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ON THE ASYMPTOTIC EXACTNESS OF AN ERROR ESTIMATOR FOR THE LOWEST-ORDER RAVIART–THOMAS MIXED FINITE ELEMENT

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Abstract. In this paper we analyze an error estimator for the lowest-order triangular Raviart–Thomas mixed finite element which is based on solution of local problems for the error. This estimator was proposed in [Alonso, Error estimators for a mixed method, Numer. Math. 74 (1996), 385–395] and has a similar concept to that of Bank and Weiser. We show that it is asymptotically exact for the Poisson equation if the underlying triangulations are uniform and the exact solution is regular enough.

1. Introduction

In this work we consider the Poisson equation

$$\begin{cases} -\Delta u = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{1.1}$$

on a bounded polygonal domain $\Omega \subset \mathbb{R}^2$ with the boundary $\partial\Omega$. By introducing the vector variable $\boldsymbol{\sigma} = -\nabla u$, the Poisson equation may be

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rewritten as the system of first-order PDEs

$$\boldsymbol{\sigma} + \nabla u = 0 \quad \text{and} \quad \operatorname{div} \boldsymbol{\sigma} - f = 0 \qquad \text{in } \Omega.$$

The mixed variational formulation then seeks $(\boldsymbol{\sigma}, u) \in H(\operatorname{div}, \Omega) \times L^2(\Omega)$ such that

$$\begin{aligned} (\boldsymbol{\sigma}, \boldsymbol{\tau})_\Omega - (\operatorname{div} \boldsymbol{\tau}, u)_\Omega &= 0 \qquad \forall \boldsymbol{\tau} \in H(\operatorname{div}, \Omega), \\ (\operatorname{div} \boldsymbol{\sigma}, w)_\Omega - (f, w)_\Omega &= 0 \qquad \forall w \in L^2(\Omega), \end{aligned}$$

where $(\cdot, \cdot)_S$ is the standard L^2 inner product over a domain $S \subset \mathbb{R}^2$ and

$$H(\operatorname{div}, \Omega) = \{\boldsymbol{\tau} \in (L^2(\Omega))^2 : \operatorname{div} \boldsymbol{\tau} \in L^2(\Omega)\}.$$

Several a posteriori error estimators have been developed and analyzed for the mixed finite element methods based on the above formulation; see, for example, [2,4,7,16] for some early works and [1,10–12,15] for more recent ones. The performance of an error estimator is commonly measured by the effectivity index which is the ratio of the estimated error to the actual error. In particular, we say that the error estimator is *asymptotically exact* if the effectivity index approaches unity as the mesh size goes to zero. For the $P1$ conforming finite element method, the asymptotic exactness was established for the error estimator based on gradient recovery in [17, 18] and for the error estimator of Bank and Weiser in [9, 13]. It should be noted that the proof of the asymptotic exactness usually depends on super-closeness between the finite element solution and the Lagrange interpolant of the exact solution which has been shown to be valid only if the underlying triangulations are uniform (or small perturbations of uniform ones) and the exact solution is regular enough.

In this paper we consider one of the error estimators proposed by Alonso [2] for the lowest-order triangular Raviart–Thomas mixed finite element which is based on solution of local problems similar to those of Bank and Weiser [3]. This estimator yields an estimate of the vector error in the L^2 norm and was shown to be equivalent to the actual error under a saturation assumption. By adapting the proof of [9] and using the superconvergence result of [5], we show that it is asymptotically exact under the usual assumption that the underlying triangulations are uniform and the exact solution is regular enough.

The rest of the paper is organized as follows. In Section 2 we state some preliminary results and define the mixed finite element method and the error estimator proposed by Alonso. Then we establish the asymptotic exactness of this error estimator in Section 3. Finally, in Section 4, some numerical results are presented to confirm the theoretical result.

2. Mixed Finite Element Method and Error Estimator

We introduce some notation and preliminary results before defining the mixed finite element method and the error estimator for it.

Assume that a family of shape-regular triangulations $\{\mathcal{T}_h\}_{h>0}$ of Ω made of triangles is given with the mesh size $h = \max_{T\in\mathcal{T}_h} h_T$, where h_T is the diameter of T. For an element $T \in \mathcal{T}_h$, we denote the set of three edges of T by $\mathcal{E}_T$, the unit outward normal vector to ∂T by $\boldsymbol{n}_T$, and the union of all elements of $\mathcal{T}_h$ sharing at least one edge with T by

$$T^* = \bigcup\{T' : T \text{ and } T' \text{ share an edge}\}.$$

For a vector function $\boldsymbol{\tau} = (\tau_1, \tau_2)$ and a scalar function v defined over T, the following integration-by-parts formula holds

$$(\operatorname{rot}\boldsymbol{\tau}, v)_T - (\boldsymbol{\tau}, \mathbf{curl}\, v)_T = \langle \boldsymbol{\tau}\cdot\boldsymbol{t}_T, v\rangle_{\partial T},$$

where

$$\operatorname{rot}\boldsymbol{\tau} = \frac{\partial\tau_2}{\partial x_1} - \frac{\partial\tau_1}{\partial x_2}, \qquad \mathbf{curl}\, v = \left(\frac{\partial v}{\partial x_2}, -\frac{\partial v}{\partial x_1}\right)$$

and $\boldsymbol{t}_T = (-n_2, n_1)$ when $\boldsymbol{n}_T = (n_1, n_2)$. The tangential jump of a vector function $\boldsymbol{\tau}$ across an interior edge $e = \partial T \cap \partial T'$ is defined as

$$[\![\boldsymbol{\tau}\cdot\boldsymbol{t}]\!]|_e = \boldsymbol{\tau}|_T\cdot\boldsymbol{t}_T + \boldsymbol{\tau}|_{T'}\cdot\boldsymbol{t}_{T'}.$$

We simply set $[\![\boldsymbol{\tau}\cdot\boldsymbol{t}]\!]|_e = 2(\boldsymbol{\tau}\cdot\boldsymbol{t}_T)|_e$ for $e \subset \partial T \cap \partial\Omega$.

Let $\mathbb{P}_k(T)$ be the space of all polynomials on T of total degree at most k and let

$$\mathbb{P}_2^0(T) = \{\varphi \in \mathbb{P}_2(T) : \varphi \text{ vanishes at all vertices of } T\}.$$

For $T \in \mathcal{T}_h$ and $e \in \mathcal{E}_T$, the following estimates are easily derived by the scaling argument

$$\|v\|_{0,e} \lesssim h_T^{-1/2}\|v\|_{0,T} \qquad \forall v \in \mathbb{P}_1(T), \tag{2.1}$$

$$\|\varphi\|_{0,T} + h_T^{1/2}\|\varphi\|_{0,e} \lesssim h_T\|\,\mathbf{curl}\,\varphi\|_{0,T} \qquad \forall\varphi \in \mathbb{P}_2^0(T). \tag{2.2}$$

Here and throughout the paper, we will frequently use the notation $a \lesssim b$ for positive quantities a and b to indicate that the inequality $a \le Cb$

holds with constant $C > 0$ independent of the mesh size h. In addition, the notation $\|\cdot\|_{k,S}$ will be used for the norm of the Sobolev space $H^k(S)$.

Now the lowest-order triangular Raviart–Thomas mixed finite element method for the problem (1.1) is defined as follows: find $(\boldsymbol{\sigma}_h, u_h) \in \boldsymbol{V}_h \times W_h$ such that

$$(\boldsymbol{\sigma}_h, \boldsymbol{\tau}_h)_\Omega - (\operatorname{div} \boldsymbol{\tau}_h, u_h)_\Omega = 0 \qquad \forall \boldsymbol{\tau} \in \boldsymbol{V}_h, \tag{2.3}$$

$$(\operatorname{div} \boldsymbol{\sigma}_h, w_h)_\Omega - (f, w_h)_\Omega = 0 \qquad \forall w_h \in W_h, \tag{2.4}$$

where

$$\boldsymbol{V}_h = \{\boldsymbol{\tau} \in H(\operatorname{div}, \Omega) : \boldsymbol{\tau}|_T \in (\mathbb{P}_0(T))^2 + \boldsymbol{x}\mathbb{P}_0(T) \ \ \forall T \in \mathcal{T}_h\},$$
$$W_h = \{w \in L^2(\Omega) : w|_T \in \mathbb{P}_0(T) \ \ \forall T \in \mathcal{T}_h\}.$$

The following optimal a priori error estimates are well established (cf. [6, 14])

$$\|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{0,\Omega} \le Ch\|\boldsymbol{\sigma}\|_{1,\Omega}, \qquad \|u - u_h\|_{0,\Omega} \le Ch(\|\boldsymbol{\sigma}\|_{1,\Omega} + \|u\|_1).$$

Finally, we present the error estimator proposed by Alonso [2] for the mixed finite element method (2.3)–(2.4). Using the integration by parts and the equality $\operatorname{rot} \boldsymbol{\sigma}_h = 0$, one can easily derive the following error equation for the test function $\boldsymbol{\tau} = \mathbf{curl}\, \varphi \in H(\operatorname{div}, \Omega)$

$$\begin{aligned}(\boldsymbol{\sigma} - \boldsymbol{\sigma}_h, \boldsymbol{\tau})_\Omega = -(\boldsymbol{\sigma}_h, \mathbf{curl}\, \varphi)_\Omega &= \sum_{e \in \mathcal{E}_h} \int_e [\![\boldsymbol{\sigma}_h \cdot \boldsymbol{t}]\!] \varphi \, ds \\ &= \sum_{T \in \mathcal{T}_h} \frac{1}{2} \int_{\partial T} [\![\boldsymbol{\sigma}_h \cdot \boldsymbol{t}]\!] \varphi \, ds.\end{aligned}$$

Based on the idea of Bank and Weiser [3], this leads us to consider the following local problem.

DEFINITION 2.1. *For every $T \in \mathcal{T}_h$, define $\psi_T \in \mathbb{P}_2^0(T)$ to be the solution of*

$$(\mathbf{curl}\, \psi_T, \mathbf{curl}\, \varphi)_T = \frac{1}{2} \int_{\partial T} [\![\boldsymbol{\sigma}_h \cdot \boldsymbol{t}]\!] \varphi \, ds \qquad \forall \varphi \in \mathbb{P}_2^0(T) \tag{2.5}$$

and set

$$\eta = \left(\sum_{T \in \mathcal{T}_h} \| \mathbf{curl}\, \psi_T \|_{0,T}^2 \right)^{1/2}. \tag{2.6}$$

Note that (2.5) gives rise to a 3×3 matrix system, since $\mathbb{P}_2^0(T)$ is spanned by three quadratic edge bubble functions on T. By comparing the estimator η with the standard edge residual, we obtain the following local lower bound for η.

LEMMA 2.2. *For every $T \in \mathcal{T}_h$, we have*

$$\|\,\mathbf{curl}\,\psi_T\|_{0,T} \lesssim \|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{0,T^*}. \tag{2.7}$$

Proof. Taking $\varphi = \psi_T$ in (2.5) and applying (2.2), we obtain

$$\|\,\mathbf{curl}\,\psi_T\|_{0,T} \lesssim h_T^{1/2}\|[\![\boldsymbol{\sigma}_h \cdot \boldsymbol{t}]\!]\|_{0,\partial T}.$$

Now the desired result follows from the local lower bound for the standard edge residual [7]. □

3. Asymptotic Exactness of Error Estimator

In this section we establish the asymptotic exactness of the error estimator η defined in the previous section. Now, throughout the paper, *the triangulation $\mathcal{T}_h$ is assumed to be uniform in the sense that every pair of adjacent triangles of $\mathcal{T}_h$ forms a parallelogram.* Under this condition, the following superconvergence result was shown in [5, 8]

$$\|\Pi_h\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{0,\Omega} \lesssim h^{\frac{3}{2}}\|\boldsymbol{\sigma}\|_{2,\Omega}, \tag{3.1}$$

where $\Pi_h : (H^1(\Omega))^2 \to \boldsymbol{V}_h$ denotes the Raviart–Thomas projection defined by

$$\int_e \Pi_h\boldsymbol{\tau} \cdot \boldsymbol{n}_T\, ds = \int_e \boldsymbol{\tau} \cdot \boldsymbol{n}_T\, ds \qquad \forall e \in \mathcal{E}_T,\ T \in \mathcal{T}_h$$

for $\boldsymbol{\tau} \in (H^1(\Omega))^2$. By adapting the proof of [9] and using this result, we are able to derive the following super-closeness result.

THEOREM 3.1. *Suppose that the triangulation $\mathcal{T}_h$ is uniform. Then we have*

$$\left(\sum_{T\in\mathcal{T}_h} \|(\boldsymbol{\sigma} - \boldsymbol{\sigma}_h) - \mathbf{curl}\,\psi_T\|_{0,T}^2\right)^{1/2} \lesssim h^{\frac{3}{2}}\|\boldsymbol{\sigma}\|_{2,\Omega}.$$

Proof. Let $\mathcal{T}_h^\partial \subset \mathcal{T}_h$ be the set of all triangles with at least one edge on $\partial\Omega$ and let $\Omega_h^\partial = \bigcup_{T\in\mathcal{T}_h^\partial} T^*$. Using the local lower bound (2.7) and the well-known estimate $\|\boldsymbol{\sigma} - \Pi_h\boldsymbol{\sigma}\|_{0,T} \lesssim h_T\|\boldsymbol{\sigma}\|_{1,T}$ for $T \in \mathcal{T}_h$, we obtain

$$\begin{aligned}\sum_{T\in\mathcal{T}_h^\partial} \|(\boldsymbol{\sigma} - \boldsymbol{\sigma}_h) - \mathbf{curl}\,\psi_T\|_{0,T}^2 &\lesssim \|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{0,\Omega_h^\partial}^2 \\ &\lesssim \|\boldsymbol{\sigma} - \Pi_h\boldsymbol{\sigma}\|_{0,\Omega_h^\partial}^2 + \|\Pi_h\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{0,\Omega_h^\partial}^2 \\ &\lesssim h^2\|\boldsymbol{\sigma}\|_{1,\Omega_h^\partial}^2 + \|\Pi_h\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{0,\Omega_h^\partial}^2,\end{aligned}$$

which yields by Lemma 2.2 of [5]

$$\sum_{T\in\mathcal{T}_h^\partial} \|(\boldsymbol{\sigma} - \boldsymbol{\sigma}_h) - \mathbf{curl}\,\psi_T\|_{0,T}^2 \lesssim h^3\|\boldsymbol{\sigma}\|_{2,\Omega}^2 + \|\Pi_h\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{0,\Omega}^2. \tag{3.2}$$

Motivated by [9], we define $w_{\boldsymbol{\tau}}|_T \in \mathbb{P}_2^0(T)$ on each element $T \in \mathcal{T}_h \setminus \mathcal{T}_h^\partial$ (which means that T has three neighbors) to be the solution of

$$(\mathbf{curl}\,w_{\boldsymbol{\tau}}, \mathbf{curl}\,\varphi)_T = (\mathrm{rot}\,\boldsymbol{\tau}, \varphi)_T + \frac{1}{2}\int_{\partial T} [\![\Pi_h\boldsymbol{\tau}\cdot\boldsymbol{t}]\!]\varphi\,ds \qquad \forall\varphi \in \mathbb{P}_2^0(T) \tag{3.3}$$

for a given $\boldsymbol{\tau} \in (H^1(\Omega))^2$. By the triangle inequality we have

$$\begin{aligned}&\sum_{T\in\mathcal{T}_h\setminus\mathcal{T}_h^\partial} \|(\boldsymbol{\sigma} - \boldsymbol{\sigma}_h) - \mathbf{curl}\,\psi_T\|_{0,T}^2 \\ &\quad\lesssim \sum_{T\in\mathcal{T}_h\setminus\mathcal{T}_h^\partial} \big(\|(\boldsymbol{\sigma} - \Pi_h\boldsymbol{\sigma}) - \mathbf{curl}\,w_{\boldsymbol{\sigma}}\|_{0,T}^2 + \|\mathbf{curl}(w_{\boldsymbol{\sigma}} - \psi_T)\|_{0,T}^2\big) \\ &\qquad+ \|\Pi_h\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{0,\Omega}^2.\end{aligned}$$

The first term is handled by Lemma 3.3 given below, which results in

$$\sum_{T\in\mathcal{T}_h\setminus\mathcal{T}_h^\partial} \|(\boldsymbol{\sigma} - \Pi_h\boldsymbol{\sigma}) - \mathbf{curl}\,w_{\boldsymbol{\sigma}}\|_{0,T}^2 \lesssim h^4\|\boldsymbol{\sigma}\|_{2,\Omega}^2.$$

To bound the second term, we note that $\mathrm{rot}\,\boldsymbol{\sigma} = 0$ in (3.3) and then use the estimates (2.1)–(2.2) to obtain for $\varphi \in \mathbb{P}_2^0(T)$

$$\begin{aligned}(\mathbf{curl}(w_{\boldsymbol{\sigma}} - \psi_T), \mathbf{curl}\,\varphi)_T &= \frac{1}{2}\int_{\partial T} [\![(\Pi_h\boldsymbol{\sigma} - \boldsymbol{\sigma}_h)\cdot\boldsymbol{t}]\!]\varphi\,ds \\ &\le \frac{1}{2}\|[\![(\Pi_h\boldsymbol{\sigma} - \boldsymbol{\sigma}_h)\cdot\boldsymbol{t}]\!]\|_{0,\partial T}\|\varphi\|_{0,\partial T} \\ &\lesssim \|\Pi_h\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{0,T^*}\|\mathbf{curl}\,\varphi\|_{0,T}.\end{aligned}$$

Taking $\varphi = w_{\boldsymbol{\sigma}} - \psi_T$ and summing over all $T \in \mathcal{T}_h \setminus \mathcal{T}_h^\partial$ gives

$$\sum_{T\in\mathcal{T}_h\setminus\mathcal{T}_h^\partial} \|\,\mathbf{curl}(w_{\boldsymbol{\sigma}} - \psi_T)\|_{0,T}^2 \lesssim \|\Pi_h\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{0,\Omega}^2.$$

Therefore it follows that

$$\sum_{T\in\mathcal{T}_h\setminus\mathcal{T}_h^\partial} \|(\boldsymbol{\sigma} - \boldsymbol{\sigma}_h) - \mathbf{curl}\,\psi_T\|_{0,T}^2 \lesssim h^4\|\boldsymbol{\sigma}\|_{2,\Omega}^2 + \|\Pi_h\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{0,\Omega}^2. \tag{3.4}$$

The desired result is derived by combining (3.2) and (3.4) with the superconvergence result (3.1). □

To complete the proof of Theorem 3.1, we prove the following two lemmas.

Lemma 3.2. *Suppose that $T \cup T'$ is a parallelogram with the center $\boldsymbol{m}$. If $\boldsymbol{\tau} \in (\mathbb{P}_1(T \cup T'))^2$, then*

$$\boldsymbol{\tau}(\boldsymbol{m}) = \frac{1}{2}(\Pi_h\boldsymbol{\tau}|_T(\boldsymbol{m}) + \Pi_h\boldsymbol{\tau}|_{T'}(\boldsymbol{m})).$$

Proof. Following the argument of [5], we may take $\boldsymbol{m}$ as the origin and assume that $\boldsymbol{\tau}(\boldsymbol{m}) = 0$, since $\boldsymbol{\tau} = \Pi_h\boldsymbol{\tau}$ over $T \cup T'$ if $\boldsymbol{\tau} \in (\mathbb{P}_0(T \cup T'))^2$. Thus $\boldsymbol{\tau}$ is odd over $T \cup T'$ and so is $\Pi_h\boldsymbol{\tau}$, which implies that $\frac{1}{2}(\Pi_h\boldsymbol{\tau}|_T(\boldsymbol{m}) + \Pi_h\boldsymbol{\tau}|_{T'}(\boldsymbol{m})) = 0 = \boldsymbol{\tau}(\boldsymbol{m})$. This completes the proof. □

Lemma 3.3. *Assume that $T \in \mathcal{T}_h$ has three neighbors and let $w_{\boldsymbol{\sigma}} \in \mathbb{P}_2^0(T)$ be defined by (3.3). Then*

$$\|(\boldsymbol{\sigma} - \Pi_h\boldsymbol{\sigma}) - \mathbf{curl}\,w_{\boldsymbol{\sigma}}\|_{0,T} \lesssim h_T^2\|\boldsymbol{\sigma}\|_{2,T^*}.$$

Proof. For $\boldsymbol{\tau} \in (\mathbb{P}_1(T^*))^2$ and $\varphi \in \mathbb{P}_2^0(T)$, it holds that (since $\operatorname{rot}\Pi_h\boldsymbol{\tau}|_T = 0$)

$$(\boldsymbol{\tau} - \Pi_h\boldsymbol{\tau}, \mathbf{curl}\,\varphi)_T = (\operatorname{rot}\boldsymbol{\tau}, \varphi)_T - \int_{\partial T}(\boldsymbol{\tau} - \Pi_h\boldsymbol{\tau}) \cdot \boldsymbol{t}_T\varphi\,ds.$$

Moreover, for each edge $e \in \mathcal{E}_T$ with the midpoint $\boldsymbol{m}_e$, we get by Simpson's rule and Lemma 3.2

$$\begin{aligned}\int_e(\boldsymbol{\tau} - \Pi_h\boldsymbol{\tau}) \cdot \boldsymbol{t}_T\varphi\,ds &= \frac{2}{3}(\boldsymbol{\tau} - \Pi_h\boldsymbol{\tau})|_T(\boldsymbol{m}_e) \cdot \boldsymbol{t}_T\varphi(\boldsymbol{m}_e)\\ &= -\frac{2}{3}\cdot\frac{1}{2}[\![\Pi_h\boldsymbol{\tau}\cdot\boldsymbol{t}]\!](\boldsymbol{m}_e)\varphi(\boldsymbol{m}_e)\\ &= -\frac{1}{2}\int_e[\![\Pi_h\boldsymbol{\tau}\cdot\boldsymbol{t}]\!]\varphi\,ds,\end{aligned}$$

which gives

$$(\boldsymbol{\tau} - \Pi_h\boldsymbol{\tau}, \mathbf{curl}\,\varphi)_T = (\mathrm{rot}\,\boldsymbol{\tau}, \varphi)_T + \frac{1}{2}\int_{\partial T} [\![\Pi_h\boldsymbol{\tau}\cdot\boldsymbol{t}]\!]\varphi\,ds.$$

Now observe that $(\boldsymbol{\tau} - \Pi_h\boldsymbol{\tau})|_T$ belongs to $\mathbf{curl}\,\mathbb{P}_2^0(T)$ if $\boldsymbol{\tau} \in (\mathbb{P}_1(T^*))^2$, since $\mathrm{div}(\boldsymbol{\tau} - \Pi_h\boldsymbol{\tau})|_T = 0$ by the commuting property of Π_h and $\int_e(\boldsymbol{\tau} - \Pi_h\boldsymbol{\tau})\cdot\boldsymbol{n}_T\,ds = 0$ for every edge $e \in \mathcal{E}_T$. In view of the definition (3.3), we conclude that $\boldsymbol{\tau} - \Pi_h\boldsymbol{\tau} = \mathbf{curl}\,w_{\boldsymbol{\tau}}$ on T if $\boldsymbol{\tau} \in (\mathbb{P}_1(T^*))^2$. It is also easy to see that the following bound holds for $\boldsymbol{\tau} \in (H^2(\Omega))^2$

$$\|(\boldsymbol{\tau} - \Pi_h\boldsymbol{\tau}) - \mathbf{curl}\,w_{\boldsymbol{\tau}}\|_{0,T} \lesssim \|\boldsymbol{\tau}\|_{1,T} + \|\,\mathbf{curl}\,w_{\boldsymbol{\tau}}\|_{0,T} \lesssim \|\boldsymbol{\tau}\|_{1,T^*} \le \|\boldsymbol{\tau}\|_{2,T^*}.$$

Now the proof is completed by applying the Bramble–Hilbert lemma. □

As a direct consequence of Theorem 3.1, we get the asymptotic exactness of the error estimator η defined by (2.6).

THEOREM 3.4. *Under the assumption of Theorem 3.1, we have*

$$\|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{0,\Omega} = \eta + O(h^{\frac{3}{2}}).$$

Moreover,

$$\left|\frac{\eta}{\|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{0,\Omega}} - 1\right| = O(h^{\frac{1}{2}}),$$

provided that there exists a constant $C > 0$ independent of the mesh size h such that

$$\|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{0,\Omega} \ge Ch.$$

Proof. By Theorem 3.1, we have

$$\big|\|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{0,\Omega} - \eta\big| \le \left(\sum_{T\in\mathcal{T}_h} \|(\boldsymbol{\sigma} - \boldsymbol{\sigma}_h) - \mathbf{curl}\,\psi_T\|_{0,T}^2\right)^{1/2} \lesssim h^{\frac{3}{2}},$$

and thus

$$\left|\frac{\eta}{\|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{0,\Omega}} - 1\right| \lesssim \frac{h^{\frac{3}{2}}}{h} = h^{\frac{1}{2}}.$$

□

REMARK 3.5. *It is straightforward to extend the above results to the mixed boundary condition*

$$u = 0 \quad \text{on } \Gamma_D \qquad \text{and} \qquad \nabla u\cdot\boldsymbol{n} = g_N \quad \text{on } \Gamma_N,$$

where $\partial\Omega = \overline{\Gamma_D \cup \Gamma_N}$, $\Gamma_D \cap \Gamma_N = \emptyset$ and $\boldsymbol{n}$ is the unit outward normal vector to $\partial\Omega$. (In this case the tangential jump $[\![\boldsymbol{\tau}\cdot\boldsymbol{t}]\!]|_e$ is set to be zero for $e \subset \Gamma_N$).

Remark 3.6. *The proof of Theorem 3.1 crucially depends on some super-closeness results which have been derived only for uniform triangulations (assuming that the exact solution is regular enough). However, as suggested by numerical results presented in the next section and some theoretical results derived for primal finite elements [13,17], it is expected that the asymptotic exactness of η would be valid as well for non-uniform meshes satisfying the so-called (α, σ)-condition [17].*

4. Numerical Experiments

To confirm the theoretical result established in the previous section, some numerical experiments are carried out for the Poisson equation

$$\begin{cases} -\Delta u = f & \text{in } \Omega = (0,1)^2 \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

with the function f determined by the exact solution

$$u(x,y) = x(1-x)\sin(\pi y).$$

In the first experiment the computation is performed on a sequence of uniform triangulations $\{\mathcal{T}_h\}$ with the horizontal mesh size $h = 1/2^k$ ($k = 2, 3, 4, \cdots$) as depicted in Fig. 1. For the vector approximation $\boldsymbol{\sigma}_h$ computed by the mixed finite element method (2.3)–(2.4) over each triangulation $\mathcal{T}_h$, we report the values of the actual error $\|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{0,\Omega}$, the error estimator η defined by (2.6) and the effectivity index $\theta = \eta / \|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{0,\Omega}$ in Table 1. It is clearly observed that the effectivity index converges to unity, in agreement with the theoretical result (more quickly than $O(h^{\frac{1}{2}})$ predicted by Theorem 3.4). The actual error $\|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{0,\Omega}$ shown in the second column of Table 1 is calculated by applying a high-order quadrature locally on each element of the given triangulation $\mathcal{T}_h$ and summing the local errors.

Next we consider a sequence of triangulations obtained by perturbing each interior vertex (x, y) of the uniform triangulation $\mathcal{T}_h$ by $(\delta x, \delta y)$, where $\delta x = \delta y = 0.5h^{1.2}\sin(\pi xy)$. The perturbed triangulations corresponding to the uniform ones of Fig. 1 are depicted in Fig. 2 and the numerical results are reported in Table 2 in the same fashion as Table 1. Currently, Theorem 3.4 does not cover this kind of triangulations but it seems that the effectivity index converges to unity as the mesh is refined (although at a much slower rate than in the first case).

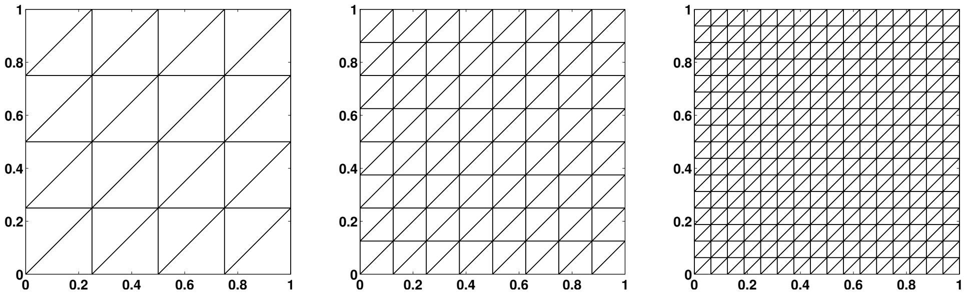

FIGURE 1. Uniform triangulations with $1/h = 4$, 8 and 16

TABLE 1. Actual errors, error estimators and effectivity indices on uniform triangulations

$1/h$	$\|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{0,\Omega}$	η	θ
4	1.329221e–1	1.322683e–1	0.995081
8	6.809937e–2	6.827401e–2	1.002565
16	3.426935e–2	3.430849e–2	1.001142
32	1.716268e–2	1.716862e–2	1.000346
64	8.584860e–3	8.585665e–3	1.000094
128	4.292870e–3	4.292975e–3	1.000024
256	2.146490e–3	2.146504e–3	1.000006
512	1.073252e–3	1.073254e–3	1.000002

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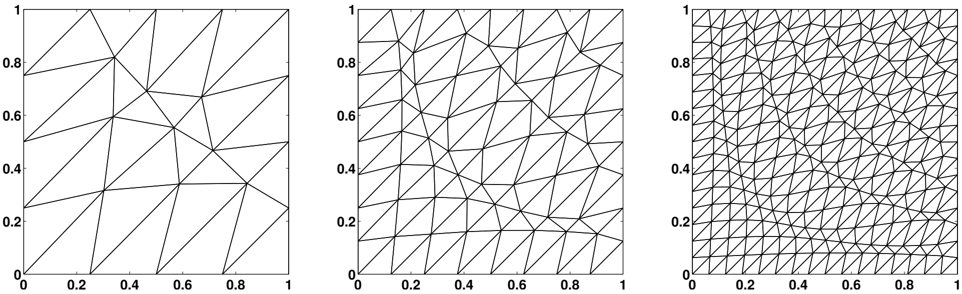

FIGURE 2. $O(h^{1.2})$-perturbed triangulations corresponding to uniform ones of Fig. 1

TABLE 2. Actual errors, error estimators and effectivity indices on $O(h^{1.2})$-perturbed triangulations

$1/h$	$\|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{0,\Omega}$	η	θ
4	1.490591e–1	1.486892e–1	0.997519
8	7.905254e–2	8.685740e–2	1.098730
16	4.016750e–2	4.700896e–2	1.170323
32	1.971184e–2	2.290443e–2	1.161963
64	9.619483e–3	1.095612e–2	1.138951
128	4.699532e–3	5.228679e–3	1.112596
256	2.303969e–3	2.509490e–3	1.089203
512	1.133726e–3	1.212537e–3	1.069516

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