpha_{q+1}(1 - k_1) \\
&\quad + \alpha_{q+2}(k_2 - 1)||z|^{n-q-1} + |k'_1\beta_{q+1} - k'_2\beta_{q+2} + \beta_{q+1}(1 - k'_1) \\
&\quad + \beta_{q+2}(k'_2 - 1)||z|^{n-q-1} + \cdots + |k_{p-q-1}\alpha_{p-1} - k_{p-q}\alpha_p + \alpha_{p-1}(1 - k_{p-q-1}) \\
&\quad + \alpha_p(k_{p-q} - 1)||z|^{n-p+1} + |k'_{p-q-1}\beta_{p-1} - k'_{p-q}\beta_p + \beta_{p-1}(1 - k'_{p-q-1}) \\
&\quad \left. + \beta_p(k'_{p-q} - 1)||z|^{n-p+1} + |a_p - a_{p+1}||z|^{n-p} + \cdots + |a_{n-1} - a_n||z| + |a_n| \right) \\
&\geq |z|^n \left[|a_0||z| - \left(|a_0 - a_1| + \frac{|a_1 - a_2|}{|z|} + \cdots + \frac{|a_{q-1} - a_q|}{|z|^{q-1}} + \frac{k_1\alpha_{q+1} - \rho\alpha_q}{|z|^q} \right. \right. \\
&\quad + \frac{|\alpha_q|(1 - \rho)}{|z|^q} + \frac{|\alpha_{q+1}|(k_1 - 1)}{|z|^q} + \frac{k'_1\beta_{q+1} - \rho'\beta_q}{|z|^q} + \frac{|\beta_q|(1 - \rho')}{|z|^q} \\
&\quad + \frac{|\beta_{q+1}|(k'_1 - 1)}{|z|^q} + \frac{k_2\alpha_{q+2} - k_1\alpha_{q+1}}{|z|^{q+1}} + \frac{|\alpha_{q+1}|(k_1 - 1)}{|z|^{q+1}} + \frac{|\alpha_{q+2}|(k_2 - 1)}{|z|^{q+1}} \\
&\quad + \frac{k'_2\beta_{q+2} - k'_1\beta_{q+1}}{|z|^{q+1}} + \frac{|\beta_{q+1}|(k'_1 - 1)}{|z|^{q+1}} + \frac{|\beta_{q+2}|(k'_2 - 1)}{|z|^{q+1}} + \cdots + \\
&\quad + \frac{k_{p-q}\alpha_p - k_{p-q-1}\alpha_{p-1}}{|z|^{p-1}} + \frac{|\alpha_{p-1}|(k_{p-q-1} - 1)}{|z|^{p-1}} + \frac{|\alpha_p|(k_{p-q} - 1)}{|z|^{p-1}} + \\
&\quad + \frac{k'_{p-q}\beta_p - k'_{p-q-1}\beta_{p-1}}{|z|^{p-1}} + \frac{|\beta_{p-1}|(k'_{p-q-1} - 1)}{|z|^{p-1}} + \frac{|\beta_p|(k'_{p-q} - 1)}{|z|^{p-1}} + \frac{|a_p - a_{p+1}|}{|z|^p} \\
&\quad \left. \left. + \cdots + \frac{|a_{n-1} - a_n|}{|z|^{n-1}} + \frac{|a_n|}{|z|^n} \right) \right].
\end{aligned}$$

Now for $|z| > 1$, implies $\frac{1}{|z|^{n-s}} < 1$, $0 \leq s \leq n - 1$, so that we have

$$\begin{aligned}
 |R(z)| &\geq |z|^n \left[|a_0||z| - \left(|a_0 - a_1| + |a_1 - a_2| + \cdots + |a_{q-1} - a_q| + (k_{p-q}\alpha_p - \rho\alpha_q) \right. \right. \\
 &\quad \left. \left. + (1 - \rho)|\alpha_q| + 2 \sum_{s=1}^{p-q} (k_s - 1)|\alpha_{q+s}| + (k'_{p-q}\beta_p - \rho'\beta_q) + (1 - \rho')|\beta_q| \right. \right. \\
 &\quad \left. \left. + 2 \sum_{s=1}^{p-q} (k'_s - 1)|\beta_{q+s}| + |a_p - a_{p+1}| + \cdots + |a_{n-1} - a_n| + |a_n| \right) \right] \\
 &= |z|^n \left[|a_0||z| - \left(\sum_{s=1}^q |a_{s-1} - a_s| + (k_{p-q}\alpha_p - \rho\alpha_q) + (1 - \rho)|\alpha_q| \right. \right. \\
 &\quad \left. \left. + 2 \sum_{s=1}^{p-q} (k_s - 1)|\alpha_{q+s}| + (k'_{p-q}\beta_p - \rho'\beta_q) + (1 - \rho')|\beta_q| \right. \right. \\
 &\quad \left. \left. + 2 \sum_{s=1}^{p-q} (k'_s - 1)|\beta_{q+s}| + \sum_{s=p+1}^n |a_{s-1} - a_s| + |a_n| \right) \right] \\
 &= |z|^n \left[|a_0||z| - \left(N_q + (k_{p-q}\alpha_p - \rho\alpha_q) + (1 - \rho)|\alpha_q| + 2 \sum_{s=1}^{p-q} (k_s - 1)|\alpha_{q+s}| \right. \right. \\
 &\quad \left. \left. + (k'_{p-q}\beta_p - \rho'\beta_q) + (1 - \rho')|\beta_q| + 2 \sum_{s=1}^{p-q} (k'_s - 1)|\beta_{q+s}| + M_p + |a_n| \right) \right] \\
 &> 0,
 \end{aligned}$$

if

$$\begin{aligned}
 |z| &> \frac{1}{|a_0|} \left(N_q + (k_{p-q}\alpha_p - \rho\alpha_q) + (1 - \rho)|\alpha_q| + 2 \sum_{s=1}^{p-q} (k_s - 1)|\alpha_{q+s}| \right. \\
 &\quad \left. + (k'_{p-q}\beta_p - \rho'\beta_q) + (1 - \rho')|\beta_q| + 2 \sum_{s=1}^{p-q} (k'_s - 1)|\beta_{q+s}| + M_p + |a_n| \right).
 \end{aligned}$$

Therefore all the zeros of $R(z)$ whose modulus is greater than 1 lie in

$$\begin{aligned}
 |z| &\leq \frac{1}{|a_0|} \left(N_q + (k_{p-q}\alpha_p - \rho\alpha_q) + (1 - \rho)|\alpha_q| + 2 \sum_{s=1}^{p-q} (k_s - 1)|\alpha_{q+s}| \right. \\
 &\quad \left. + (k'_{p-q}\beta_p - \rho'\beta_q) + (1 - \rho')|\beta_q| + 2 \sum_{s=1}^{p-q} (k'_s - 1)|\beta_{q+s}| + M_p + |a_n| \right).
 \end{aligned}$$

This gives all zeros of $R(z)$ and hence of $T(z)$ lie in

$$|z| \leq \max \left\{ 1, \left(N_q + (k_{p-q}\alpha_p - \rho\alpha_q) + (1 - \rho)|\alpha_q| + 2 \sum_{s=1}^{p-q} (k_s - 1)|\alpha_{q+s}| \right. \right. \\ \left. \left. + (k'_{p-q}\beta_p - \rho'\beta_q) + (1 - \rho')|\beta_q| + 2 \sum_{s=1}^{p-q} (k'_s - 1)|\beta_{q+s}| + M_p + |a_n| \right) / |a_0| \right\}.$$

Hence all the zeros of $P(z)$ lie in

$$|z| \geq \min \left\{ 1, |a_0| / \left(N_q + (k_{p-q}\alpha_p - \rho\alpha_q) + (1 - \rho)|\alpha_q| + 2 \sum_{s=1}^{p-q} (k_s - 1)|\alpha_{q+s}| \right. \right. \\ \left. \left. + (k'_{p-q}\beta_p - \rho'\beta_q) + (1 - \rho')|\beta_q| + 2 \sum_{s=1}^{p-q} (k'_s - 1)|\beta_{q+s}| + M_p + |a_n| \right) \right\},$$

as claimed. □

We now give a proof of Theorem 2.2.

Proof. Let $P(z) = \sum_{s=0}^n a_s z^s$, $a_n \neq 0$ be the given polynomial satisfying the stated hypotheses of Theorem 2.2. We can assume without loss of generality that $\mu = 0$. Consider

$$P(z)(1 - z) = a_0 + \sum_{s=1}^n (a_s - a_{s-1})z^s = f(z) - a_n z^{n+1}$$

For $|z| = 1$, we have

$$\begin{aligned} |f(z)| &= \left| a_0 + \sum_{s=1}^n (a_s - a_{s-1})z^s \right| \\ &\leq |a_0| + \sum_{s=1}^n |a_s - a_{s-1}| \\ &= |a_0| + \sum_{s=1}^q |a_s - a_{s-1}| + |a_{q+1} - a_q| + \dots + |a_{p-1} - a_{p-2}| + |a_p - a_{p-1}| \\ &\quad + \sum_{s=p+1}^n |a_s - a_{s-1}| \\ &= |a_0| + N_q + |a_{q+1} - a_q| + \dots + |a_{p-1} - a_{p-2}| + |a_p - a_{p-1}| + M_p, \end{aligned}$$

where $N_q = \sum_{s=1}^q |a_s - a_{s-1}|$ and $M_p = \sum_{s=p+1}^n |a_s - a_{s-1}|$. Therefore for $|z| = 1$

$$\begin{aligned} |f(z)| &\leq |a_0| + N_q + |a_{q+1} - a_q| + \cdots + |a_{p-1} - a_{p-2}| + |a_p - a_{p-1}| + M_p \\ &= |a_0| + N_q + |k_1 a_{q+1} - \rho a_q + a_q(\rho - 1) + a_{q+1}(1 - k_1)| + \\ &\quad |k_2 a_{q+2} - k_1 a_{q+1} + a_{q+1}(k_1 - 1) + a_{q+2}(1 - k_2)| + \cdots + \\ &\quad |k_{p-q} a_p - k_{p-q-1} a_{p-1} + a_{p-1}(k_{p-q-1} - 1) + a_p(1 - k_{p-q})| + M_p \\ &\leq |a_0| + N_q + |k_1 a_{q+1} - \rho a_q| + |a_q|(1 - \rho) + |a_{q+1}|(k_1 - 1) + \\ &\quad |k_2 a_{q+2} - k_1 a_{q+1}| + |a_{q+1}|(k_1 - 1) + |a_{q+2}|(k_2 - 1) + \cdots + \\ &\quad |k_{p-q} a_p - k_{p-q-1} a_{p-1}| + |a_{p-1}|(k_{p-q-1} - 1) + |a_p|(k_{p-q} - 1) + M_p \end{aligned}$$

Applying Lemma 3.1, we get for $|z| = 1$,

$$\begin{aligned} |f(z)| &\leq |a_0| + N_q + (k_1 |a_{q+1}| - |\rho a_q|) \cos \theta + (k_1 |a_{q+1}| + |\rho a_q|) \sin \theta + \\ &\quad |a_q|(1 - \rho) + 2 \sum_{s=1}^{p-q} (k_s - 1) |a_{q+s}| + (k_2 |a_{q+2}| - k_1 |a_{q+1}|) \cos \theta + \\ &\quad (k_2 |a_{q+2}| + k_1 |a_{q+1}|) \sin \theta + \cdots + (k_{p-q} |a_p| - k_{p-q-1} |a_{p-1}|) \cos \theta + \\ &\quad (k_{p-q} |a_p| + k_{p-q-1} |a_{p-1}|) \sin \theta + M_p \\ &\leq |a_0| + N_q + |a_q| + 2 \sum_{s=1}^{p-q-1} (k_s - 1) |a_{q+s}| + \rho |a_q| (\sin \theta - \cos \theta - 1) \\ &\quad + 2 \sin \theta \sum_{s=1}^{p-q-1} k_s |a_{q+s}| + k_{p-q} |a_p| (\sin \theta + \cos \theta + 1) - |a_p| + M_p. \end{aligned}$$

Hence also,

$$\begin{aligned} \left| z^n f\left(\frac{1}{z}\right) \right| &\leq |a_0| + N_q + |a_q| + 2 \sum_{s=1}^{p-q-1} (k_s - 1) |a_{q+s}| + \rho |a_q| (\sin \theta - \cos \theta - 1) \\ &\quad + 2 \sin \theta \sum_{s=1}^{p-q-1} k_s |a_{q+s}| + k_{p-q} |a_p| (\sin \theta + \cos \theta + 1) - |a_p| + M_p. \end{aligned}$$

Therefore by Maximum Modulus Theorem

$$\begin{aligned} \left| z^n f\left(\frac{1}{z}\right) \right| &\leq |a_0| + N_q + |a_q| + 2 \sum_{s=1}^{p-q-1} (k_s - 1) |a_{q+s}| + \rho |a_q| (\sin \theta - \cos \theta - 1) \\ &\quad + 2 \sin \theta \sum_{s=1}^{p-q-1} k_s |a_{q+s}| + k_{p-q} |a_p| (\sin \theta + \cos \theta + 1) - |a_p| + M_p \end{aligned}$$

holds inside $|z| \leq 1$ as well.

If $R > 1$, then $\frac{1}{R} e^{-i\omega}$ lies inside the unit disc for every real ω . Hence it follows

that

$$|f(Re^{i\omega})| \leq \left(|a_0| + N_q + |a_q| + 2 \sum_{s=1}^{p-q-1} (k_s - 1)|a_{q+s}| + \rho|a_q|(\sin \theta - \cos \theta - 1) \right. \\ \left. + 2 \sin \theta \sum_{s=1}^{p-q-1} k_s|a_{q+s}| + k_{p-q}|a_p|(\sin \theta + \cos \theta + 1) - |a_p| + M_p \right) R^n$$

for every $R \geq 1$ and real ω . Thus for every $|z| = R > 1$,

$$|P(z)(1-z)| = |-a_n z^{n+1} + f(z)| \\ \geq |a_n|R^{n+1} - |f(z)| \\ \geq |a_n|R^{n+1} - \left(|a_0| + N_q + |a_q| + 2 \sum_{s=1}^{p-q-1} (k_s - 1)|a_{q+s}| + \rho|a_q|(\sin \theta - \right. \\ \left. \cos \theta - 1) + 2 \sin \theta \sum_{s=1}^{p-q-1} k_s|a_{q+s}| + k_{p-q}|a_p|(\sin \theta + \cos \theta + 1) - \right. \\ \left. |a_p| + M_p \right) R^n \\ = R^n \left[|a_n|R - \left(|a_0| + N_q + |a_q| + 2 \sum_{s=1}^{p-q-1} (k_s - 1)|a_{q+s}| + \rho|a_q|(\sin \theta - \right. \right. \\ \left. \left. \cos \theta - 1) + 2 \sin \theta \sum_{s=1}^{p-q-1} k_s|a_{q+s}| + k_{p-q}|a_p|(\sin \theta + \cos \theta + 1) - \right. \right. \\ \left. \left. |a_p| + M_p \right) \right] \\ > 0,$$

if

$$R > \frac{1}{|a_n|} \left(|a_0| + N_q + |a_q| + 2 \sum_{s=1}^{p-q-1} (k_s - 1)|a_{q+s}| + \rho|a_q|(\sin \theta - \right. \\ \left. \cos \theta - 1) + 2 \sin \theta \sum_{s=1}^{p-q-1} k_s|a_{q+s}| + k_{p-q}|a_p|(\sin \theta + \cos \theta + 1) - \right. \\ \left. |a_p| + M_p \right).$$

Therefore all the zeros of $P(z)$ lie in

$$|z| \leq \frac{1}{|a_n|} \left(|a_0| + N_q + |a_q| + 2 \sum_{s=1}^{p-q-1} (k_s - 1)|a_{q+s}| + \rho|a_q|(\sin \theta - \right. \\ \left. \cos \theta - 1) + 2 \sin \theta \sum_{s=1}^{p-q-1} k_s|a_{q+s}| + k_{p-q}|a_p|(\sin \theta + \cos \theta + 1) - \right. \\ \left. |a_p| + M_p \right)$$

as claimed.

Next, consider the polynomial

$$T(z) = z^n P\left(\frac{1}{z}\right) = \sum_{s=0}^n a_s z^{n-s}.$$

Let

$$\begin{aligned} R(z) &= (1-z)T(z) \\ &= -a_0 z^{n+1} + (a_0 - a_1)z^n + (a_1 - a_2)z^{n-1} + \dots + (a_q - a_{q+1})z^{n-q} + \dots \\ &\quad + (a_p - a_{p+1})z^{n-p} + \dots + (a_{n-1} - a_n)z + a_n. \end{aligned}$$

This gives

$$\begin{aligned} |R(z)| &\geq |a_0||z|^{n+1} - \left(|a_0 - a_1||z|^n + |a_1 - a_2||z|^{n-1} + \dots + |a_q - a_{q+1}||z|^{n-q} + \dots \right. \\ &\quad \left. + |a_p - a_{p+1}||z|^{n-p} + \dots + |a_{n-1} - a_n||z| + |a_n| \right) \\ &= |a_0||z|^{n+1} - \left(|a_0 - a_1||z|^n + |a_1 - a_2||z|^{n-1} + \dots + |a_{q-1} - a_q||z|^{n-q+1} + \right. \\ &\quad | \rho a_q - k_1 a_{q+1} + a_q(1 - \rho) + a_{q+1}(k_1 - 1) ||z|^{n-q} + |k_1 a_{q+1} - k_2 a_{q+2} \\ &\quad + a_{q+1}(1 - k_1) + a_{q+2}(k_2 - 1) ||z|^{n-q-1} + \dots + |k_{p-q-1} a_{p-1} - k_{p-q} a_p \\ &\quad + a_{p-1}(1 - k_{p-q-1}) + a_p(k_{p-q} - 1) ||z|^{n-p+1} + |a_p - a_{p+1}||z|^{n-p} + \dots + \\ &\quad \left. |a_{n-1} - a_n||z| + |a_n| \right) \\ &= |z|^n \left[|a_0||z| - \left(|a_0 - a_1| + |a_1 - a_2| \frac{1}{|z|^{n-1}} + \dots + |a_{q-1} - a_q| \frac{1}{|z|^{q-1}} + \right. \right. \\ &\quad | \rho a_q - k_1 a_{q+1} + a_q(1 - \rho) + a_{q+1}(k_1 - 1) | \frac{1}{|z|^q} + |k_1 a_{q+1} - k_2 a_{q+2} \\ &\quad + a_{q+1}(1 - k_1) + a_{q+2}(k_2 - 1) | \frac{1}{|z|^{q+1}} + \dots + |k_{p-q-1} a_{p-1} - k_{p-q} a_p \\ &\quad + a_{p-1}(1 - k_{p-q-1}) + a_p(k_{p-q} - 1) | \frac{1}{|z|^{p-1}} + |a_p - a_{p+1}| \frac{1}{|z|^p} + \dots + \\ &\quad \left. \left. |a_{n-1} - a_n| \frac{1}{|z|^{n-1}} + |a_n| \frac{1}{|z|^n} \right) \right]. \end{aligned}$$

Now for $|z| > 1$, implies $\frac{1}{|z|^{n-s}} < 1$, $0 \leq s < n$, so that we have

$$|R(z)| \geq |z|^n \left[|a_0||z| - \left(|a_0 - a_1| + |a_1 - a_2| + \dots + |a_{q-1} - a_q| + \right. \right. \\ \left. \left. |\rho a_q - k_1 a_{q+1} + a_q(1 - \rho) + a_{q+1}(k_1 - 1)| + |k_1 a_{q+1} - k_2 a_{q+2} \right. \right. \\ \left. \left. + a_{q+1}(1 - k_1) + a_{q+2}(k_2 - 1) \right) + \dots + |k_{p-q-1} a_{p-1} - k_{p-q} a_p \right. \\ \left. + a_{p-1}(1 - k_{p-q-1}) + a_p(k_{p-q} - 1) \right) + |a_p - a_{p+1}| + \dots + \\ \left. |a_{n-1} - a_n| + |a_n| \right].$$

Which with the help of Lemma 3.1 as done above yields

$$|R(z)| \geq |z|^n \left[|a_0||z| - \left(N_q + |a_q| + 2 \sum_{s=1}^{p-q-1} (k_s - 1)|a_{q+s}| + \rho|a_q|(\sin \theta - \cos \theta - 1) \right. \right. \\ \left. \left. + 2 \sin \theta \sum_{s=1}^{p-q-1} k_s |a_{q+s}| + k_{p-q}|a_p|(\sin \theta + \cos \theta + 1) - |a_p| + M_p + |a_n| \right) \right] \\ > 0,$$

if

$$|z| > \frac{1}{|a_0|} \left(N_q + |a_q| + 2 \sum_{s=1}^{p-q-1} (k_s - 1)|a_{q+s}| + \rho|a_q|(\sin \theta - \cos \theta - 1) \right. \\ \left. + 2 \sin \theta \sum_{s=1}^{p-q-1} k_s |a_{q+s}| + k_{p-q}|a_p|(\sin \theta + \cos \theta + 1) - |a_p| + M_p + |a_n| \right).$$

Therefore all the zeros of $R(z)$ whose modulus is greater than 1 lie in

$$|z| \leq \frac{1}{|a_0|} \left(N_q + |a_q| + 2 \sum_{s=1}^{p-q-1} (k_s - 1)|a_{q+s}| + \rho|a_q|(\sin \theta - \cos \theta - 1) \right. \\ \left. + 2 \sin \theta \sum_{s=1}^{p-q-1} k_s |a_{q+s}| + k_{p-q}|a_p|(\sin \theta + \cos \theta + 1) - |a_p| + M_p + |a_n| \right).$$

This gives all zeros of $R(z)$ and hence of $T(z)$ lie in

$$|z| \leq \max \left\{ 1, \left(N_q + |a_q| + 2 \sum_{s=1}^{p-q-1} (k_s - 1)|a_{q+s}| + \rho|a_q|(\sin \theta - \cos \theta - 1) \right. \right. \\ \left. \left. + 2 \sin \theta \sum_{s=1}^{p-q-1} k_s |a_{q+s}| + k_{p-q}|a_p|(\sin \theta + \cos \theta + 1) - |a_p| + M_p + |a_n| \right) / |a_0| \right\}.$$

Hence all the zeros of $P(z)$ lie in

$$|z| \geq \min \left\{ 1, |a_0| / \left(N_q + |a_q| + 2 \sum_{s=1}^{p-q-1} (k_s - 1) |a_{q+s}| + \rho |a_q| (\sin \theta - \cos \theta - 1) \right. \right. \\ \left. \left. + 2 \sin \theta \sum_{s=1}^{p-q-1} k_s |a_{q+s}| + k_{p-q} |a_p| (\sin \theta + \cos \theta + 1) - |a_p| + M_p + |a_n| \right) \right\}.$$

This completes the proof of Theorem 2.2 \square

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References

- [1] Aziz, A. and Zargar, B. A., *Bounds for the zeros of a polynomial with restricted coefficients*, Applied Mathematics, **3** (2012), 30–33.
<http://dx.doi.org/10.4236/am.2012.31005>
- [2] Chan, T. and Malik, M., *On Erdős-Lax Theorem*, Proceedings of the Indian Academy of Science, **250** (1983), 191–193.
- [3] Eneström, G., *Härledning af en allmän formel för antalet pensionärer, som vid en godtycklig tidpunkt förefinnas inom en sluten pensionskassa*, Övfers. Vetensk.-Akad. Förhh., **50** (1983), 405–415.
- [4] Gardner, R. and Govil, N. K., *On the Location of the Zeros of a Polynomial*, Journal of Approximation Theory, **78** (1994), 286–292 .
- [5] Gardner, R. and Govil, N. K., *The Eneström-Keakeya Theorem and Some of Its Generalizations*, Current Topics in Pure and Computational Complex Analysis, S. Joshi, M. Dorff, and I. Lahiri, New Delhi: Springer-Verlag (2014), 171-200.
<https://faculty.etsu.edu/gardnerr/pubs/GardnerGovilF.pdf>
- [6] Govil, N. K. and Rahman, Q. I., *On the Eneström-Keakeya Theorem*, Tôhoku Math. J. **20** (1968), 126–136.
- [7] Joyal, A. ; Labelle, G. and Rahman, Q. I., *On the Location of Zeros of Polynomials.*, Canadian Mathematical Bulletin **10**(1) (1967), 53–63 .
- [8] Keakeya, S., *On the limits of the roots of an algebraic equation with positive coefficient*, Tôhoku Math. J. **2** (1912-1913), 140–142.
- [9] Shah, M. A.; Swroop, R.; Sofi, H. M. and Nisar, I., *A generalisation of Eneström-Keakeya Theorem and a zero free region of a polynomial*. Journal of Applied Mathematics and Physics, **9**(2021), 1271–1277.
<https://doi.org/10.4236/jamp.2021.96087>
- [10] Marden, M., *Geometry of Polynomials.*, Math. Surveys, No. **3**, Amer. Math. Soc.

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