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A (k,μ)-CONTACT METRIC MANIFOLD AS AN η−EINSTEIN SOLITON

Arup Kumar Mallick* and Arindam Bhattacharyya

Abstract. The aim of the paper is to study an η-Einstein soliton on $(2n+1)$-dimensional (k,μ)-contact metric manifold. At first, we establish various results related to $(2n+1)$-dimensional (k,μ)-contact metric manifold that exhibit an η-Einstein soliton. Next we study some curvature conditions admitting an η-Einstein soliton on $(2n+1)$-dimensional (k,μ)-contact metric manifold. Furthermore, we consider specific conditions associated with an η-Einstein soliton on $(2n+1)$-dimensional (k,μ)-contact metric manifold. Finally, we show the existance of an η-Einstein soliton on (k,μ)-contact metric manifold.

1. Introduction

In 1995, Blair et al. [4] introduced the notion of contact metric manifold with characteristic vector field ξ belonging to the (k,μ) distribution and such type of manifold is called (k,μ)- contact metric manifold. They obtained several results and a full classification of this manifold has been given by Boeckx [8].
A contact metric manifold is known [13] to exist where the curvature tensor R, in the direction of the characteristic vector field ξ, satisfies the equation $R(X,Y)\xi=0$ for any tangent vector field X,Y. For instance, the tangent sphere bundle of a flat Riemannian manifold possesses such a structure [5]. By applying a D-homothetic deformation [21] on M^{2n+1} with the equation $R(X,Y)\xi=0$, A novel class of contact metric manifolds that fulfills the condition

$$(1)\qquad R(X,Y)\xi = k\left\{\eta(Y)X-\eta(X)Y\right\}+\mu\left\{\eta(Y)hX-\eta(X)hY\right\}, k,\mu\in R$$

where h represents the Lie differentiation of ϕ in the direction of ξ and R is the curvature tensor. A notable characteristic of this class is that the equation's type remains unchanged under a D-homothetic deformation.
A contact metric manifold that satisfies the aforementioned relation (1) is known as a (k,μ)- contact metric manifold. This class of manifolds encompasses both Sasakian and non-Sasakian manifolds. In the case of Sasakian manifolds, $k=1$, resulting in $h=0$. However, for non-Sasakian manifolds, $k<1$. Examples of such manifolds can be found in all dimensions. Notably, the tangent sphere bundles of Riemannian

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∗ Corresponding author.

manifolds with constant sectional curvature c, excluding $c = 1$, serve as characteristic examples of non-Sasakian (k, μ)- contact metric manifolds. Particularly in the 3-dimensional case, this class includes the Lie group $SO(3)$, $SL(2, R)$, $SU(2)$, $O(1, 2)$, $E(2)$, $E(1, 1)$ with a left invariant metric [4]. For additional examples and a comprehensive classification of such manifolds, we refer to the mentioned paper [4]. It is worth noting that the papers also discuss contact metric manifolds with ξ belonging to the (k, μ)- nullity distribution [7, 18, 19, 22, 23], along with numerous other studies on this topic.

For the real constants k, μ, the (k, μ)- nullity distribution of a contact metric manifold forms a distribution [7]

$$\begin{aligned} N(k,\mu) : p \to N_p(k,\mu) = [Z \in TpM : R(X,Y)Z \\ = k\{g(Y,Z)X - g(X,Z)Y\} \\ + \mu\{g(Y,Z)hX - g(X,Z)hY\}], \end{aligned} \tag{2}$$

for each $X, Y \in T_pM$.
Consequently, if the characteristic vector field ξ belongs to the (k, μ)- nullity distribution, the above relation holds true. If $\xi \in N(k)$, we classify the manifold as an $N(k)$ contact metric manifold [3]. For $k = 1$, then the manifold is Sasakian, and if $k = 0$, the manifold is locally isometric to the product $E^{n+1}(0) \times S^n(4)$ for $n > 1$ and flat for $n = 1$ [6], where n is the dimension of the manifolds. In a (k, μ)- contact metric manifold, the manifold becomes an $N(k)$- contact manifold for $\mu = 0$.

In 1982, R.S. Hamilton [15, 16] introduced the concept of the Ricci flow as means to determine a canonical metric on a smooth manifold. The Ricci flow is an evolution equation that applies to a Riemannian metric $g(t)$ on a smooth manifold M. It is defined by the following equation:

$$\frac{\partial g}{\partial t} = -2S, \tag{3}$$

where S is the Ricci tensor of the metric $g\,(t)$.
A smooth manifold M, equipped with a Riemannian metric g, is known as a Ricci soliton if it moves only by a one parameter family of diffeomorphism and scaling. The Ricci soliton there exists a constant λ and a smooth vector field V on M that satisfies the following equation:

$$£_V g + 2S = 2\lambda g, \tag{4}$$

where $£_V$ denotes the Lie derivative along the direction of the vector field V and λ is a constant. The Ricci soliton exhibits shrinking, steady and expanding behaviour depending on $\lambda > 0, \lambda = 0, \lambda < 0$ respectively.
A Ricci soliton is a generalization of an Einstein metric which moves only by an one-parameter group diffeomorphisms and scaling [15].
A.E. Fischer [14] in 2005, developed the concept of conformal Ricci flow equation which is a variation of the classical Ricci flow equation that modifies the unit volume constraint of that equation to a scalar curvature constraint. The conformal Ricci flow on M is defined by the equation [14]

$$\frac{\partial g}{\partial t} + 2\left(S + \frac{g}{n}\right) = -pg, \quad r(g) = -1, \tag{5}$$

where M is considered as a smooth closed connected oriented manifold, p is a non-dynamical(time dependent) scalar field, $r(g)$ is the scalar curvature of the manifold and n is the dimension of manifold.
In 2015, N. Basu and A. Bhattacharyya [1] introduced the notion of conformal Ricci soliton as a generalization of the Ricci soliton and the equation is given by

$$(\pounds_V g) + 2S = \left[2\lambda - \left(p + \frac{2}{n}\right)\right] g, \tag{6}$$

where p is the conformal pressure.
The concept of $\eta-$Ricci soliton introduced by J.T. Cho and M. Kimura [11], and later C. Calin and M. Crasmareanu [9] studied it on Hopf hyper- surfaces in complex space forms. A Riemannian manifold is said to admit an η-Ricci soliton if for a smooth vector field V, the metric g satisfies the following equation

$$(\pounds_V g) + 2S + 2\lambda g + 2\mu\eta \otimes \eta = 0, \tag{7}$$

where $\pounds_V$ is the Lie derivative along the direction of V.

In 2018, M.D. Siddiqui [12] introduced the concept of a conformal $\eta-$Ricci soliton and the equation is given by

$$(\pounds_V g) + 2S + \left[2\lambda - \left(p + \frac{2}{n}\right)\right] g + 2\mu\eta \otimes \eta = 0. \tag{8}$$

In 2018, A.M. Blaga [2] proposed that a Riemannian manifold admits an $\eta-$Einstein soliton if the equation satisfies

$$\pounds_V g + 2S + (2\lambda - r)\, g + 2\mu\eta \otimes \eta = 0, \tag{9}$$

For $\mu = 0$, the data (g, ξ, λ) is called Einstein soliton [10].

The outline of the paper is organized as follows:
The introduction provides an overview and motivation for the study of an $\eta-$Einstein solitons on (k,μ)-contact metric manifolds. Section 2 presents fundamental tools and concepts related to $(2n+1)$-dimensional (k,μ)-contact metric manifolds. Section 3 focuses on $(2n+1)$-dimensional (k,μ)-contact metric manifold that admit an $\eta-$Einstein soliton. Section 4 investigates an $\eta-$Einstein soliton on $(2n+1)$-dimensional (k,μ)-contact metric manifolds satisfying $R(X,Y).S = 0$. Section 5 is devoted to the study of an $\eta-$Einstein soliton on (k,μ)-contact metric manifolds satisfying curvature condition $C(\xi, X).S = 0$. The investigation continues in Section 6, which delves into torse-forming vector field on (k,μ)-contact metric manifolds admitting an $\eta-$Einstein solitons. In section 7, a specific example of $(2n+1)$-dimensional (k,μ)-contact metric manifold possesses an $\eta-$Einstein soliton is presented.

2. Preliminaries

A $(2n+1)$-dimensional smooth manifold (M^{2n+1}, g) is called an almost contact manifold with structure (ϕ, ξ, η), where ϕ is a tensor field of type (1,1), ξ is a vector field, η is a 1-form and a Riemannian metric g if

$$\phi^2(X) = -X + \eta(X)\xi, \quad \eta(\xi) = 1, \quad \eta(\phi X) = 0, \quad \phi\xi = 0, \tag{10}$$

for any $X, Y \in \chi(M)$.
Let g be a conformable Riemannian metric with structure (ϕ, ξ, η, g), i.e.,

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \quad g(X, \xi) = \eta(X), \quad g(\xi, \xi) = 1. \tag{11}$$

Then M^{2n+1} becomes an almost contact metric manifold furnish with an almost contact metric structure (ϕ, ξ, η, g), i.e.,

$$g(X, \phi Y) = -g(\phi X, Y), \tag{12}$$

for every $X, Y \in \chi(M)$.
An almost contact metric structure enhance a contact metric structure if

$$d\eta(X, Y) = g(X, \phi Y), \tag{13}$$

for every $X, Y \in \chi(M)$.
In a contact metric manifold M^{2n+1}, we define the (1,1)-tensor field h by $2hX = (\pounds_\xi \phi)(X)$, where $\pounds_\xi$ denotes Lie differentiation in the direction of the vector field ξ. The tensor h is symmetric, such that

$$h\xi = 0, \quad h\phi = -\phi h, \quad tr(h) = 0, \quad tr(\phi h) = 0, \tag{14}$$

$$\nabla_X \xi = -\phi X - \phi h X. \tag{15}$$

$$(\nabla_X \eta) Y = g(X, \phi Y) - g(X, \phi h Y). \tag{16}$$

In a (k, μ)-contact metric manifold the following results hold [4, 5]:

$$(\nabla_X \phi) Y = g(X + hX, Y)\xi - \eta(Y)(X + hX), \tag{17}$$

$$h^2 = (k-1)\phi^2 \tag{18}$$

and

$$rank\phi = 2n. \tag{19}$$

Also in a $(2n+1)$-dimensional (k, μ)-contact metric manifold, we have the following relations hold from [4, 8]

$$\begin{aligned} \eta(R(X,Y)Z) = &k[g(Y,Z)\eta(X) - g(X,Z)\eta(Y)] \\ &+ \mu[g(hY,Z)\eta(X) - g(hX,Z)\eta(Y)], \end{aligned} \tag{20}$$

$$S(\phi X, \phi Y) = S(X,Y) - 2nk\eta(X)\eta(Y) - 2(2n-2+\mu)g(hX,Y), \tag{21}$$

$$\begin{aligned} S(X,Y) = &(2n-2-n\mu)\, g(X,Y) + (2-2n+2nk+n\mu)\,\eta(X)\eta(Y) \\ &+ (2n-2+\mu)\, g(hX,Y), \end{aligned} \tag{22}$$

$$S(X, \xi) = 2nk\eta(X), \tag{23}$$

$$S(\xi, \xi) = 2nk, \tag{24}$$

$$R(\xi, X)Y = k[g(X,Y)\xi - \eta(Y)X] + \mu[g(hX,Y)\xi - \eta(Y)hX], \tag{25}$$

$$R(\xi, X)\xi = k[\eta(X)\xi - X] + \mu[\eta(hX) - hX], \tag{26}$$

$$r = (2n - 2 + k - n\mu), \tag{27}$$

$$Q\xi = 2nk\xi. \tag{28}$$

Now, we give the following definitions:

Definition 2.1. [20] A Riemannian manifold is said to have Ricci-recurrent if it satisfies the following relation

$$(\nabla_X S)(Y,Z) = B(X)S(Y,Z),$$

for all vector fields $X,Y,Z \in \chi(M)$, where B is a 1-form on M. If the 1-form B is identically zero on M , then the Ricci-recurrent manifold is said to be a Ricci-symmetric manifold, that is, the Ricci tensor is covariant constant.

Definition 2.2. The concircular curvature tensor in a $(2n+1)$-dimensional (k,μ)-contact metric manifold is defined by [24]

$$C(X,Y)Z = R(X,Y)Z - \frac{r}{2n(2n+1)}\left[g(Y,Z)X - g(X,Z)Y\right], \tag{29}$$

for each vector fields $X,Y,Z \in \chi(M)$. The manifold (M^{2n+1}, g) is called ξ-concircularly flat if $C(X,Y)\xi = 0$ for each vector fields $X,Y \in \chi(M)$.

Definition 2.3. A vector field V on a $(2n+1)-$dimensional (k,μ)-contact metric manifold is said to be torse-forming vector field [25] if

$$\nabla_Y V = fY + \gamma(Y)V, \tag{30}$$

where f is a smooth function and γ is a 1-form.

Definition 2.4. A $(2n+1)$-dimensional (k,μ)-contact metric manifold is said to be an η-Einstein manifold if its Ricci tensor S is of the form

$$S(X,Y) = ag(X,Y) + b\eta(X)\eta(Y),$$

for all $X,Y \in \chi(M)$ and smooth functions a,b on M. If $b=0$, then the manifold is said to be an Einstein manifold.

3. $(2n+1)-$dimensional (k,μ)-contact metric manifold admitting an $\eta-$Einstein Soliton

Here we consider (k,μ)-contact metric manifold (M^{2n+1}, g) admitting an $\eta-$Einstein soliton. In the first part, we try to characterize the nature of the soliton by calculating the condition under which an $\eta-$Einstein soliton is shrinking, steady or expanding on a $(2n+1)-$dimensional (k,μ)-contact metric manifold.

Now, we state the following theorem :

Theorem 3.1. *If a (2n+1)-dimensional (k,μ)-contact metric manifold (M^{2n+1}, g) is Ricci symmetric (i.e., $\nabla S = 0$) and admits an η-Einstein soliton (g,ξ,λ,μ), then $\mu = 0$ and the constant scalar curvature $r = 2\lambda + 4kn$. Furthermore, the soliton is shrinking, steady and expanding for $r < 4kn$, $r = 4kn$ and $r > 4kn$, respectively.*

Proof. Let us consider a (k,μ)-contact metric manifold (M^{2n+1}, g) admitting an η-Einstein soliton (g,ξ,λ,μ). Then from the equation (9), we have

$$(\pounds_\xi g)(X,Y) + 2S(X,Y) + (2\lambda - r)\, g(X,Y) + 2\mu\eta(X)\eta(Y) = 0, \tag{31}$$

for all vector fields $X,Y \in \chi(M)$. From (31), we get

$$2S(X,Y) = -(\pounds_\xi g)(X,Y) - (2\lambda - r)\, g(X,Y) - 2\mu\eta(X)\eta(Y). \tag{32}$$

Now, with the help of (15), we have

$$(\pounds_{\xi} g)(X,Y) = -2g(\phi hX, Y). \tag{33}$$

From (32) and (33), we obtain

$$S(X,Y) = \left(\frac{r}{2} - \lambda\right) g(X,Y) - \mu\eta(X)\eta(Y) + g(\phi hX, Y). \tag{34}$$

Putting $Y = \xi$ in (34), we get

$$S(X,\xi) = \left(\frac{r}{2} - \lambda - \mu\right)\eta(X). \tag{35}$$

Comparing the equations (35) and (23), we have

$$2kn\eta(X) = \left(\frac{r}{2} - \lambda - \mu\right)\eta(X).$$

Since η is a non-zero 1-form, it becomes

$$r = 2\lambda + 2\mu + 4kn. \tag{36}$$

It is well known that,

$$(\nabla_X S)(Y,Z) = X(S(Y,Z)) - S(\nabla_X Y, Z) - S(Y, \nabla_X Z), \tag{37}$$

for every vector fields X, Y, Z on M^{2n+1}.
Using the equation (34) and (37), we achieve

$$(\nabla_X S)(Y,Z) = -\mu[\eta(Z)(\nabla_X\eta)Y + \eta(Y)(\nabla_X\eta)Z], \tag{38}$$

for every vector fields X, Y, Z on M^{2n+1}.
Using equation (16), the above equation becomes

$$(\nabla_X S)(Y,Z) = -\mu\left[\eta(Z)\left(g(X,\phi Y) - g(X,\phi hY)\right) + \eta(Y)\left(g(X,\phi Z) - g(X,\phi hZ)\right)\right]. \tag{39}$$

If the manifold M^{2n+1} is Ricci symmetric, then $\nabla S = 0$.
Therefore the equation (39) reduces to

$$-\mu\left[\eta(Z)\left(g(X,\phi Y) - g(X,\phi hY)\right) + \eta(Y)\left(g(X,\phi Z) - g(X,\phi hZ)\right)\right] = 0, \tag{40}$$

for all vector fields $X, Y, Z \in \chi(M)$.
Putting $Z = \xi$ in the equation (40), we have

$$\mu\left[g(X,\phi Y) - g(X,\phi hY)\right] = 0, \tag{41}$$

for any $X, Y \in \chi(M)$. Then μ=0 as $g(\phi X, Y) \neq g(X, \phi hY)$.
Equation (36) reduce to

$$r = 2\lambda + 4kn. \tag{42}$$

From (42), we can conclude the following :
(i) If $\lambda < 0$, then $r < 4kn$ implies the soliton is shrinking.
(ii) If $\lambda = 0$, then $r = 4kn$ implies the soliton is steady.
(iii) If $\lambda > 0$, then $r > 4kn$ implies the soliton is expanding.
This completes the proof.

□

THEOREM 3.2. *If the metric of a (2n+1)-dimensional (k,μ)-contact metric manifold is an $\eta-$Einstein soliton and the Ricci tensor is $\eta-$Recurrent (i.e. ∇S=$\eta \otimes S$), then the constant scalar curvature $r = 2(\lambda + \mu)$*

Proof. Let us have a look the Ricci tensor is $\eta-$Recurrent, then we get

$$\nabla S = \eta \otimes S, \tag{43}$$

that is,

$$(\nabla_X S)(Y,Z) = \eta(X)S(Y,Z), \tag{44}$$

for all vector fields X, Y, Z on M.
From equations (39) and (44), we obtain

(45)
$$-\mu\left[\eta(Z)\left(g(X,\phi Y) - g(X,\phi hY)\right) + \eta(Y)\left(g(X,\phi Z) - g(X,\phi hZ)\right)\right] = \eta(X)S(Y,Z).$$

Putting $Y = Z = \xi$ in the equation (45) and using the equation (35), we obtain

$$\left(\frac{r}{2} - \lambda - \mu\right)\eta(X) = 0. \tag{46}$$

Since η is 1-form, the above equation becomes

$$r = 2(\lambda + \mu).$$

This completes the proof. □

Theorem 3.3. *If a (2n+1)-dimensional (k,μ)- contact metric manifold (M^{2n+1}, g) admits an η-Einstein soliton (g,ν,λ,μ) such that the vector field ν is pointwise collinear with ξ (i.e ν is a constant multiple of ξ), then the manifold (M^{2n+1}, g) becomes an η-Einstein manifold of constant scalar curvature $r = 2\lambda + 2\mu + 4kn$.*

Proof. Considering a (k,μ)- contact metric manifold (M^{2n+1}, g) that admits an η-Einstein soliton (g,ν,λ,μ) such that ν is parallel to ξ, that is, $\nu = c\xi$ for some function c, and using this in equation (9), it follows that

$$(\pounds_{c\xi} g)(X,Y) + 2S(X,Y) + (2\lambda - r)g(X,Y) + 2\mu\eta(X)\eta(Y) = 0,$$

which gives

$$\begin{aligned} &cg(\nabla_X \xi, Y) + (Xc)\eta(Y) + cg(\nabla_Y \xi, X) + (Yc)\eta(X) \\ &+ 2S(X,Y) + (2\lambda - r)g(X,Y) + 2\mu\eta(X)\eta(Y) = 0. \end{aligned} \tag{47}$$

Using (15) in the equation (47), we get

$$\begin{aligned} &- cg(\phi X, Y) - cg(\phi hX, Y) + (Xc)\eta(Y) - cg(\phi Y, X) - cg(\phi hY, X) + (Yc)\eta(X) \\ &+ 2S(X,Y) + (2\lambda - r)g(X,Y) + 2\mu\eta(X)\eta(Y) = 0. \end{aligned} \tag{48}$$

Substituting $Y = \xi$ in (48), we have

$$(Xc) + (2\lambda - r + \xi c + 4kn + 2\mu)\eta(X) = 0. \tag{49}$$

If

$$(2\lambda - r + \xi c + 4kn + 2\mu) = 0,$$

then $Xc = 0$, that is, c is constant. This implies $\xi c = 0$. From equation (49), we obtain

$$r = 2\lambda + 2\mu + 4kn. \tag{50}$$

Since c is constant, equation (48) becomes

$$S(X,Y)=\left(\frac{r}{2}-\lambda\right)g(X,Y)-\mu\eta(X)\eta(Y), \tag{51}$$

for all $X,Y\in\chi(M)$.
Hence the result. □

4. η-Einstein soliton on $(2n+1)-$dimensional (k,μ)-contact metric manifold satisfying $R(X,Y).S=0$

In this section, first we consider a (k,μ)-contact metric manifold (M^{2n+1},g) that admits an η-Einstein soliton (g,ξ,λ,μ) and the manifold satisfies the curvature condition $R(X,Y).S=0$, then

$$S(R(X,Y)Z,W)+S(Z,R(X,Y)W)=0, \tag{52}$$

for all $X,Y,Z,W\in\chi(M)$.
we can state the following theorem:

THEOREM 4.1. *Let $(2n+1)$-dimensional (k,μ)-contact metric manifold admits an η-Einstein soliton (g,ξ,λ,μ). If the manifold satisfies the curvature condition $R(X,Y).S=0$, then the manifold admit a constant scalar curvature $r=2\lambda+4kn$ and the soliton is shrinking, steady and expanding as*
(i) $r<4kn$,
(ii) $r=4kn$,
(iii) $r>4kn$.

Proof. Setting $W=\xi$ in (52), we obtain

$$S(R(X,Y)Z,\xi)+S(Z,R(X,Y)\xi)=0, \tag{53}$$

for all $X,Y,Z\in\chi(M)$.
Using equations (1), (20) and (23) in (53), we get

$$\begin{aligned}&2nk(k\left[g(Y,Z)\eta(X)-g(X,Z)\eta(Y)\right]+\mu\left[g(hY,Z)\eta(X)-g(hX,Z)\eta(Y)\right])\\&+S(Z,k\left\{\eta(Y)X-\eta(X)Y\right\}+\mu\left\{\eta(Y)hX-\eta(X)hY\right\})=0,\end{aligned} \tag{54}$$

which implies,

$$\begin{aligned}&[2nk^2g(Y,Z)-kS(Y,Z)+2nk\mu g(hY,Z)-\mu S(hY,Z)]\eta(X)\\&+[kS(X,Z)-2nk^2g(X,Z)+\mu S(Z,hX)-2nk\mu g(hX,Z)]\eta(Y)=0.\end{aligned} \tag{55}$$

Taking $X=\xi$ in the above equation, then it reduces to

$$kS(Y,Z)+\mu S(hY,Z)=2nk^2g(Y,Z)+2nk\mu g(hY,Z). \tag{56}$$

Now, X replace by hX in (22), we get

$$\begin{aligned}S(hX,Y)=&(2n-2-n\mu)\,g(hX,Y)-(k-1)\,(2n-2+\mu)\,g(X,Y)\\&+(k-1)\,(2n-2+\mu)\,\eta(X)\eta(Y).\end{aligned} \tag{57}$$

From (56) and (57), we obtain

$$S(Y,Z) = \left[2kn + \frac{k-1}{k}(2n-2+\mu)\mu\right] g(Y,Z) + \left[2n\mu - \frac{1}{k}(2n-2-n\mu)\mu\right] g(hY,Z)$$
$$- \left(\frac{k-1}{k}\right)(2n-2+\mu)\mu\eta(Y)\eta(Z). \tag{58}$$

If $\left[2n\mu - \frac{1}{k}(2n-2-n\mu)\mu\right] = 0$,
that is, $\mu = 0$ and $\left[2n - \frac{1}{k}(2n-2-n\mu)\right] \neq 0$, then (58) becomes

$$S(Y,Z) = 2kng(Y,Z), \tag{59}$$

for all $Y, Z \in \chi(M)$.
Let us assume that the Einstein semi-symmetric $(2n+1)$-dimensional (k,μ)- contact metric manifold admits an η-Einstein soliton (g,ξ,λ,μ). Then equation (34) holds and combining (34) with the equation (59), we obtain

$$2kn(2n+1) = (2n+1)\left(\frac{r}{2} - \lambda\right), \tag{60}$$

that is,

$$r = 2\lambda + 4kn, \tag{61}$$

for any $X \in \chi(M)$. From (61), we can conclude the following :
(i) If $\lambda < 0$, then $r < 4kn$ implies the soliton is shrinking.
(ii) If $\lambda = 0$, then $r = 4kn$ implies the soliton is steady.
(iii) If $\lambda > 0$, then $r > 4kn$ implies the soliton is expanding.
This completes the proof. □

Theorem 4.2. *Let $(2n+1)-$dimensional (k,μ)-contact metric manifold admits an η-Einstein soliton (g,ξ,λ,μ). If the manifold is Ricci semi-symmetric, then the manifold is locally isometric to the Riemannian product $E^{n+1}(0) \times S^n(4)$ for $n > 1$ and flat for $n = 1$.*

Proof. Again from (56) and (57), we obtain

$$k(2n-2-n\mu)g(Y,Z) + k(2-2n+2nk+n\mu)\eta(Y)\eta(Z) + k(2n-2+\mu)g(hY,Z)$$
$$= \left[2k^2n + (k-1)(2n-2+\mu)\mu\right] g(Y,Z) + \left[2kn\mu + (2n-2-n\mu)\mu\right] g(hY,Z)$$
$$- (k-1)(2n-2+\mu)\mu\eta(Y)\eta(Z). \tag{62}$$

Comparing the both sides, we get
$\mu = 0, k = 0$.
Hence the manifold is locally isometric to the Riemannian product $E^{n+1}(0) \times S^n(4)$ for $n > 1$ and flat for $n = 1$.
Hence the result. □

5. η-Einstein soliton on $(2n+1)-$dimensional (k,μ)-contact metric manifold satisfying $C(\xi,X).S=0$

In this section, we consider a (k,μ)-contact metric manifold (M^{2n+1},g) that admits an η-Einstein soliton (g,ξ,λ,μ) and the manifold satisfies the curvature condition $C(\xi,X).S=0$, then

$$S(C(\xi,X)Y,Z)+S(Y,C(\xi,X)Z)=0. \tag{63}$$

Now we can state the following theorem.

THEOREM 5.1. *Let* $(2n+1)$*-dimensional (k,*μ*)-contact metric manifold admits an* η*-Einstein soliton* (g,ξ,λ,μ)*. If the manifold satisfies the curvature condition* $C(\xi,X).S=0$*, then the manifold admit a constant scalar curvature* $r=2\lambda+4kn$*.*

Proof. From equation (29), we find

$$C(\xi,X)Y=R(\xi,X)Y-\frac{r}{2n(2n+1)}\left[g(X,Y)\xi-\eta(Y)X\right]. \tag{64}$$

Using (25) in (64), we have

$$C(\xi,X)Y=\left[k-\frac{r}{2n(2n+1)}\right]\left[g(X,Y)\xi-\eta(Y)X\right]+\mu\left[g(hX,Y)\xi-\eta(Y)hX\right]. \tag{65}$$

Similarly,

$$C(\xi,X)Z=\left[k-\frac{r}{2n(2n+1)}\right]\left[g(X,Z)\xi-\eta(Z)X\right]+\mu\left[g(hX,Z)\xi-\eta(Z)hX\right]. \tag{66}$$

Using equations (65), (66) in (63), we obtain

$$\begin{aligned}&\left[k-\frac{r}{2n(2n+1)}\right]S(\left[g(X,Y)\xi-\eta(Y)X\right],Z)+S(\mu\left[g(hX,Y)\xi-\eta(Y)hX\right],Z)+\\&\left[k-\frac{r}{2n(2n+1)}\right]S(\left[g(X,Z)\xi-\eta(Z)X\right],Y)+S(\mu\left[g(hX,Z)\xi-\eta(Z)hX\right],Y)=0,\end{aligned} \tag{67}$$

which implies

$$\begin{aligned}&\left[k-\frac{r}{2n(2n+1)}\right]\left[2kng(X,Y)\eta(Z)-S(X,Z)\eta(Y)+2kng(X,Z)\eta(Y)-S(X,Y)\eta(Z)\right]\\&+\mu\left[2kng(hX,Y)\eta(Z)-S(hX,Z)\eta(Y)+2kng(hX,Z)\eta(Y)-S(hX,Y)\eta(Z)\right]=0.\end{aligned} \tag{68}$$

Setting $Z=\xi$ in (68) and using (23), we get

$$\left[k-\frac{r}{2n(2n+1)}\right]\left[2kng(X,Y)-S(X,Y)\right]+\mu\left[2kng(hX,Y)-S(hX,Y)\right]=0. \tag{69}$$

Using equation (57) in (69), we have

$$\begin{aligned}\left[k-\frac{r}{2n(2n+1)}\right]S(X,Y)=&\left\{2kn\left[k-\frac{r}{2n(2n+1)}\right]+(k-1)(2n-2+\mu)\mu\right\}g(X,Y)\\&+(2kn-2n+2+n\mu)\mu g(hX,Y)-(k-1)(2n-2+\mu)\mu\eta(X)\eta(Y).\end{aligned} \tag{70}$$

If $[2kn-2n+2+n\mu]\,\mu=0$,
that is, $\mu=0$ and $[2kn-2n+2+n\mu)]\neq 0$, then (70) becomes

$$S(X,Y)=2kng(X,Y), \tag{71}$$

for all $X,Y\in\chi(M)$.
Let us assume that the Einstein semi-symmetric $(2n+1)-$dimensional (k,μ)- contact metric manifold admits an $\eta-$Einstein soliton (g,ξ,λ,μ). Then equation (34) holds and combining (34) with the equation (70), we obtain

$$2kn(2n+1)=(2n+1)\left(\frac{r}{2}-\lambda\right), \tag{72}$$

that is,

$$r=2\lambda+4kn. \tag{73}$$

This completes the proof. □

6. η-Einstein soliton on $(2n+1)-$dimensional (k,μ)-contact metric manifold with torse-forming vector field

In this section we prove the following theorem.

THEOREM 6.1. *Let $(2n+1)-$dimensional (k,μ)-contact metric manifold admits an η-Einstein soliton (g,ξ,λ,μ) with torse-forming vector field ξ, then the manifold becomes an $\eta-$Einstein manifold.*

Proof. Let us consider a (k,μ)-contact metric manifold (M^{2n+1},g) admitting an η-Einstein soliton (g,ξ,λ,μ) and assume that Reeb vector field ξ of the manifold is a torse-forming vector field. Then ξ being a torse-forming vector field, from equation (30), we infer that

$$\nabla_Y\xi=fY+\gamma(Y)\xi, \tag{74}$$

for each $Y\in\chi(M)$.
Using equation (15) and taking inner product with ξ, we obtain

$$g(\nabla_Y\xi,\xi)=-(\phi+\phi h)\eta(Y). \tag{75}$$

Taking inner product in equation (74), with ξ we have

$$g(\nabla_Y\xi,\xi)=f\eta(Y)+\gamma(Y). \tag{76}$$

The equations (75) and (76), give us

$$\gamma=-(\phi+\phi h+f). \tag{77}$$

Thus for a torse-forming vector field ξ in $(k,\,\mu)$-contact metric manifold, we obtain

$$\nabla_Y\xi=f(Y-\eta(Y)\xi)-(\phi+\phi h)\eta(Y)\xi. \tag{78}$$

Since (g,ξ,λ,μ) is an $\eta-$Einstein soliton, from equation (9), we have

$$g(\nabla_X\xi,Y)+g(\nabla_Y\xi,X)+2S(X,Y)+(2\lambda-r)g(X,Y)+2\mu\eta(X)\eta(Y)=0, \tag{79}$$

for all vector fields $X,Y\in\chi(M)$.
Using (78) in the above equation, we obtain

$$S(X,Y)=\left[\frac{r}{2}-(\lambda+f)\right]g(X,Y)+(\phi+\phi h+f-\mu)\eta(X)\eta(Y). \tag{80}$$

This means that the manifold is an $\eta-$Einstein manifold. □

Now, we give an example of a (k, μ)-contact metric manifold:

7. Example of a (k, μ)-contact metric manifold admitting an $\eta-$Einstein soliton

Let us consider $M = \{(x, y, z) \in \mathbf{R}^3, (x, y, z) \neq (0, 0, 0)\}$ be a three-dimensional manifold [17] admitting an $\eta-$Einstein soliton (g, ξ, λ, μ). The vector fields e_1, e_2, e_3 are linearly independent in R^3 so as

$[e_1, e_2] = (1 + \beta)e_3$, $[e_3, e_1] = (1 - \beta)e_2$, $[e_2, e_3] = 2e_1$,

where $\beta = \pm\sqrt{1 - k}$ is a real number.
We define the Riemannian metric g by
$g(e_1, e_2) = g(e_2, e_3) = g(e_1, e_3) = 0$ and $g(e_1, e_1) = g(e_2, e_2) = g(e_3, e_3) = 1$.
Let 1-form η defined by
$\eta(X) = g(X, e_1)$,
for each $X \in \chi(M)$. The (1,1) tensor field ϕ is defined as

$$\phi(e_1) = 0, \phi(e_2) = e_3, \phi(e_3) = -e_2.$$

Using the linearity of ϕ and g, we have

$$\eta(e_1) = 1,$$
$$\phi^2(X) = -X + \eta(X)e_1$$

and

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y),$$

for each $X, Y \in \chi(M)$. Furthermore
$he_1 = 0, he_2 = \beta e_2$, and $he_3 = -\beta e_3$.
By using Koszul's formula for the Riemannian metric g, we can calculate

$$\nabla_{e_1} e_1 = 0, \nabla_{e_1} e_2 = 0, \nabla_{e_1} e_3 = 0,$$
$$\nabla_{e_2} e_1 = -(1 + \beta)e_3, \nabla_{e_2} e_2 = 0, \nabla_{e_2} e_3 = (1 + \beta)e_1,$$
$$\nabla_{e_3} e_1 = (1 - \beta)e_2, \nabla_{e_3} e_2 = -(1 - \beta)e_1, \nabla_{e_3} e_3 = 0.$$

Using these we can verify $\nabla_X \xi = -\phi X - \phi hX$ for $e_1 = \xi$. Hence the manifold is a contact metric manifold with the contact structure (ϕ, ξ, η, g) .
Also from the relation of Riemmanian curvature tensor we can calculate the following components

$$R(e_1, e_1)e_1 = 0, R(e_1, e_2)e_1 = -(1 - \beta^2)e_2, R(e_1, e_2)e_2 = (1 - \beta^2)e_1,$$
$$R(e_1, e_2)e_3 = 0, R(e_2, e_3)e_1 = 0, R(e_2, e_3)e_3 = -(1 - \beta^2)e_2,$$
$$R(e_1, e_3)e_1 = (1 - \beta^2)e_3, R(e_1, e_3)e_2 = 0, R(e_1, e_3)e_3 = (1 - \beta^2)e_1,$$
$$R(e_2, e_1)e_1 = -(1 - \beta^2)e_2, R(e_3, e_1)e_1 = (1 - \beta^2)e_3, R(e_2, e_3)e_2 = (1 - \beta^2)e_3.$$

From these curvature tensors, we can calculate the components of Ricci tensors as follows:

$$S(e_1, e_1) = 2(1 - \beta^2), S(e_2, e_2) = 0, S(e_3, e_3) = 0.$$

From equation (59), we can obtain

$$S(e_3, e_3) = 2kng(e_3, e_3) = 2kn.$$

By equating both the values of $S(e_3, e_3)$, we get

$$k = 0.$$

Hence the manifold (R^3, g) is locally isometric to the product $E^2(0) \times S^1(4)$.
Again, we can calculate equation(34)

$$S(e_3, e_3) = \left[\frac{r}{2} - (\lambda + \mu)\right].$$

Therefore,

$$\left[\frac{r}{2} - (\lambda + \mu)\right] = 0,$$

which implies that,

$$r = 2(\lambda + \mu).$$

Since $k = 0$, equation(36) reduces to

$$r = 2(\lambda + \mu).$$

Hence the constants λ and μ satisfies equation (36) and so g defines an $\eta-$Einstein soliton on (k, μ)-contact manifold M.
Further, putting $k = 0$ in (42), we can calculate

$$\lambda = \frac{r}{2}.$$

Thus the soliton (g, ξ, λ) on (k, μ)-contact manifold is shrinking, steady and expanding as $r < 0$, $r = 0$ and $r > 0$, respectively.

References

[1] N. Basu and A. Bhattacharyya, *Conformal Ricci soliton in Kenmotsu manifold*, Global Journal of Advanced Research on classical and Modern Geometrices, **4** (1) (2015), 15–21.

[2] A. M. Blaga, *On gradient $\eta-$Einstein solitons*, Kragujevac Journal of Mathematics **42** (2) (2018), 229–237.
https://doi.org/10.5937/KGJMATH1802229B

[3] D. E. Blair, J. S. Kim and M. M. Tripathi, *On the concircular curvature tensor of a contact metric manifold*, J.Korean Math.Soc. **42** (2005), 883–992.
https://doi.org/10.4134/JKMS.2005.42.5.883

[4] D. E. Blair, T. Koufogiorgos and B. J. Papantoniou, *Contact metric manifolds satisfying a nullity condition*, Israel J. Math. **91** (1995), 189–214.

[5] D .E. Blair, *Contact manifolds in Riemannian geometry*, Lecture notes in Mathematics, Springer- Verlag, Berlin, Vol.509 (1976).

[6] D. E. Blair, *Two remarks on contact metric structures*, Tohoku Math.J. **29** (1977), 319–324.
https://doi.org/10.2748/TMJ/1178240602

[7] E. Boeckx, P. Bueken and L. Vanhecke, *ϕ-symmetric contact metric spaces*, Glasgow Math. J. **41** (1999), 409–416.
[8] E. Boecks, *A full classification of contact metric (k,μ)-spaces*, Illinois J. Math. **44** (2000), 212–219.
[9] C. Calin and M. Crasmareanu, *$\eta-$Ricci solitons on Hopf hypersurfaces in a complex space forms*, Revue Roumaine de Math.Pures et Appl. **57** (1) (2012), 53–63.
[10] G. Catino and I. Mazzieri, *Gradient Einstein solitons,* Nonlinear Anal. **132** (2016), 66–94. https://doi.org/10.1016/j.na.2015.10.021
[11] J.T. Cho and M. Kimura, *Ricci solitons and real hypersurfaces in a complex space form*, Tohoku Math. J **61** (2) (2009), 205–212. https://doi.org/10.2748/tmj/1245849443
[12] Mohd. Danish Siddiqi, *Conformal $\eta-$Ricci solitons in $\delta-$Lorentzian trans Sasakian manifolds*, International Journal of Maps in Mathematics **1** (1) (2018), 15–34.
[13] U. C. De and A. A. Shaikh, *Complex Manifolds and Contact Manifolds*, 142–143.
[14] A. E. Fischer, *An introduction to conformal Ricci flow*, class. Quantum. Grav. **21** (2004), 171–218. https://doi.org/10.1088/0264-9381/21/3/011
[15] R. S. Hamilton, *The formation of singularities in the Ricci flow*, Surveys in Differential Geometry (Cambridge, MA, 1993) **2** 7-136, International Press, Cambridge, MA, 1995.
[16] R. S. Hamilton, *Three manifolds with positive Ricci curvature*, J.Differential Geom. (1982), Vol.17, isu.2, 255–306. https://doi.org/10.1086/wp.17.4.1180866
[17] D. Kar, P. Majhi and U. C. De, *η-Ricci solitons on 3-dimensional $N(k)$-contact metric manifolds*, Acta Univ. Apulensis **54** (2018), 71–88.
[18] T. Koufogiorgos, *Contact metric manifolds*, Annals of Global Analysis and Geometry **11** (1993), 25–34. https://doi.org/10.1007/BF00773361
[19] T. Koufogiorgos, *On a class of contact Riemannian 3-manifolds*, Results in Mathematics **27** (1995), 51–62. https://doi.org/10.1007/BF03322269
[20] E. M. Patterson, *Some theorems on Ricci recurrent spaces*, J. London Math. Soc. **27** (1952), 287–295.
[21] S. Tanno, *The topology of contact Riemannian manifolds*, Illinois J. Math. **12** (1968), 700–717. https://doi.org/10.1215/IJM/1256053971
[22] M. M. Tripathi and J. S. Kim, *On the concircular curvature tensor of a (k,μ)-manifold*, Balkan J Geo Appl **2** (2004), 104–114.
[23] C. Udriste, *On contact 3-structures*, Bull. Univ. Brasov **16** (1974), 85–92. https://doi.org/10.1007/s002090100279
[24] K. Yano, *Concircular geometry I. Concircular transformations*, Proc. Impe. Acad., Tokyo., **16** (1940), 195–200. https://doi.org/10.3792/PIA/1195579139
[25] K. Yano, *On torse-forming directions in Riemannian spaces*, Proc. Impe. Acad., Tokyo., **20** (1944), 701–705. https://doi.org/10.3792/PIA/1195572958

Arup Kumar Mallick
Department of Mathematics, Jadavpur University, Kolkata-700032, India
E-mail: arupkm14@gmail.com

Arindam Bhattacharyya
Department of Mathematics, Jadavpur University, Kolkata-700032, India
E-mail: bhattachar1968@yahoo.co.in