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SEMILOCAL CONVERGENCE ANALYSIS OF THE THIRD ORDER NEWTON-LIKE METHOD IN RIEMANNIAN MANIFOLDS

Chandresh Prasad and Pradip Kumar Parida*

Abstract. In this paper, we present the semilocal convergence analysis of the third order Newton-like method in Riemannian manifolds. We study the convergence analysis of our method under Lipschitz continuity condition on the first order covariant derivative of a vector field. Using normal coordinates the order of convergence is derived. Finally, a numerical example is given to show the effectiveness of our results.

1. Introduction

Many problems in civil engineering, chemical engineering, mechanics and numerical optimization can be solved through nonlinear equation

$$\mathfrak{E}(x) = 0, \tag{1}$$

where $\mathfrak{E}$ is a nonlinear operator defined in an open convex subset Θ of a Banach space B into itself. The exact solution of these equations are difficult to find so that we use iterative methods to solve these equations. Some study the convergence of iterative methods usually based on semilocal and local convergence analysis. If the convergence analysis which gives information about a solution and estimates the radius of the convergence ball, then it is said to be local convergence where as if the convergence analysis tells information about an initial point, then it is said to be semilocal convergence. There are thousands of papers that can be found in literature which devoted to study of so many iterative methods in Banach spaces [1–3]. The third order Newton-like method [6, 7] in Banach space to solve (1) is defined as:

$$\left.\begin{aligned} y_n &= x_n - \mathfrak{E}'(x_n)^{-1}\mathfrak{E}(x_n), \\ x_{n+1} &= x_n - \Big(\frac{\mathfrak{E}'(x_n) + \mathfrak{E}'(y_n)}{2}\Big)^{-1}\mathfrak{E}(x_n), \text{ for each } n = 0, 1, 2, \ldots, \end{aligned}\right\} \tag{2}$$

where $\mathfrak{E}'$ is first Fréchet derivative of $\mathfrak{E}$. As is known, the iterative methods of higher order to find the zero of a nonlinear equation (1) have been studied in the past years. The most important iterative methods are Newton's, Halley, Super-Halley, Chebyshev, Chebyshev-Halley, and the two-steps. The convergence of these methods

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∗ Corresponding author.

in Banach space started with the work of Kantorovich in the 1960's. There are several proofs for the convergence and uniqueness of these methods under different assumptions and the initial points considered, see [11–15]. The Newton-like method is probably the best known whose iterative procedure is cubic for solving nonlinear equations, in the same way as in Banach spaces [6]. Recently, there has been a growing interest in studying iterative methods in Riemannian manifolds, since there are many numerical problems in manifolds that arise in many contexts (see for instance [5] and the references therein). In developing iterative method from Banach space to Riemannian manifold creates some difficulties which did not happen in Banach spaces. Some difficulties are, in Banach spaces the space B and T_aB at a point a are isomorphic so that it is legitimate to sum points and vectors but in Riemannian manifold this does not happen. For this reason we have used the exponential function. Another difficulty is ordinary derivative as ordinary derivative is sufficient to define a iterative method in Banach space but in manifold ordinary derivative is not necessary tangent to the manifold because of this we have used covariant derivative to define a iterative method in manifold. Other kind of difficulty is unlike Banach spaces geodesics are straight lines but in manifolds open balls are not necessarily geodesically convex. The result obtained in this paper is basically to find the singular point of a vector fields in Riemannian manifolds. We will also prove the convergence and uniqueness of our iterative method using recurrence relations. To do this we will use some basic results of differential geometry.

The paper is divided into six sections as follows: Section 1 is the introduction. In Section 2, we discuss some basic properties of differential geometry. In Section 3, we present the semilocal convergence analysis of the third order Newton-like method under Lipschitz continuity condition on the first order covariant derivative of a vector field to approximate the zero of a vector field in Riemannian manifolds. In Section 4, order of convergence using normal coordinates are given. In Section 5, an example is given to show the effectiveness of our results. Finally, in Section 6, conclusion is given.

2. Preliminaries

In this section, we discuss some basic results of differential geometry (see [8], [18]).

Definition 2.1. Let Z be a real n dimensional Riemannian manifold. The tangent space of Z at a is denoted by T_aZ. The inner product $\langle .,. \rangle_a$ on T_aZ induces the norm $\|.\|_a$. The tangent bundle of Z is denoted by TZ and is defined by

$$TZ := \{(a, v); a \in Z \text{ and } v \in T_aZ\} = \bigcup_{a \in Z} T_aZ.$$

Let $a, t \in Z$, and let $\varrho : [0, 1] \to Z$ be a piecewise smooth curve joining the points a and t. Then the arc length of ϱ is defined by

$$l(\varrho) = \int_0^1 \|\varrho'(x)\| dx = \int_0^1 \left\langle \frac{d\varrho}{dx}, \frac{d\varrho}{dx} \right\rangle^{\frac{1}{2}} dx$$

and the Riemannian distance from a to t is defined by

$$d(a, t) = \inf_{\varrho} l(\varrho),$$

where the infimum is taken over all the piecewise smooth curves ϱ connecting a and t.

Definition 2.2. A vector field X is a map $X : Z \to TZ$ such that $X(a) \in T_aZ$ for any $a \in Z$.

Definition 2.3. Let $\chi(Z)$ be the set of all vector fields of class C^∞ on Z and $D(Z)$ the ring of real-valued functions of class C^∞ defined on Z. An affine connection ∇ on Z is a mapping

$$\begin{aligned} \nabla : \chi(Z) \times \chi(Z) &\to \chi(Z) \\ (X, \mathfrak{U}) &\mapsto \nabla_X \mathfrak{U} \end{aligned}$$

that satisfies the following properties

$$\begin{aligned} &\text{(i) } \nabla_{fX+g\mathfrak{U}}\mathfrak{V} = f\nabla_X\mathfrak{V} + g\nabla_{\mathfrak{U}}\mathfrak{V}. \\ &\text{(ii) } \nabla_X(\mathfrak{U}+\mathfrak{V}) = \nabla_X\mathfrak{U} + \nabla_X\mathfrak{V}. \\ &\text{(iii) } \nabla_X(f\mathfrak{U}) = f\nabla_X\mathfrak{U} + X(f)\mathfrak{U}, \end{aligned}$$

where $X,\ \mathfrak{U},\ \mathfrak{V} \in \chi(Z)$ and $f\ , g \in D(Z)$.

Definition 2.4. Let $\mathfrak{U}$ be a vector field of class C^1 on Z, the covariant derivative of $\mathfrak{U}$ is determined by the connection ∇ which defines on each $a \in Z$ a linear application of T_aZ itself

$$\begin{aligned} D\mathfrak{U}(a) : T_aZ &\to T_aZ \\ v \mapsto D\mathfrak{U}(a)(v) &= \nabla_X\mathfrak{U}(a), \end{aligned}$$

where X is a vector field satisfying $X(a) = v$.

Definition 2.5. A geodesic in Z connecting the points a and t is called a minimizing geodesic if its arc length equals its Riemannian distance between a and t.

Definition 2.6. A parametrized curve $\varrho : I \subseteq \mathbb{R} \to Z$ is a geodesic at $p_0 \in I$, if $\nabla_{\varrho'(p)}\varrho'(p) = 0$ at the point p_0. If ϱ is a geodesic for all $p \in I$, we say ϱ is a geodesic. If $[x, y] \subseteq I$, ϱ is a geodesic segment joining $\varrho(x)$ to $\varrho(y)$.

A basic property of geodesic is that $\varrho'(p)$ is parallel along $\varrho(p)$; this implies that $\|\varrho'(p)\|$ is constant.

Let $U(a, s)$ and $U[a, s]$ be an open and a closed geodesic ball with centre a and radius s respectively. By the Hopf-Rinow theorem (see [17]), if Z is a complete metric space, then for any $a, t \in Z$ there exists a geodesic ϱ, called the minimizing geodesic ϱ joining a to t with

$$l(\varrho) = d(a, t).$$

Moreover, if $v \in T_aZ$ then there exists a unique minimizing geodesic ϱ such that $\varrho(0) = a$ and $\varrho'(0) = v$. The point $\varrho(1)$ is called the image of v by the exponential map at a, i.e.

$$\exp_a : T_aZ \to Z,$$

such that $\exp_a(v) = \varrho(1)$ and $\varrho(p) = \exp_a(pv)$, for any $p \in [0, 1]$.

Definition 2.7. Let ϱ be a piecewise smooth curve. Then for any $x, y \in \mathbb{R}$, the parallel transport along ϱ is denoted by $R_{\varrho,\cdot,\cdot}$ and given by

$$\begin{aligned} R_{\varrho,x,y} : T_{\varrho(x)}Z &\to T_{\varrho(y)}Z \\ v &\mapsto V(\varrho(y)), \end{aligned}$$

where V is the unique vector field along ϱ such that $\nabla_{\varrho'(p)}V = 0$ and $V(\varrho(x)) = v$.

Definition 2.8. Let $j \in \mathbb{N}$ and $\mathfrak{U}$ be a vector field of class C^k. Then the covariant derivative of order j of $\mathfrak{U}$ is denoted by $D^j\mathfrak{U}$ and defined as the multilinear map

$$D^j\mathfrak{U} : \underbrace{C^k(TZ) \times C^k(TZ) \times \cdots \times C^k(TZ)}_{\text{j-times}} \to C^{k-j}(TZ)$$

which is given by

$$\begin{aligned} D^j\mathfrak{U}(A_1, A_2, \ldots, A_{j-1}, A) &= \nabla_A D^{j-1}\mathfrak{U}(A_1, A_2, \ldots, A_{j-1}) \\ &- \sum_{i=1}^{j-1} D^{j-1}\mathfrak{U}(A_1, A_2, \ldots, \nabla_A A_i, \ldots, A_{j-1}), \end{aligned} \tag{3}$$

for all $A_1, A_2, \ldots, A_{j-1} \in C^k(TZ)$.

Definition 2.9. A set H in a Riemannian manifold Z is called convex if for any pair of points $a,\ t \in H$ any minimizing geodesic $[a, t]$ lies in H.

Definition 2.10. Let Z be a Riemannian manifold, $\Omega \subseteq Z$ be an open convex set, and $\mathfrak{U} \in \chi(Z)$. Then the covariant derivative $D\mathfrak{U} = \nabla_{(.)}\mathfrak{U}$ is Lipschitz with constant $K > 0$, if for any geodesic ϱ and $x, y \in \mathbb{R}$ such that $\varrho[x, y] \subseteq \Omega$, it satisfies the inequality

$$\|R_{\varrho,y,x} D\mathfrak{U}(\varrho(y)) R_{\varrho,x,y} - D\mathfrak{U}(\varrho(x))\| \leq K \int_x^y \|\varrho'(p)\| dp.$$

We will write $D\mathfrak{U} \in Lip_K(\Omega)$. If $Z = \mathbb{R}^n$, then the above definition is the same as the usual Lipschitz condition for the operator $D\mathfrak{U} : Z \to Z$.

Lemma 2.11. (Banach's lemma ([9])) Let P be an invertible bounded linear operator in a Banach space B and Q be a bounded linear operator in B. If

$$\|P^{-1}Q - I\| < 1$$

then Q is invertible and

$$\|Q^{-1}\| \leq \frac{\|P^{-1}\|}{1 - \|P^{-1}Q - I\|} \leq \frac{\|P^{-1}\|}{1 - \|P^{-1}\|\|Q - P\|}.$$

Moreover,

$$\|Q^{-1}P\| \leq \frac{1}{1 - \|P^{-1}Q - I\|} \leq \frac{1}{1 - \|P^{-1}\|\|Q - P\|}.$$

Now, we take some theorems from [8] which are useful to prove our convergence of iterative method and the proofs are given there.

Theorem 2.12. *Let ϱ be a geodesic in Z and let $\mathfrak{U}$ be a C^1-vector field on Z. Then,*

$$R_{\varrho,t,0}\mathfrak{U}(\varrho(t)) = \mathfrak{U}(\varrho(0)) + \int_0^t R_{\varrho,\theta,0} D\mathfrak{U}(\varrho(\theta))\varrho'(\theta) d\theta. \tag{4}$$

THEOREM 2.13. *Let ϱ be a geodesic in Z and let $\mathfrak{U}$ be a C^2-vector field on Z. Then,*

$$R_{\varrho,t,0}D\mathfrak{U}(\varrho(t))\varrho'(t) = D\mathfrak{U}(\varrho(0))\varrho'(0) + \int_0^t R_{\varrho,\theta,0}D^2\mathfrak{U}(\varrho(\theta))(\varrho'(\theta),\varrho'(\theta))d\theta.$$

3. Third order Newton-like method in Riemannian manifolds

In this section, we will establish the semilocal convergence analysis of the third order Newton-like method in Riemannian manifolds. The third order Newton-like method (2) in Riemannian manifolds has the form

$$\left.\begin{aligned} f_n &= -\Gamma_n\mathfrak{U}(a_n),\\ b_n &= \exp_{a_n}(f_n),\\ \Psi_n(t) &= \exp_{a_n}(tf_n),\\ g_n &= -\left(\frac{D\mathfrak{U}(a_n) + R_{\Psi_n,1,0}D\mathfrak{U}(b_n)R_{\Psi_n,0,1}}{2}\right)^{-1}\mathfrak{U}(a_n),\\ a_{n+1} &= \exp_{a_n}(g_n),\ \text{for each } n = 0,1,2,\ldots, \end{aligned}\right\} \tag{5}$$

where $D\mathfrak{U}(a_n) = \nabla_{(.)}\mathfrak{U}(a_n)$ and $\Gamma_n = D\mathfrak{U}(a_n)^{-1}$. Let $a_0 \in \Omega \subseteq Z$ and assume that

1. $\|\Gamma_0\| \le \xi,\ \xi > 0,$
2. $\|\Gamma_0\mathfrak{U}(a_0)\| \le \zeta,\ \zeta > 0,$
3. $\|R_{\varrho,b,a}D\mathfrak{U}(\varrho(b))R_{\varrho,a,b} - D\mathfrak{U}(\varrho(a))\| \le K\int_a^b \|\varrho'(x)\|dx,\ K > 0,$ where ϱ is the geodesic such that $\varrho[a,b] \subseteq \Omega$.

Let $x_0 = K\xi\zeta$ and for all $n \ge 1$, we define the sequences

$$y_n = \mathcal{A}(x_n)\mathcal{B}(x_n),\ x_{n+1} = x_n\mathcal{A}(x_n)y_n, \tag{6}$$

where

$$\mathcal{A}(a) = \frac{2-a}{2-3a}, \tag{7}$$

$$\mathcal{B}(a) = \frac{a(4-a)}{(2-a)^2}. \tag{8}$$

Let $\mathbf{b}_0$ be the smallest positive zero of the polynomial $\mathcal{C}(a) = 5a^2 - 8a + 2$, then $\mathbf{b}_0 = 0.310102\ldots$ Before studying the convergence of (5), we will first analyze some properties of real functions given in (7)-(8) and then about the sequence $\{x_n\}$.

LEMMA 3.1. *Let $\mathcal{A}(a)$ and $\mathcal{B}(a)$ be the real valued functions, which is given by* (7) *and* (8)*, then, for $a \in (0, \mathbf{b}_0]$*

1. $\mathcal{A}$, $\mathcal{B}$ are increasing and $\mathcal{A}(a) > 1$,
2. $\mathcal{A}(\lambda a) < \mathcal{A}(a)$ and $\mathcal{B}(\lambda a) < \lambda\mathcal{B}(a)$, for $\lambda \in (0,1)$.

Proof. It is easy to prove and hence omitted. □

LEMMA 3.2. *Let $x_0 \in (0, \mathbf{b}_0]$, then,*

1. $y_n\mathcal{A}(x_n) \le 1$,
2. $\{x_n\}$, $\{y_n\}$ are decreasing sequence and $x_n < 1$, for all $n \ge 1$.

Proof. See [6]. □

LEMMA 3.3. *Suppose that all the assumptions of Lemma 3.2 holds and define* $\varsigma = x_1/x_0$, *then, for* $n \geq 1$, *we have*

1. $x_n \leq \varsigma^{2^{n-1}} x_{n-1} \leq \varsigma^{2^{n-1}} x_0$, *for* $n \geq 2$,
2. $y_n < \varsigma^{2^n}/\mathcal{A}(x_0)$.

Proof. See [6]. □

LEMMA 3.4. *Let* $\mathfrak{U}$ *be a vector field of class* C^1 *and let* $\Upsilon_n(t) = \exp_{a_n}(tg_n)$ *be a family of geodesically convex neighborhoods with minimizing geodesics, then, for all* $n \geq 0$, *we have*

$$\Gamma_n R_{\Upsilon_n,1,0}\mathfrak{U}(a_{n+1}) = \Big(\frac{R_{\Psi_n,1,0}D\mathfrak{U}(b_n)R_{\Psi_n,0,1} + D\mathfrak{U}(a_n)}{2}\Big)^{-1}\Big[\frac{R_{\Psi_n,1,0}D\mathfrak{U}(b_n)R_{\Psi_n,0,1} - D\mathfrak{U}(a_n)}{2}\Big] \times \Gamma_n\mathfrak{U}(a_n) + \int_0^1 \Gamma_n\big[R_{\Upsilon_n,t,0}D\mathfrak{U}(\Upsilon_n(t))R_{\Upsilon_n,0,t} - D\mathfrak{U}(a_n)\big]\Upsilon_n'(0)dt.$$

Proof. We have

$$\begin{aligned}
\Gamma_n R_{\Upsilon_n,1,0}\mathfrak{U}(a_{n+1}) =&\Gamma_n\mathfrak{U}(a_n) + \Gamma_n R_{\Upsilon_n,1,0}\mathfrak{U}(a_{n+1}) - \Gamma_n\mathfrak{U}(a_n)\\
=&\Gamma_n\mathfrak{U}(a_n) + \int_0^1 \Gamma_n R_{\Upsilon_n,t,0}D\mathfrak{U}(\Upsilon_n(t))R_{\Upsilon_n,0,t}\Upsilon_n'(0)dt\\
=&\Gamma_n\mathfrak{U}(a_n) + \Upsilon_n'(0) + \int_0^1 \big(\Gamma_n R_{\Upsilon_n,t,0}D\mathfrak{U}(\Upsilon_n(t))R_{\Upsilon_n,0,t} - I_{a_n}\big)\Upsilon_n'(0)dt\\
=&\Gamma_n\mathfrak{U}(a_n) - \Big(\frac{D\mathfrak{U}(a_n) + R_{\Psi_n,1,0}D\mathfrak{U}(b_n)R_{\Psi_n,0,1}}{2}\Big)^{-1}\mathfrak{U}(a_n)\\
&+ \int_0^1 \Gamma_n\Big[R_{\Upsilon_n,t,0}D\mathfrak{U}(\Upsilon_n(t))R_{\Upsilon_n,0,t} - D\mathfrak{U}(a_n)\Big]\Upsilon_n'(0)dt\\
=&\Big[I_{a_n} - \Big(\frac{D\mathfrak{U}(a_n) + R_{\Psi_n,1,0}D\mathfrak{U}(b_n)R_{\Psi_n,0,1}}{2}\Big)^{-1}D\mathfrak{U}(a_n)\Big]\Gamma_n\mathfrak{U}(a_n)\\
&+ \int_0^1 \Gamma_n\Big[R_{\Upsilon_n,t,0}D\mathfrak{U}(\Upsilon_n(t))R_{\Upsilon_n,0,t} - D\mathfrak{U}(a_n)\Big]\Upsilon_n'(0)dt\\
=&\Big(\frac{D\mathfrak{U}(a_n) + R_{\Psi_n,1,0}D\mathfrak{U}(b_n)R_{\Psi_n,0,1}}{2}\Big)^{-1}\\
&\times\Big[\frac{R_{\Psi_n,1,0}D\mathfrak{U}(b_n)R_{\Psi_n,0,1} + D\mathfrak{U}(a_n)}{2} - D\mathfrak{U}(a_n)\Big]\Gamma_n\mathfrak{U}(a_n)\\
&+ \int_0^1 \Gamma_n\Big[R_{\Upsilon_n,t,0}D\mathfrak{U}(\Upsilon_n(t))R_{\Upsilon_n,0,t} - D\mathfrak{U}(a_n)\Big]\Upsilon_n'(0)dt\\
=&\Big(\frac{D\mathfrak{U}(a_n) + R_{\Psi_n,1,0}D\mathfrak{U}(b_n)R_{\Psi_n,0,1}}{2}\Big)^{-1}\\
&\times\Big[\frac{R_{\Psi_n,1,0}D\mathfrak{U}(b_n)R_{\Psi_n,0,1} - D\mathfrak{U}(a_n)}{2}\Big]\Gamma_n\mathfrak{U}(a_n)\\
&+ \int_0^1 \Gamma_n\Big[R_{\Upsilon_n,t,0}D\mathfrak{U}(\Upsilon_n(t))R_{\Upsilon_n,0,t} - D\mathfrak{U}(a_n)\Big]\Upsilon_n'(0)dt.
\end{aligned}$$

□

Now, we are going to prove our theorem.

THEOREM 3.5. *Let Z be a complete Riemannian manifold, $\Omega \subseteq Z$ be an open convex set, and $\mathfrak{U} \in \chi(Z)$ satisfies the conditions* (1) – (3) *with:*

$$0 < x_0 \leq \boldsymbol{b}_0, \ M\xi K\zeta < \frac{1}{2}, \ V[a_0, M\zeta] \subset \Omega,$$

where $\kappa = \frac{1}{\mathcal{A}(x_0)}$ and $M = \frac{2}{(2-x_0)(1-\varsigma\kappa)}$. Then, the method given by (5) *converges to a singular point a^* of the vector field $\mathfrak{U}$ and the singularity a^*, the iterations a_n, b_n belong to $V[a_0, M\zeta]$, and a^* is unique in $V[a_0, M\zeta]$.*

Proof. Firstly, we shall prove that the following statements are true for $n \geq 1$, by using the principle of mathematical induction:

(I) $\|\Gamma_n R_{\Upsilon_{n-1},0,1}\Gamma_{n-1}^{-1}\| \leq \mathcal{A}(x_{n-1})$,
(II) $d(b_n, a_n) \leq y_{n-1} d(b_{n-1}, a_{n-1})$,
(III) $K\|\Gamma_n\| d(b_n, a_n) \leq x_n$,
(IV) $d(a_{n+1}, a_n) \leq 2/(2-x_n) d(b_n, a_n)$,
(V) $d(a_{n+1}, b_n) \leq (4-x_n)/(2-x_n) d(b_n, a_n)$.

Before proving the inequalities $(I)-(V)$, we will prove some results and later will be used to prove $(I)-(V)$. Now, we get $K\|\Gamma_0\| d(b_0, a_0) \leq K\xi\zeta = x_0$ as Γ_0 exists and we have

$$\begin{aligned}\left\| I_{a_0} - \Gamma_0 \frac{R_{\Psi_0,1,0} D\mathfrak{U}(b_0) R_{\Psi_0,0,1} + D\mathfrak{U}(a_0)}{2} \right\| &= \left\| \Gamma_0 \frac{D\mathfrak{U}(a_0) - R_{\Psi_0,1,0} D\mathfrak{U}(b_0) R_{\Psi_0,0,1}}{2} \right\| \\ &\leq \frac{1}{2} K \|\Gamma_0\| d(b_0, a_0) \leq \frac{x_0}{2} < 1.\end{aligned}$$

By Banach's lemma, $\left(\frac{R_{\Psi_0,1,0} D\mathfrak{U}(b_0) R_{\Psi_0,0,1} + D\mathfrak{U}(a_0)}{2}\right)^{-1} D\mathfrak{U}(a_0)$ exists and

$$\left\| \left(\frac{R_{\Psi_0,1,0} D\mathfrak{U}(b_0) R_{\Psi_0,0,1} + D\mathfrak{U}(a_0)}{2} \right)^{-1} D\mathfrak{U}(a_0) \right\| \leq \frac{2}{2-x_0}.$$

Since $a_{n+1} = \exp_{a_n}(g_n)$, for $n = 0$, we have

$$\begin{aligned} d(a_1, a_0) &= \|g_0\| \\ &\leq \left\| \left(\frac{R_{\Psi_0,1,0} D\mathfrak{U}(b_0) R_{\Psi_0,0,1} + D\mathfrak{U}(a_0)}{2} \right)^{-1} D\mathfrak{U}(a_0) \right\| \left\| D\mathfrak{U}(a_0)^{-1} \mathfrak{U}(a_0) \right\| \\ &\leq \frac{2}{2-x_0} d(b_0, a_0) \end{aligned}$$

and

$$d(a_1, b_0) \leq d(a_1, a_0) + d(a_0, b_0) \leq \frac{4-x_0}{2-x_0} d(b_0, a_0).$$

Now, we will prove the conditions $(I)-(V)$ for $n \geq 1$. We have

$$\begin{aligned} &\|I_{a_0} - \Gamma_0 R_{\Upsilon_0,1,0} D\mathfrak{U}(a_1) R_{\Upsilon_0,0,1}\| \\ =&\|\Gamma_0\| \, \|D\mathfrak{U}(a_0) - R_{\Upsilon_0,1,0} D\mathfrak{U}(a_1) R_{\Upsilon_0,0,1}\| \leq K \|\Gamma_0\| d(a_1, a_0) \\ \leq&\frac{2}{2-x_0} K\xi d(b_0, a_0) \leq \frac{2x_0}{2-x_0} < 1. \end{aligned}$$

By Banach's lemma , Γ_1 exists and

$$\|\Gamma_1 R_{\Upsilon_0,0,1}\Gamma_0^{-1}\| \leq \frac{2-x_0}{2-3x_0} = \mathcal{A}(x_0).$$

Now, by Lemma 3.4,

$$\begin{aligned}
&\|\Gamma_0 R_{\Upsilon_0,1,0}\mathfrak{U}(a_1)\| \\
=&\Bigg\|\Big(\frac{R_{\Psi_0,1,0}D\mathfrak{U}(b_0)R_{\Psi_0,0,1}+D\mathfrak{U}(a_0)}{2}\Big)^{-1}\Big[\frac{R_{\Psi_0,1,0}D\mathfrak{U}(b_0)R_{\Psi_0,0,1}-D\mathfrak{U}(a_0)}{2}\Big] \\
&\times \Gamma_0\mathfrak{U}(a_0) + \int_0^1 \Gamma_0\big[R_{\Upsilon_0,t,0}D\mathfrak{U}(\Upsilon_0(t))R_{\Upsilon_0,0,t} - D\mathfrak{U}(a_0)\big]\Upsilon_0'(0)dt\Bigg\| \\
\leq&\frac{2}{2-x_0}\|\Gamma_0\|\frac{K}{2}d(b_0,a_0)^2 + \|\Gamma_0\|\frac{K}{2}d(a_1,a_0)^2 \\
\leq&\frac{K}{2-x_0}\|\Gamma_0\|d(b_0,a_0)^2 + \|\Gamma_0\|\frac{2K}{(2-x_0)^2}d(b_0,a_0)^2 \\
\leq&\frac{x_0(4-x_0)}{(2-x_0)^2}d(b_0,a_0) = \mathcal{B}(x_0)d(b_0,a_0).
\end{aligned}$$

So that

$$\begin{aligned}
d(b_1,a_1) =&\|f_1\| = \|\Gamma_1\mathfrak{U}(a_1)\| = \|\Gamma_1 R_{\Upsilon_0,0,1}\Gamma_0^{-1}\Gamma_0 R_{\Upsilon_0,1,0}\mathfrak{U}(a_1)\| \\
\leq&\|\Gamma_1 R_{\Upsilon_0,0,1}\Gamma_0^{-1}\|\|\Gamma_0 R_{\Upsilon_0,1,0}\mathfrak{U}(a_1)\| \leq \mathcal{A}(x_0)\mathcal{B}(x_0)d(b_0,a_0) = y_0 d(b_0,a_0).
\end{aligned}$$

Also

$$\|\Gamma_1\| \leq \frac{\|\Gamma_0\|}{1-2x_0/2-x_0} = \mathcal{A}(x_0)\|\Gamma_0\|,$$

$$K\|\Gamma_1\|d(b_1,a_1) \leq K\|\Gamma_0\|\mathcal{A}(x_0)^2\mathcal{B}(x_0)d(b_0,a_0) = x_0\mathcal{A}(x_0)y_0 = x_1.$$

Again as Γ_1 exists , therefore

$$\begin{aligned}
\Big\|I_{a_1} - \Gamma_1\frac{R_{\Psi_1,1,0}D\mathfrak{U}(b_1)R_{\Psi_1,0,1}+D\mathfrak{U}(a_1)}{2}\Big\| =& \Big\|\Gamma_1\frac{D\mathfrak{U}(a_1)-R_{\Psi_1,1,0}D\mathfrak{U}(b_1)R_{\Psi_1,0,1}}{2}\Big\| \\
\leq&\frac{1}{2}K\|\Gamma_1\|d(b_1,a_1) \leq \frac{x_1}{2} < 1.
\end{aligned}$$

By Banach's lemma, $\Big(\frac{R_{\Psi_1,1,0}D\mathfrak{U}(b_1)R_{\Psi_1,0,1}+D\mathfrak{U}(a_1)}{2}\Big)^{-1}D\mathfrak{U}(a_1)$ exists and

$$\Bigg\|\Big(\frac{R_{\Psi_1,1,0}D\mathfrak{U}(b_1)R_{\Psi_1,0,1}+D\mathfrak{U}(a_1)}{2}\Big)^{-1}D\mathfrak{U}(a_1)\Bigg\| \leq \frac{2}{2-x_1}.$$

We obtain that

$$\begin{aligned}
d(a_2,a_1) =&\|g_1\| \leq \Bigg\|\Big(\frac{R_{\Psi_1,1,0}D\mathfrak{U}(b_1)R_{\Psi_1,0,1}+D\mathfrak{U}(a_1)}{2}\Big)^{-1}D\mathfrak{U}(a_1)\Bigg\|\,\|D\mathfrak{U}(a_1)^{-1}\mathfrak{U}(a_1)\| \\
\leq&\frac{2}{2-x_1}d(b_1,a_1)
\end{aligned}$$

and

$$d(a_2,b_1) \leq d(a_2,a_1) + d(a_1,b_1) \leq \frac{4-x_1}{2-x_1}d(b_1,a_1).$$

Therefore all the conditions $(I)-(V)$ for $n=1$ hold. Suppose it holds for $n=2,3,4,\dots,k$. Then, we will prove it for $n=k+1$. As $K\|\Gamma_k\|d(b_k,a_k)\le x_k$. We have

$$\begin{aligned}&\|I_{a_k}-\Gamma_k R_{\Upsilon_k,1,0}D\mathfrak{U}(a_{k+1})R_{\Upsilon_k,0,1}\|\\ =&\|\Gamma_k\|\,\|D\mathfrak{U}(a_k)-R_{\Upsilon_k,1,0}D\mathfrak{U}(a_{k+1})R_{\Upsilon_k,0,1}\|\le K\|\Gamma_k\|d(a_{k+1},a_k)\\ \le&\frac{2}{2-x_k}K\|\Gamma_k\|d(b_k,a_k)\le\frac{2x_k}{2-x_k}<1.\end{aligned}$$

By Banach's lemma , Γ_{k+1} exists and

$$\|\Gamma_{k+1}R_{\Upsilon_k,0,1}\Gamma_k^{-1}\|\le\frac{2-x_k}{2-3x_k}=\mathcal{A}(x_k).$$

Now, by Lemma 3.4,

$$\begin{aligned}&\|\Gamma_k R_{\Upsilon_k,1,0}\mathfrak{U}(a_{k+1})\|\\ =&\left\|\Big(\frac{R_{\Psi_k,1,0}D\mathfrak{U}(b_k)R_{\Psi_k,0,1}+D\mathfrak{U}(a_k)}{2}\Big)^{-1}\Big[\frac{R_{\Psi_k,1,0}D\mathfrak{U}(b_k)R_{\Psi_k,0,1}-D\mathfrak{U}(a_k)}{2}\Big]\right.\\ &\left.\times\Gamma_k\mathfrak{U}(a_k)+\int_0^1\Gamma_k\big[R_{\Upsilon_k,t,0}D\mathfrak{U}(\Upsilon_k(t))R_{\Upsilon_k,0,t}-D\mathfrak{U}(a_k)\big]\Upsilon_k'(0)dt\right\|\\ \le&\frac{2}{2-x_k}\|\Gamma_k\|\frac{K}{2}d(b_k,a_k)^2+\|\Gamma_k\|\frac{K}{2}d(a_{k+1},a_k)^2\\ \le&\frac{K}{2-x_k}\|\Gamma_k\|d(b_k,a_k)^2+\|\Gamma_k\|\frac{2K}{(2-x_k)^2}d(b_k,a_k)^2\\ \le&\frac{x_k(4-x_k)}{(2-x_k)^2}d(b_k,a_k)=\mathcal{B}(x_k)d(b_k,a_k).\end{aligned}$$

So that

$$\begin{aligned}d(b_{k+1},a_{k+1})=&\|f_{k+1}\|\\ =&\|\Gamma_{k+1}\mathfrak{U}(a_{k+1})\|=\|\Gamma_{k+1}R_{\Upsilon_k,0,1}\Gamma_k^{-1}\Gamma_kR_{\Upsilon_k,1,0}\mathfrak{U}(a_{k+1})\|\\ \le&\|\Gamma_{k+1}R_{\Upsilon_k,0,1}\Gamma_k^{-1}\|\|\Gamma_kR_{\Upsilon_k,1,0}\mathfrak{U}(a_{k+1})\|\\ \le&\mathcal{A}(x_k)\mathcal{B}(x_k)d(b_k,a_k)=y_kd(b_k,a_k).\end{aligned}$$

Also

$$\|\Gamma_{k+1}\|\le\frac{\|\Gamma_k\|}{1-2x_k/2-x_k}=\mathcal{A}(x_k)\|\Gamma_k\|,$$

$$K\|\Gamma_{k+1}\|d(b_{k+1},a_{k+1})\le K\|\Gamma_k\|\mathcal{A}(x_k)^2\mathcal{B}(x_k)d(b_k,a_k)=x_k\mathcal{A}(x_k)y_k=x_{k+1}.$$

Now,

$$\begin{aligned}&\left\|I_{a_{k+1}}-\Gamma_{k+1}\frac{R_{\Psi_{k+1},1,0}D\mathfrak{U}(b_{k+1})R_{\Psi_{k+1},0,1}+D\mathfrak{U}(a_{k+1})}{2}\right\|\\ =&\left\|\Gamma_{k+1}\frac{D\mathfrak{U}(a_{k+1})-R_{\Psi_{k+1},1,0}D\mathfrak{U}(b_{k+1})R_{\Psi_{k+1},0,1}}{2}\right\|\\ \le&\frac{1}{2}K\|\Gamma_{k+1}\|d(b_{k+1},a_{k+1})\le\frac{x_{k+1}}{2}<1.\end{aligned}$$

By Banach's lemma, $\Big(\frac{R_{\Psi_{k+1},1,0}D\mathfrak{U}(b_{k+1})R_{\Psi_{k+1},0,1}+D\mathfrak{U}(a_{k+1})}{2}\Big)^{-1}D\mathfrak{U}(a_{k+1})$ exists and

$$\Big\|\Big(\frac{R_{\Psi_{k+1},1,0}D\mathfrak{U}(b_{k+1})R_{\Psi_{k+1},0,1}+D\mathfrak{U}(a_{k+1})}{2}\Big)^{-1}D\mathfrak{U}(a_{k+1})\Big\| \leq \frac{2}{2-x_{k+1}}.$$

Thus, we have

$$\begin{aligned}
&d(a_{k+2},a_{k+1}) = \|g_{k+1}\| \\
&\leq \Big\|\Big(\frac{R_{\Psi_{k+1},1,0}D\mathfrak{U}(b_{k+1})R_{\Psi_{k+1},0,1}+D\mathfrak{U}(a_{k+1})}{2}\Big)^{-1}D\mathfrak{U}(a_{k+1})\Big\| \, \big\|D\mathfrak{U}(a_{k+1})^{-1}\mathfrak{U}(a_{k+1})\big\| \\
&\leq \frac{2}{2-x_{k+1}}d(b_{k+1},a_{k+1})
\end{aligned}$$

and

$$d(a_{k+2},b_{k+1}) \leq d(a_{k+2},a_{k+1}) + d(a_{k+1},b_{k+1}) \leq \frac{4-x_{k+1}}{2-x_{k+1}}d(b_{k+1},a_{k+1}).$$

Hence by mathematical induction it holds for all $n \in \mathbb{N}$. Now, to prove the sequence $\{a_n\}$ is convergent, it will be sufficient to show that it is Cauchy sequence. By Lemma 3.2, for $x_0 = \mathbf{b}_0,$ we have $\mathcal{C}(x_0) = 0$ and $\mathcal{A}(x_0)y_0 = 1$. We have from (6), $x_n = x_{n-1} = \cdots = x_0$ and $y_n = y_{n-1} = \cdots = y_0$. From condition (II), we have

$$d(b_n,a_n) \leq y_{n-1}d(b_{n-1},a_{n-1}) = y_0 d(b_{n-1},a_{n-1}) \leq \cdots \leq y_0^n d(b_0,a_0) \leq \kappa^n\zeta$$

and

$$d(a_{n+1},a_n) \leq \frac{2}{2-x_n}d(b_n,a_n) \leq \frac{2}{2-x_0}\kappa^n\zeta.$$

Thus, we have

$$\begin{aligned}
d(a_{m+n},a_m) &\leq d(a_{m+n},a_{m+n-1}) + \cdots + d(a_{m+1},a_m) \\
&\leq \frac{2}{2-x_0}\big[\kappa^{m+n-1} + \cdots + \kappa^m\big]\zeta = \frac{2\kappa^m}{2-x_0}\Big(\frac{1-\kappa^n}{1-\kappa}\Big)\zeta. \qquad (9)
\end{aligned}$$

From (9), $\{a_n\}$ is a Cauchy sequence as $\kappa < 1$. Let $0 < x_0 < \mathbf{b}_0$ and $\mathcal{C}(x_0) > 0$. Now from conditions $(I)-(V)$ and Lemma 3.3(2), for $n \geq 1$, we have

$$d(b_n,a_n) \leq y_{n-1}d(b_{n-1},a_{n-1}) \leq \cdots \leq \prod_{i=0}^{n-1}(y_i)d(b_0,a_0) \leq \prod_{i=0}^{n-1}(\varsigma^{2^i}\kappa)\zeta = \varsigma^{2^n-1}\kappa^n\zeta,$$

where $\varsigma = x_1/x_0 < 1$ and $\kappa = 1/\mathcal{A}(x_0) < 1$. We obtain that

$$\begin{aligned}
d(a_{n+m},a_m) &\leq d(a_{n+m},a_{m+n-1}) + \cdots + d(a_{m+1},a_m) \\
&\leq \frac{2}{2-x_{m+n-1}}d(b_{n+m-1},a_{m+n-1}) + \cdots + \frac{2}{2-x_m}d(b_m,a_m) \\
&\leq \frac{2}{2-x_{m+n-1}}\varsigma^{2^{m+n-1}-1}\kappa^{m+n-1}\zeta + \cdots + \frac{2}{2-x_m}\varsigma^{2^m-1}\kappa^m\zeta \\
&\leq \frac{2\kappa^m}{2-x_m}\big[\varsigma^{2^{m+n-1}-1}\kappa^{n-1} + \cdots + \varsigma^{2^m-1}\big]\zeta \\
&\leq \frac{2\varsigma^{2^m-1}\kappa^m}{2-\varsigma^{2^m-1}x_0}\big[\varsigma^{2^m[2^{n-1}-1]}\kappa^{n-1} + \cdots + \varsigma^{2^m[2-1]}\kappa + 1\big]\zeta.
\end{aligned}$$

We know by Bernoulli's inequality that for every $z \in \mathbb{R} > -1$ and $k \in \mathbb{Z} \geq 0$, we have $(1+z)^k - 1 \geq kz$. Therefore if we take $z = 1$ we have the following

$$\begin{aligned}&\left[\varsigma^{2^m[2^{n-1}-1]}\kappa^{n-1} + \cdots + \varsigma^{2^m[2-1]}\kappa + 1\right]\\ \leq&\left[\left(\varsigma^{2^m}\kappa\right)^{n-1} + \left(\varsigma^{2^m}\kappa\right)^{n-2} + \cdots + 1\right] = \frac{1-\varsigma^{2^m n}\kappa^n}{1-\varsigma^{2^m}\kappa}.\end{aligned}$$

Hence, we get

$$d(a_{n+m}, a_m) \leq \frac{2\varsigma^{2^m-1}\kappa^m}{2-\varsigma^{2^m-1}x_0} \times \frac{1-\varsigma^{2^m n}\kappa^n}{1-\varsigma^{2^m}\kappa}\zeta. \tag{10}$$

As $\varsigma < 1$ and $M = 2/((2-x_0)(1-\varsigma\kappa))$, then from (10), we have for $m = 0$,

$$d(a_n, a_0) \leq \frac{2}{2-x_0} \times \frac{1-\varsigma^n\kappa^n}{1-\varsigma\kappa}\zeta \leq M\zeta. \tag{11}$$

Therefore $a_n \in V[a_0, M\zeta]$. Also $b_n \in V[a_0, M\zeta]$, as

$$\begin{aligned}d(b_{n+1}, a_0) \leq& d(b_{n+1}, a_{n+1}) + d(a_{n+1}, a_n) + \cdots + d(a_1, a_0)\\ \leq& d(b_{n+1}, a_{n+1}) + \frac{2}{2-x_n}d(b_n, a_n) + \cdots + \frac{2}{2-x_0}d(b_0, a_0)\\ \leq& \frac{2}{2-x_{n+1}}d(b_{n+1}, a_{n+1}) + \cdots + \frac{2}{2-x_0}d(b_0, a_0)\\ \leq& \cdots \leq \frac{2}{2-x_0}\frac{1-\varsigma^{n+1}\kappa^{n+1}}{1-\varsigma\kappa}\zeta \leq M\zeta.\end{aligned}$$

Taking $n \to \infty$ in (9) or (11), we get $a^* \in V[a_0, M\zeta]$. Now, to prove a^* is a singular point of $\mathfrak{U}$, we find the bounds of $\|D\mathfrak{U}(a_n)\|$. For this let δ be a geodesically convex neighborhoods with minimizing geodesic such that $\delta(0) = a_0$ and $\delta(1) = a_n$. By using Theorem 2.13, we have

$$\begin{aligned}\|D\mathfrak{U}(a_n)\| =& \|R_{\delta,1,0}D\mathfrak{U}(a_n)R_{\delta,0,1} + D\mathfrak{U}(a_0) - D\mathfrak{U}(a_0)\|\\ \leq& \|R_{\delta,1,0}D\mathfrak{U}(a_n)R_{\delta,0,1} - D\mathfrak{U}(a_0)\| + \|D\mathfrak{U}(a_0)\|\\ \leq& \|D\mathfrak{U}(a_0)\| + KM\zeta.\end{aligned}$$

Then, from (5), we have

$$\|\mathfrak{U}(a_n)\| = \|D\mathfrak{U}(a_n)f_n\| \leq \|D\mathfrak{U}(a_n)\|\|f_n\| = \|D\mathfrak{U}(a_n)\|d(b_n, a_n).$$

Since $\|D\mathfrak{U}(a_n)\|$ is bounded and $d(b_n, a_n) \to 0$, when $n \to \infty$, we obtain that $\|\mathfrak{U}(a^*)\| \leq 0$, thus $\mathfrak{U}(a^*) = 0$. Now, we will prove the singularity is unique, before that we will find the bounds of $\|\Gamma_n\|$. We have

$$\|R_{\delta,1,0}D\mathfrak{U}(a_n)R_{\delta,0,1} - D\mathfrak{U}(a_0)\| \leq K\int_0^1 \|\delta'(0))\|ds = Kd(a_n, a_0) \leq KM\zeta$$

and

$$\|\Gamma_0\|\|R_{\delta,1,0}D\mathfrak{U}(a_n)R_{\delta,0,1} - D\mathfrak{U}(a_0)\| \leq \xi KM\zeta < 1.$$

So that $R_{\delta,1,0}D\mathfrak{U}(a_n)R_{\delta,0,1}$ is invertible by Banach's lemma and

$$\begin{aligned}\|\Gamma_n\| =& \|R_{\delta,1,0}\Gamma_n R_{\delta,0,1}\|\\ \leq& \frac{\|\Gamma_0\|}{1-\|\Gamma_0\|\|R_{\delta,1,0}D\mathfrak{U}(a_n)R_{\delta,0,1}-D\mathfrak{U}(a_0)\|}\\ \leq& \frac{\xi}{1-K\xi M\zeta}.\end{aligned}$$

Now, let a^{**} be an another singularity of $\mathfrak{U}$ in $V[a_0, M\zeta]$ and ϑ be a geodesically convex neighborhoods with minimizing geodesic such that $\vartheta(0) = a^*$ and $\vartheta(1) = a^{**}$. Then we have

$$\begin{aligned}\|R_{\vartheta,t,0}D\mathfrak{U}(\vartheta(t))R_{\vartheta,0,t} - D\mathfrak{U}(a^*)\| \leq K\int_0^t \|\vartheta'(0)\|ds\\ =Ktd(a^*, a^{**}) \leq Kt\Big(d(a_0, a^*) + d(a_0, a^{**})\Big).\end{aligned}$$

Hence

$$\begin{aligned}&\|D\mathfrak{U}(a^*)^{-1}\| \int_0^1 \|R_{\vartheta,t,0}D\mathfrak{U}(\vartheta(t))R_{\vartheta,0,t}dt - D\mathfrak{U}(a^*)\|dt\\ \leq& \frac{\xi}{(1-K\xi M\zeta)}\int_0^1 Kt\Big(d(a_0, a^*) + d(a_0, a^{**})\Big)dt\\ \leq& \frac{\xi}{(1-K\xi M\zeta)}\frac{K}{2}(M\zeta + M\zeta) < 1.\end{aligned}$$

By Banach's lemma, the operator $\int_0^1 R_{\vartheta,t,0}D\mathfrak{U}(\vartheta(t))R_{\vartheta,0,t}dt$ is invertible and we have

$$0 = R_{\vartheta,1,0}\mathfrak{U}(a^{**}) - \mathfrak{U}(a^*) = \int_0^1 R_{\vartheta,t,0}D\mathfrak{U}(\vartheta(t))R_{\vartheta,0,t}(\vartheta'(0))dt.$$

Therefore $\vartheta'(0) = 0$. As $0 = \|\vartheta'(0)\| = d(a^*, a^{**})$, we get $a^* = a^{**}$. Hence the proof is complete. □

4. Order of convergence of the Newton-like method in Riemannian manifolds

In this section, we will study the order of convergence of the Newton-like method in Riemannian manifolds.

Definition 4.1. ([10]) Let Z be a complete Riemannian manifold and $\{a_n\}$ be a sequence in Z converges to a^*. If there is a chart (U, h) of a^* and constants $l > 0$ and $\mathfrak{T} \geq 0$ such that

$$\|h^{-1}(a_{n+1}) - h^{-1}(a^*)\| \leq \mathfrak{T}\|h^{-1}(a_n) - h^{-1}(a^*)\|^l \tag{12}$$

holds for all sufficiently large n, we say that $\{a_n\}$ converges to a^* with order at least l.

Remark 4.2. The above definition do not depend on the choice of the chart i.e. if (Y, z) is another chart of a^*, then (12) holds changing h by z and constant $\mathfrak{T}$ by

$\mathfrak{K}$ [16]. So that, we can assume U is a normal neighborhood of each of its points, see Theorem 3.7 in [17]. Since in a totally normal neighborhood U of a^*,

$$d(a,b) = \| \exp_{a_n}^{-1}(a) - \exp_{a_n}^{-1}(b)\|, \tag{13}$$

for all $a, b \in U$ and for all sufficiently large n, we can rewrite (12) as

$$d(a_{n+1}, a^*) \leq \mathfrak{T} d(a_n, a^*)^l.$$

Now, we will prove the order of convergence of the Newton-like method in Riemannian manifolds. Before that, we will prove the order of convergence of the iterative method defined by

$$\left.\begin{aligned} g_n &= -\Big(\frac{D\mathfrak{U}(a_n) + R_{\Psi_n,1,0} D\mathfrak{U}(b_n) R_{\Psi_n,0,1}}{2}\Big)^{-1} \mathfrak{U}(a_n), \\ a_{n+1} &= \exp_{a_n}(g_n), \text{ for each } n = 0,1,2,\ldots \end{aligned}\right\} \tag{14}$$

.For this, we will prove a lemma.

Lemma 4.3. *Let Z be a complete Riemannian manifold, $\Omega \subseteq Z$ be an open convex set, $\mathfrak{U} \in \chi(Z)$, and $\alpha(t) = \exp_{a_n}(tv)$.*
Then,

$$R_{\alpha,t,0}\mathfrak{U}(\alpha(t)) = \mathfrak{U}(a_n) + t\Big(\frac{R_{\Psi_n,1,0} D\mathfrak{U}(b_n) R_{\Psi_n,0,1} + D\mathfrak{U}(a_n)}{2}\Big) v + R(t)$$

with

$$\|R(t)\| \leq \frac{K}{2}\big(t\|v\| + d(a_n, b_n)\big) t\|v\|.$$

Proof. From (4), we have

$$R_{\alpha,t,0}\mathfrak{U}(\alpha(t)) = \mathfrak{U}(a_n) + \int_0^t R_{\alpha,s,0} D\mathfrak{U}(\alpha(s)) R_{\alpha,0,s}(v) ds.$$

Thus

$$\begin{aligned} &R_{\alpha,t,0}\mathfrak{U}(\alpha(t)) - \mathfrak{U}(a_n) - t\Big(\frac{R_{\Psi_n,1,0} D\mathfrak{U}(b_n) R_{\Psi_n,0,1} + D\mathfrak{U}(a_n)}{2}\Big) v \\ &= \int_0^t \Big(R_{\alpha,s,0} D\mathfrak{U}(\alpha(s)) R_{\alpha,0,s}(v) - \Big(\frac{R_{\Psi_n,1,0} D\mathfrak{U}(b_n) R_{\Psi_n,0,1} + D\mathfrak{U}(a_n)}{2}\Big) v\Big) ds, \end{aligned}$$

letting

$$R(t) = \int_0^t \Big(R_{\alpha,s,0} D\mathfrak{U}(\alpha(s)) R_{\alpha,0,s}(v) - \Big(\frac{R_{\Psi_n,1,0} D\mathfrak{U}(b_n) R_{\Psi_n,0,1} + D\mathfrak{U}(a_n)}{2}\Big) v\Big) ds,$$

we obtain

$$\begin{aligned} &\|R(t)\| \leq \frac{1}{2}\int_0^t \|2R_{\alpha,s,0} D\mathfrak{U}(\alpha(s)) R_{\alpha,0,s} - D\mathfrak{U}(a_n) - R_{\Psi_n,1,0} D\mathfrak{U}(b_n) R_{\Psi_n,0,1}\| \|v\| ds \\ &= \frac{1}{2}\int_0^t \|2R_{\alpha,s,0} D\mathfrak{U}(\alpha(s)) R_{\alpha,0,s} - 2D\mathfrak{U}(a_n) + D\mathfrak{U}(a_n) - R_{\Psi_n,1,0} D\mathfrak{U}(b_n) R_{\Psi_n,0,1}\| \|v\| ds \\ &\leq \frac{K}{2}\big(t^2\|v\| + t d(a_n, b_n)\big)\|v\|. \end{aligned}$$

Therefore

$$\|R(t)\| \leq \frac{K}{2}\big(t\|v\| + d(a_n, b_n)\big) t\|v\|.$$

□

Theorem 4.4. *(Order of convergence) The iterative method given in* (14) *is of order one.*

Proof. By Lemma 4.3, if α is a geodesically convex neighborhoods with minimizing geodesic joining a_n to a^* defined by

$$\alpha(t) = \exp_{a_n}(tv_n),$$

where $v_n \in T_{a_n}Z$ and $d(a_n, a^*) = \|v_n\|$. Then,

$$R_{\alpha,t,0}\mathfrak{U}(a^*) = \mathfrak{U}(a_n) + \Big(\frac{R_{\Psi_n,1,0}D\mathfrak{U}(b_n)R_{\Psi_n,0,1} + D\mathfrak{U}(a_n)}{2}\Big)v_n + R(1) \tag{15}$$

with

$$\|R(1)\| \leq \frac{K}{2}\big(\|v_n\| + d(a_n, b_n)\big)\|v_n\|.$$

Therefore, from (15), we have

$$0 = \Big(\frac{R_{\Psi_n,1,0}D\mathfrak{U}(b_n)R_{\Psi_n,0,1} + D\mathfrak{U}(a_n)}{2}\Big)^{-1}\mathfrak{U}(a_n) + v_n + \Big(\frac{R_{\Psi_n,1,0}D\mathfrak{U}(b_n)R_{\Psi_n,0,1} + D\mathfrak{U}(a_n)}{2}\Big)^{-1}R(1).$$

Since

$$-\Big(\frac{R_{\Psi_n,1,0}D\mathfrak{U}(b_n)R_{\Psi_n,0,1} + D\mathfrak{U}(a_n)}{2}\Big)^{-1}\mathfrak{U}(a_n) = \exp^{-1}_{a_n}(a_{n+1}) \text{ and } v_n = \exp^{-1}_{a_n}(a^*),$$

we have

$$\exp^{-1}_{a_n}(a_{n+1}) - \exp^{-1}_{a_n}(a^*) = \Big(\frac{R_{\Psi_n,1,0}D\mathfrak{U}(b_n)R_{\Psi_n,0,1} + D\mathfrak{U}(a_n)}{2}\Big)^{-1}R(1),$$

we conclude that

$$\begin{aligned}
d(a_{n+1}, a^*) &= \|\exp^{-1}_{a_n}(a_{n+1}) - \exp^{-1}_{a_n}(a^*)\| \\
&= \Big\|\Big(\frac{R_{\Psi_n,1,0}D\mathfrak{U}(b_n)R_{\Psi_n,0,1} + D\mathfrak{U}(a_n)}{2}\Big)^{-1}R(1)\Big\| \\
&\leq \Big\|\Big(\frac{R_{\Psi_n,1,0}D\mathfrak{U}(b_n)R_{\Psi_n,0,1} + D\mathfrak{U}(a_n)}{2}\Big)^{-1}\Big\|\,\|R(1)\| \\
&\leq \frac{2K}{4-2a_n}\big(\|v_n\| + d(a_n, b_n)\big)\|v_n\| \\
&\leq \frac{2K}{4-2a_n}\|\Gamma_n\|\big(d(a_n, a^*) + d(a_n, b_n)\big)d(a_n, a^*) \\
&= \frac{2K}{4-2a_n}\|\Gamma_n\|\Big(\frac{d(a_n, a^*)}{d(a_n, b_n)} + 1\Big)d(a_n, b_n)d(a_n, a^*).
\end{aligned}$$

If n is sufficiently large, then $d(a_n, a^*) \leq d(a_n, b_n)$, and therefore

$$\Big(\frac{d(a_n, a^*)}{d(a_n, b_n)} + 1\Big) \leq 2$$

and then, for b_n sufficiently close to a^*,

$$d(a_{n+1}, a^*) \leq N_0 d(a_n, b_n)d(a_n, a^*),$$

with $N_0 \leq \frac{4K}{4-2a_n}\|\Gamma_n\|$. □

REMARK 4.5. If we fix a_j sufficiently close to a^*, then, the calculations made in the Theorem 4.4 become in

$$d(a_{n+1}, a^*) \leq N_j d(a_j, a^*) d(a_n, a^*),$$

with $N_j \leq \frac{4K}{4-2a_n}\|\Gamma_n\|$. Thus,

$$d(a_{n+1}, a^*) \leq N d(a_j, a^*) d(a_n, a^*), \tag{16}$$

with $N \leq \frac{4K}{4-2a_n}\|\Gamma_n\|$.

Now, we can prove the order of convergence of the Newton-like method in Riemannian manifolds.

THEOREM 4.6. *Under the hypotheses of Theorem 3.5, the method described in* (5) *converges with order of convergence* 3.

Proof. Since

$$d(a_{n+1}, a_n) \leq d(a_{n+1}, b_n) + d(b_n, a_n).$$

Now, from Theorem 3.5 the sequence $\{a_n\}$ converges to a^*. Also, the convergence order of the Newton method in Riemannian manifold is two (for proof see Lemma 28 (i) [10]) and by using (16), we have

$$d(a_{n+1}, a^*) \leq N d(a_n, a^*) d(b_n, a^*) \leq N d(a_n, a^*) C d(a_n, a^*)^2 = NC d(a_n, a^*)^3.$$

□

5. Numerical examples

In this section, a numerical example is given to show the effectiveness of our results.

EXAMPLE 5.1. Let us consider the vector field $W : \mathbb{R}^3 \to \mathbb{R}^3$ given by

$$W(a) = W\left(a_1, a_2, a_3\right)^T = \left(a_1 a_3, a_2 a_3, a_3^2 - 1.001\right)^T$$

with Frobenius norm and let $\mathfrak{U} = W|_{\boldsymbol{S}^2}$ be a vector field on $\boldsymbol{S}^2 = \{a_1, a_2, a_3 \in \mathbb{R} : a_1^2 + a_2^2 + a_3^2 = 1.001\}$. Then it can be easily verified that

$$W|_{\boldsymbol{S}^2}(a) \in T_a \boldsymbol{S}^2 \ \forall\ a \in \boldsymbol{S}^2.$$

Now, the derivative $D\mathfrak{U}$ using the algorithm given in [4] in the tangent plane of $\boldsymbol{S}^2$ is given by

$$\begin{pmatrix} a_1^4 - a_3^2 - a_1^2(-1 + a_2^2 + a_3^2) & -a_1 a_2(-1 + a_1^2 + a_2^2 + a_3^2) \\ -a_1 a_2(-1 + a_1^2 + a_2^2 + a_3^2) & a_2^4 - a_3^2 - a_2^2(-1 + a_1^2 + a_3^2) \end{pmatrix}.$$

Next, by using the method of Lagrange's multipliers, we find that

$$K = \sup\{D\mathfrak{U}(a_1, a_2, a_3) : a_1^2 + a_2^2 + a_3^2 = 1.001\} = 0.0005$$

is Lipschitz constant of $D\mathfrak{U}$. Initially for $\mathbf{a}_0 = (1/\sqrt{2}, 0.0316, 1/\sqrt{2})^T$, we get

$$0 < x_0 = 0.0014 \leq \mathbf{b}_0 = 0.310102\ldots, \ \ M\xi K\zeta = 0.00142 < \frac{1}{2}.$$

Hence all the assumptions for convergence are satisfied. By the Newton-like method on $\mathbf{S}^2$, we get the solution. The results are in the Table 1, which shows that (a_i) converges to the singularity $(0, 0, 1.0003)^T$. The numerical calculations and the error of our proposed iterative method has done by the help of MATLAB R2021a software and we have calculated the iterations with tolerance 10^{-10}.

Table 1. Results of the Newton-like method on $\mathbf{S}^2$:

Iterations	a_i	$\|\|\mathfrak{U}(a_i)\|\|$	$d(a_{i+1}, a_i)$
0	$\begin{pmatrix} 7.071068e-01 \\ 3.160000e-02 \\ 7.071068e-01 \end{pmatrix}$	7.074597e-01	0
1	$\begin{pmatrix} 9.785461e-02 \\ 4.373039e-03 \\ 9.954904e-01 \end{pmatrix}$	7.074597e-01	6.746070e-01
2	$\begin{pmatrix} 1.580080e-04 \\ 7.061243e-06 \\ 1.000295e+00 \end{pmatrix}$	9.792490e-02	9.791206e-02
3	$\begin{pmatrix} 6.576326e-13 \\ 2.938904e-14 \\ 1.000295e+00 \end{pmatrix}$	6.109799e-04	1.581657e-04

Table 2. Results of the Newton's method on $\mathbf{S}^2$:

Iterations	a_i	$\|\|\mathfrak{U}(a_i)\|\|$	$d(a_{i+1}, a_i)$
0	$\begin{pmatrix} 7.071068e-01 \\ 3.160000e-02 \\ 7.071068e-01 \end{pmatrix}$	7.074597e-01	0
1	$\begin{pmatrix} -2.125536e-01 \\ -9.498837e-03 \\ 9.772522e-01 \end{pmatrix}$	7.074597e-01	9.593971e-01
2	$\begin{pmatrix} 3.340510e-03 \\ 1.492846e-04 \\ 1.000133e+00 \end{pmatrix}$	2.127349e-01	2.173174e-01
3	$\begin{pmatrix} -1.244017e-08 \\ -5.559408e-10 \\ 1.000139e+00 \end{pmatrix}$	3.354875e-03	3.343862e-03
4	$\begin{pmatrix} 1.654361e-24 \\ 1.033976e-25 \\ 1.000139e+00 \end{pmatrix}$	2.774689e-04	1.245259e-08

Algorithm (Newton-like method Algorithm on the sphere $\mathbf{S}^2$)

1. To define $\mathfrak{U} = (h_1, h_2, h_3)$ and $a_k = (a_{1k}, a_{2k}, a_{3k})$.
2. To calculate $\mathfrak{U}(a_k) = (h_1(a_k), h_2(a_k), h_3(a_k))$.
3. To calculate

$$
\begin{aligned}
c_{1,1}(a_k) =& a_{1k}\Big(h_{1,x_3} - \sum_{m=1}^{3} a_{mk}h_{m,x_3}a_{1k}\Big) - a_{3k}\Big(h_{1,x_1} - \sum_{m=1}^{3} a_{mk}h_{m,x_1}a_{1k}\Big)\\
c_{1,2}(a_k) =& a_{2k}\Big(h_{1,x_3} - \sum_{m=1}^{3} a_{mk}h_{m,x_3}a_{1k}\Big) - a_{3k}\Big(h_{1,x_2} - \sum_{m=1}^{3} a_{mk}h_{m,x_2}a_{1k}\Big)\\
c_{2,1}(a_k) =& a_{1k}\Big(h_{2,x_3} - \sum_{m=1}^{3} a_{mk}h_{m,x_3}a_{2k}\Big) - a_{3k}\Big(h_{2,x_1} - \sum_{m=1}^{3} a_{mk}h_{m,x_1}a_{2k}\Big)\\
c_{2,2}(a_k) =& a_{2k}\Big(h_{2,x_3} - \sum_{m=1}^{3} a_{mk}h_{m,x_3}a_{2k}\Big) - a_{3k}\Big(h_{2,x_2} - \sum_{m=1}^{3} a_{mk}h_{m,x_2}a_{2k}\Big),
\end{aligned}
$$

where h_{i,x_j} is the partial derivative of h_i with respect to x_j at the point a_k.

4. To calculate
$$\begin{pmatrix} f_{1k} \\ f_{2k} \end{pmatrix} = -\begin{pmatrix} c_{1,1}(a_k) & c_{1,2}(a_k) \\ c_{2,1}(a_k) & c_{2,2}(a_k) \end{pmatrix}^{-1} \begin{pmatrix} h_1(a_k) \\ h_2(a_k) \end{pmatrix}.$$
5. To calculate
$$f_k = -\begin{pmatrix} -a_{3k} f_{1k} \\ -a_{3k} f_{2k} \\ a_{2k} f_{2k} + a_{1k} f_{1k} \end{pmatrix} \text{ and } \|f_k\|.$$
6. To calculate
$$b_k = \cos(\|f_k\|) a_k + \frac{1}{\|f_k\|} \sin(\|f_k\|) f_k.$$
7. To calculate
$$\begin{pmatrix} g_{1k} \\ g_{2k} \end{pmatrix} = -\begin{pmatrix} c_{1,1}(a_k) + c_{1,1}(b_k) & c_{1,2}(a_k) + c_{1,2}(b_k) \\ c_{2,1}(a_k) + c_{2,1}(b_k) & c_{2,2}(a_k) + c_{2,2}(b_k) \end{pmatrix}^{-1} \begin{pmatrix} h_1(a_k) \\ h_2(a_k) \end{pmatrix}.$$
8. To calculate
$$g_k = -\begin{pmatrix} -a_{3k} g_{1k} \\ -a_{3k} g_{2k} \\ a_{2k} g_{2k} + a_{1k} g_{1k} \end{pmatrix} \text{ and } \|g_k\|.$$
9. To calculate
$$a_{k+1} = \cos(\|g_k\|) a_k + \frac{1}{\|g_k\|} \sin(\|g_k\|) g_k.$$

6. Conclusion

In this paper, we have studied the third order Newton-like method in Riemannian manifolds to find singular point of a vector field. Recurrence relations are developed to establish the convergence of the Newton-like method in Riemannian manifolds. Using recurrence relations existence and uniqueness theorem is derived. Also we have derived the order of convergence of our proposed method. The scope of studying modified Newton methods on manifolds is a vibrant area of research with significant theoretical depth and diverse applications, particularly in optimization and machine learning. The concrete directions for future work is developing new algorithms with improved computational efficiency and faster convergence rates. Finally, a numerical example is given to show the effectiveness of our results.

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Chandresh Prasad
Department of Mathematics, Central University of Jharkhand,
Ranchi-835222, India
E-mail: prasadchandresh20592@gmail.com

Pradip Kumar Parida
Department of Mathematics, Central University of Jharkhand,
Ranchi-835222, India
E-mail: pkparida@cuj.ac.in