Korean J. Math. **32** (2024), No. 3, pp. 425–438
https://dx.doi.org/10.11568/kjm.2024.32.3.425

CHARACTERIZATIONS OF BIHOM-ALTERNATIVE(-LEIBNIZ) ALGEBRAS THROUGH ASSOCIATED BIHOM-AKIVIS ALGEBRAS

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Abstract. BiHom-Akivis algebras are introduced. It is shown that BiHom-Akivis algebras can be obtained either from Akivis algebras by twisting along two algebra morphisms or from a regular BiHom-algebra via the BiHom-commutator-BiHom-associator algebra. It is also proved that a BiHom-Akivis algebra associated to a regular BiHom-alternative algebra is a BiHom-Malcev algebra. Using the BiHom-Akivis algebra associated to a given regular BiHom-Leibniz algebra, a necessary and sufficient condition for BiHom-Lie admissibility of BiHom-Leibniz algebras is obtained.

1. Introduction

An Akivis algebra $(A, \{.,.\}, \{.,.,.\})$ is a vector space A together with a bilinear skew-symmetric map $(x, y) \longrightarrow \{x, y\}$ and a trilinear map $(x, y, z) \longrightarrow \{x, y, z\}$ satisfying the following so-called Akivis identity for all $x, y, z \in A$:

$$\circlearrowleft_{x,y,z} \{x, \{y, z\}\} = \circlearrowleft_{x,y,z} \{x, y, z\} - \circlearrowleft_{x,y,z} \{y, x, z\}. \tag{1}$$

Initially called "W-algebras" [3] and later Akivis algebras [11], Akivis algebras were introduced ([1], [2], [3]) to study some aspects of web geometry and its connection with loop theory.

The theory of Hom-algebras originated from Hom-Lie algebras introduced by J.T. Hartwig, D. Larsson, and S.D. Silvestrov in [10] in the study of quasi-deformations of Lie algebras of vector fields, including q-deformations of Witt algebras and Virasoro algebras. In order to generalize the construction of associative algebras from Lie algebras, the notion of Hom-associative algebras is introduced in [17], where it is shown that the commutator algebra (with the twisting map) of a Hom-associative algebra is a Hom-Lie algebra. Since then, other Hom-type algebras such as Hom-alternative algebras, Hom-Jordan algebras [16, 19] or Hom-Malcev algebras [19] are introduced and discussed. The extension in the binary-ternary case of the general theory of Hom-algebras was initiated in [12] by defining the class of Hom-Akivis algebras as a Hom-analogue of the class of Akivis algebras ([1, 2, 11]) which are a typical example of binary-ternary algebra. Later Hom-Lie-Yamaguti algebras [8] and Hom-Bol algebras [4] are also defined as other classes of binary-ternary Hom-algebras.

Received February 19, 2024. Revised April 27, 2024. Accepted August 29, 2024.
2010 Mathematics Subject Classification: 17A30, 17D10, 17D99.
Key words and phrases: Akivis algebra, BiHom-associative algebra, BiHom-Leibniz algebra, BiHom-Lie algebra, BiHom-Lie-admissibility, BiHom-Malcev algebra, BiHom-Akivis algebra.

Generalizing the approah in [6] the authors of [9] introduce BiHom-algebras, which are algebras where the identities defining the structure are twisted by two homomorphisms α and β. When the two linear maps of a BiHom-algebra are the same, it reduces to a Hom-algebra. Therefore, the class of BiHom-algebras can be viewed as an extension of the one of Hom-algebras. These algebraic structures include BiHom-associative algebras, BiHom-Lie algebras, BiHom-bialgebras, BiHom-alternative algebras, BiHom-Jordan algebras, Bihom-Malecv algebras, BiHom-Novikov-Poisson algebras [14].

As for BiHom-associative, BiHom-Lie, BiHom-alternative, BiHom-Jordan, BiHom-Malcev algebras, we consider in this paper a twisted version by two commuting linear maps of the Akivis identity which gives the so-called BiHom-Akivis algebras. This work on BiHom-Akivis algebras can be viewed as an extension in binary-ternary case of the theory of BiHom-algebras. It is known [3] that the commutator-associator algebra of a nonassociative algebra is an Akivis algebra. The Hom-version is this result can be found in [12]. This led us to consider "non-BiHom-associative algebras" i.e. BiHom-nonassociative algebras or nonassociative BiHom-algebras and we prove that the BiHom-commutator-BiHom-associator algebra of a regular non-BiHom-associative algebra has a BiHom-Akivis structure. Also the class of BiHom-Akivis algebras contains the one of BiHom-Lie algebras in the same way as the class of Akivis (resp. Hom-Akivis) algebras contains the one of Lie (resp. Hom-Le) algebras.

The rest of the present paper is organized as follows. In Section 2 we recall basic definitions and results about BiHom-algebras. Here, we prove that any two of the three conditions left BiHom-alternative, right Bihom-alternative and BiHom-flexible in a regular BiHom-algebra, imply the third (Proposition 2.8, Proposition 2.9 and Proposition 2.10). In Section 3, BiHom-Akivis algebras are considered. Two methods are used to produce BiHom-Akivis algebras starting with either a regular BiHom-algebra (Theorem 3.3) or classical Akivis algebra along with twisting maps (Corollary 3.8). BiHom-Akivis algebras are shown to be closed under twisting by two self-morphisms (Theorem 3.6). In Section 4, Generalizing the construction of Malcev (resp. Hom-Malcev) algebras from alternative [18] (resp. Hom-alternative [19]) algebras, we point out that BiHom-Akivis algebras associated to a regular BiHom-alternative algebras are BiHom-Malcev algebras (these BiHom-algebras are recently introduced [7]). In section 5, it is observed that the associated BiHom-Akivis algebra to a given regular BiHom-Leibniz algebra,leds to an additional property of BiHom-Leibniz algebras, which in turn gives a necessary and sufficient condition for BiHom-Lie admissibility of regular BiHom-Leibniz algebras.

Throughout this paper, all vector spaces and algebras are meant over a ground field $\mathbb{K}$ of characteristic 0.

2. Preliminaries

In the sequel, a BiHom-algebra refers to a quadruple (A, μ, α, β), where $\mu : A \otimes A \longrightarrow A$, $\alpha : A \longrightarrow A$ and $\beta : A \longrightarrow A$ are linear maps such that $\alpha\beta = \beta\alpha$. The composition of maps is denoted by concatenation for simplicity and the map $\tau : A^{\otimes 2} \longrightarrow A^{\otimes 2}$ denotes the twist isomorphism $\tau(a \otimes b) = b \otimes a$.

Definition 2.1. A BiHom-algebra (A, μ, α, β) is said to be regular if α and β are bijective and multiplicative if $\alpha \circ \mu = \mu \circ \alpha^{\otimes 2}$ and $\beta \circ \mu = \mu \circ \beta^{\otimes 2}$.

Definition 2.2. [9] Let (A,μ,α,β) be a BiHom-algebra.

1. A BiHom-associator of A is the trilinear map $as_{\alpha,\beta} : A^{\otimes 3} \longrightarrow A$ defined by

$$as_{\alpha,\beta} = \mu \circ (\mu \otimes \beta - \alpha \otimes \mu). \tag{2}$$

   In terms of elements, the map $as_{\alpha,\beta}$ is given by

$$as_{\alpha,\beta}(x,y,z) = \mu(\mu(x,y),\beta(z)) - \mu(\alpha(x),\mu(y,z)),\ \forall x,y,z \in A.$$

2. A BiHom-associative algebra [9] is a multiplicative BiHom-algebra (A,μ,α,β) satisfying the following BiHom-associativity condition:

$$as_{\alpha,\beta}(x,y,z) = 0,\ \text{for all } x,y,z \in A. \tag{3}$$

Note that if $\alpha = \beta = Id$, then the BiHom-associator is the usual associator denoted by as. Clearly, a Hom-associative algebra (A,μ,α) can be regarded as a BiHom-associative algebra (A,μ,α,α).

Remark 2.3. A non-BiHom-associative algebra is a BiHom-algebra (A,μ,α,β) for which there exists $x,y,z \in A$ such that $as_{\alpha,\beta}(x,y,z) \neq 0$.

Example 2.4. Let (A,μ) be the two-dimensional algebra with basis (e_1,e_2) and multiplication given by

$$\mu(e_1,e_2) = \mu(e_2,e_2) = e_1$$

and all missing products are 0. Then (A,μ) is nonassociative since, e.g., $\mu(\mu(e_1,e_2),e_2) = e_1 \neq 0 = \mu(e_1,\mu(e_2,e_2))$. Next, if we define for any $\lambda \neq -1$, linear maps $\alpha_\lambda,\beta_\lambda : A \longrightarrow A$ by

$$\alpha_\lambda(e_1) = (\lambda+1)e_1,\ \alpha_\lambda(e_2) = \lambda e_1 + e_2 \ \text{ and}$$
$$\beta_\lambda(e_1) = \frac{1}{\lambda+1}e_1,\ \beta_\lambda(e_2) = \frac{-\lambda}{\lambda+1}e_1 + e_2,$$

then

$$A_{\alpha_\lambda,\beta_\lambda} = (A,\mu_{\alpha_\lambda,\beta_\lambda} = \mu \circ (\alpha_\lambda \otimes \beta_\lambda),\alpha_\lambda,\beta_\lambda)$$

is a non BiHom-associative algebra where the non-zero products are

$$\mu_{\alpha_\lambda,\beta_\lambda}(e_1,e_2) = \mu_{\alpha_\lambda,\beta_\lambda}(e_2,e_2) = (\lambda+1)e_1$$

since e.g. $as_{\alpha_\lambda,\beta_\lambda}(e_1,e_2,e_2) = (\lambda+1)e_1 \neq 0$. Actually, $A_{\alpha_\lambda,\beta_\lambda}$ is a regular BiHom-algebra with $\beta = \alpha^{-1}$.

Let recall the notion of BiHom-alternative and BiHom-flexible algebras.

Definition 2.5. Let (A,μ,α,β) be a multiplicative BiHom-algebra.

1. (A,μ,α,β) is said to be a left BiHom-alternative (resp. right BiHom-alternative) if its satisfies the left BiHom-alternative identity,

$$as_{\alpha,\beta}(\beta(x),\alpha(y),z) + as_{\alpha,\beta}(\beta(y),\alpha(x),z) = 0, \tag{4}$$

   respectively, the right BiHom-alternative identity,

$$as_{\alpha,\beta}(x,\beta(y),\alpha(z)) + as_{\alpha,\beta}(x,\beta(z),\alpha(y)) = 0, \tag{5}$$

   for all $x,y,z \in A$. A BiHom-alternative algebra [7] is the one which is both a left and right BiHom-alternative algebra.

2. (A,μ,α,β) is said to be BiHom-flexible if its satisfies the BiHom-flexible identity,

$$as_{\alpha,\beta}(\beta^2(x),\alpha\beta(y),\alpha^2(z))+as_{\alpha,\beta}(\beta^2(z),\alpha\beta(y),\alpha^2(x))=0 \tag{6}$$

for all $x,y,z\in A$.

Observe that when $\alpha=\beta=Id$, a BiHom-alternative and a Bihom-flexible algebra reduce to an alternative and flexible algebra respectively.

Remark 2.6. 1. Any BiHom-associative algebra is a BiHom-alternative and BiHom-flexible algebra.
2. It is proved that equations (4), (5) and (6) are respectively equivalent to

$as_{\alpha,\beta}(\beta(x),\alpha(x),z)=0,\ as_{\alpha,\beta}(x,\beta(y),\alpha(z))=0$ and $as_{\alpha,\beta}(\beta^2(x),\alpha\beta(y),\alpha^2(x))=0$

for all $x,y\in A$.

Lemma 2.7. [7] *Let (A,μ,α,β) be a regular BiHom-algebra. Then, (A,μ,α,β) is a regular BiHom-alternative algebra if and only if the function $as_{\alpha,\beta}(\beta^2\otimes\alpha\beta\otimes\alpha^2)$ is alternating.*

Proposition 2.8. *Any regular BiHom-alternative algebra is BiHom-flexible.*

Proof. Follows by Lemma 2.7. □

Proposition 2.9. *Any regular left BiHom-alternative BiHom-flexible algebra is BiHom-alternative.*

Proof. Let (A,μ,α,β) be a regular left BiHom-alternative BiHom-flexible algebra. We need just to prove $as_{\alpha,\beta}(x,\beta(y),\alpha(y))=0$ since it is equivalent to (5) (see Remark 2.6). Now, let pick $x,y\in A$ then, we have:

$$\begin{aligned}
as_{\alpha,\beta}(x,\beta(y),\alpha(y)) &= as_{\alpha,\beta}(\beta^2(\beta^{-2}(x)),\alpha\beta(\alpha^{-1}(y)),\alpha^2(\alpha^{-1}(y)))\\
&= -as_{\alpha,\beta}(\beta^2(\alpha^{-1}(y)),\alpha\beta(\alpha^{-1}(y)),\alpha^2(\beta^{-2}(x)))\quad (\text{ by (6) })\\
&= -as_{\alpha,\beta}(\beta(\beta\alpha^{-1}(y)),\alpha(\beta\alpha^{-1}(y)),\alpha^2\beta^{-2}(x))=0\quad (\text{ by (4) }).
\end{aligned}$$

Hence, (A,μ,α,β) is right BiHom-alternative and therefore, it is BiHom-alternative. □

Similarly, we can prove:

Proposition 2.10. *Any regular right BiHom-alternative BiHom-flexible algebra is BiHom-alternative.*

Definition 2.11. Let $(A,[.,.],\alpha,\beta)$ be a multiplicative BiHom-algebra.

1. The BiHom-Jacobiator of A is the trilinear map $J_{\alpha,\beta}:A^{\times 3}\to A$ defined as

$$J_{\alpha,\beta}(x,y,z)=\circlearrowleft_{x,y,z}[\beta^2(x),[\beta(y),\alpha(z)]] \tag{7}$$

where $\circlearrowleft_{x,y,z}$ denotes the sum over the cylic permutation of x,y,z.

2. $(A,[.,.],\alpha,\beta)$ is said to be a BiHom-Lie algebra if
   (a) $[\beta(x),\alpha(y)]=-[\beta(y),\alpha(x)]$ (BiHom-skew-symmetry),
   (b) A satisfies the BiHom-Jacobi identity i.e.

$$J_{\alpha,\beta}(x,y,z)=0 \tag{8}$$

for all $x,y,z\in A$.

3. $(A,[.,.],\alpha,\beta)$ is said to be a BiHom-Malcev algebra if
   (a) $[\beta(x),\alpha(y)]=-[\beta(y),\alpha(x)]$ (BiHom-skew-symmetry),

(b) A satisfies a BiHom-Malcev identity i.e.

$$J_{\alpha,\beta}(\alpha\beta(x),\alpha\beta(y),[\beta(x),\alpha(z)]) = [J_{\alpha,\beta}(\beta(x),\beta(y),\beta(z)),\alpha^2\beta^2(x)] \tag{9}$$

for all $x, y, z \in A$.

Remark 2.12. 1. If $\alpha = \beta = Id$, a BiHom-Lie (resp. BiHom-Malcev) algebra reduces to a Lie (resp. Malcev) algebra.
2. Any BiHom-Lie algebra is a BiHom-Malcev algebra.

3. BiHom-Akivis algebras: Constructions and examples

In this section we twist the defining identities of Akivis algebras by two self-morphisms to obtain the so-called BiHom-Akivis algebras. These algebraic objects generalise (Hom-) Akivis algebras. We also provide an example of a BiHom-Akivis algebra and some construction methods of these BiHom-algebras (the construction from non-BiHom-associative algebras and the one from Akivis algebras).

Definition 3.1. A BiHom-Akivis algebra is a quintuple $(V, [.,.], [.,.,.], \alpha, \beta)$, where V is a vector space, $[.,.] : V \times V \to V$ a BiHom-skew-symmetric bilinear map, $[.,.,.] : V \times V \times V \to V$ a trilinear map and $\alpha, \beta : V \to V$ linear maps such that

$$J_{\alpha,\beta}(x,y,z) = \circlearrowleft_{x,y,z} [x,y,z] - \circlearrowleft_{x,y,z} [y,x,z] \tag{10}$$

for all x, y, z in V where $J_{\alpha,\beta}(x,y,z) = \circlearrowleft_{x,y,z} [\beta^2(x),[\beta(y),\alpha(z)]]$ is the BiHom-Jacobiator of $(V, [.,.], \alpha, \beta)$.

A BiHom-Akivis algebra $(V, [.,.], [.,.,.], \alpha, \beta)$ is said multiplicative if α and β preserve $[.,.]$ and $[.,.,.]$.

Similarly as Akivis and Hom-Akivis cases, let call (10) the *BiHom-Akivis identity.*

Remark 3.2. 1. If $\alpha = \beta = Id_V$, the BiHom-Akivis identity (10) is the usual Akivis identity (1).
2. The BiHom-Akivis identity (10) reduces to the BiHom-Jacobi identity (8), when $[x,y,z] = 0$, for all x, y, z in V.

The following relevant result shows how one can obtain BiHom-Akivis algebras from regular BiHom-algebras and hence from regular-BiHom-associative algebras.

Theorem 3.3. *Let (A,μ,α,β) be a multiplicative regular BiHom-algebra. Then the BiHom-commutator-BiHom-associator algebra*

$$(A, [.,.] := \mu - \mu \circ (\alpha^{-1}\beta \otimes \alpha\beta^{-1}), [.,.,] := as_{\alpha,\beta} \circ (\alpha^{-1}\beta^2 \otimes \beta \otimes \alpha))$$

of (A,μ,α,β) is a multiplicative BiHom-Akivis algebra.

Proof. Let (A,μ,α) be a multiplicative regular BiHom-algebra. For any x, y, z in A, we have $[x,y] := \mu(x,y) - \mu(\alpha^{-1}\beta(y), \alpha\beta^{-1}(x))$ and $[x,y,z] := as_{\alpha,\beta}(\alpha^{-1}\beta^2(x), \beta(y), \alpha(z))$. Then, by [7] (Lemma 2.1), we have

$$J_{\alpha,\beta}(x,y,z) = \circlearrowleft_{x,y,z} as_{\alpha,\beta}(\alpha^{-1}\beta^2(x),\beta(y),\alpha(z)) - \circlearrowleft_{x,y,z} as_{\alpha,\beta}(\alpha^{-1}\beta^2(y),\beta(x),\alpha(z))$$

i.e.

$$J_{\alpha,\beta}(x,y,z) = \circlearrowleft_{x,y,z} [x,y,z] - \circlearrowleft_{x,y,z} [y,x,z].$$

Hence, $(A, [.,.], [.,.,.], \alpha, \beta)$ is a multiplicative BiHom-Akivis algebra. □

The BiHom-Akivis algebra provided by Theorem3.3 is said *associated* (with a given regular BiHom-algebra).

Example 3.4. Consider regular non-BiHom-associative algebras $A_{\alpha_\lambda,\beta_\lambda}$ of Example 2.4. Define on A the following products:

$$[x,y]:=\mu_{\alpha_\lambda,\beta_\lambda}(x,y)-\mu_{\alpha_\lambda,\beta_\lambda}(\alpha_\lambda^{-1}\beta_\lambda(y),\alpha_\lambda\beta_\lambda^{-1}(x))$$

and

$$[x,y,z]:=as_{\alpha_\lambda,\beta_\lambda}(\alpha_\lambda^{-1}\beta_\lambda^2(x),\beta_\lambda(y),\alpha_\lambda(z)).$$

Then, By Theorem 3.3, $(A,[.,.],[.,.,.],\alpha_\lambda,\beta_\lambda)$ are multiplicative BiHom-Akivis algebras where the non-zero products are

$$[e_1,e_2]=(\lambda+1)e_1,\ [e_2,e_1]=-\frac{1}{\lambda+1}e_1,\ [e_2,e_2]=\frac{\lambda^2+2\lambda}{\lambda+1}e_1$$

and

$$[e_1,e_2,e_2]=[e_2,e_2,e_2]=\frac{1}{\lambda+1}e_1.$$

Definition 3.5. Let $(A,[.,.],[.,.,.],\alpha,\beta)$ and $(\tilde{A},\{.,.\},\{.,.,.\},\tilde{\alpha},\tilde{\beta})$ be BiHom-Akivis algebras. A morphism $\phi:A\to\tilde{A}$ of BiHom-Akivis algebras is a linear map of $\mathbb{K}$-modules A and $\tilde{A}$ such that $f\circ\alpha=\tilde{\alpha}\circ f, f\circ\beta=\tilde{\beta}\circ f$ and

$$f\circ[.,.]=\{.,.\}\circ f^{\otimes 2},\ f\circ[.,.,.]=\{.,.,.\}\circ f^{\otimes 3}.$$

For example, if we take $(A,[.,.],[.,.,.],\alpha,\beta)$ as a multiplicative BiHom-Akivis algebra, then the twisting self-maps α and β are themselfs self-morphisms of $(A,[.,.],[.,.,.],\alpha,\beta)$.

The following result holds.

Theorem 3.6. *Let $(A,[.,.],[.,.,.],\alpha,\beta)$ be a BiHom-Akivis algebra and $\varphi,\psi:A\to A$ self-morphisms of $(A,[.,.],[.,.,.],\alpha,\beta)$ such that $\varphi\psi=\psi\varphi$. Define on A a bilinear operation $[.,.]_{\varphi,\psi}$ and a trilinear operation $[.,.,.]_{\varphi,\psi}$ by*
$[x,y]_{\varphi,\psi}:=[\varphi(x),\psi(y)]$,
$[x,y,z]_{\varphi,\psi}:=\varphi\psi^2([x,y,z])$, for all $x,y,z\in A$.
Then $A_{\varphi,\psi}:=(A,[.,.]_{\varphi,\psi},[.,.,.]_{\varphi,\psi},\varphi\alpha,\psi\beta)$ is a BiHom-Akivis algebra.

Moreover, if $(A,[.,.],[.,.,.],\alpha,\beta)$ is multiplicative, then $A_{\varphi,\psi}$ is also multiplicative.

Proof. Clearly $[.,.]_{\varphi,\psi}$ (resp. $[.,.,.]_{\varphi,\psi}$) is a bilinear (resp. trilinear) map and the BiHom-skew-symmetry of $[.,.]$ in $(A,[.,.],[.,.,.],\alpha,\beta)$ implies the BiHom-skew-symmetry of $[.,.]_{\varphi,\psi}$ in $A_{\varphi,\psi}$.

Next, we have (by the BiHom-Akivis identity (10)),

$$\begin{aligned}&\circlearrowleft_{x,y,z}[(\psi\beta)^2(x),[\psi\beta(y),\varphi\alpha(z)]_{\varphi,\psi}]_{\varphi,\psi}=\circlearrowleft_{x,y,z}[\varphi\psi^2\beta^2(x),\psi([\varphi\psi\beta(y),\psi\varphi\alpha(z)])]\\&=\circlearrowleft_{x,y,z}(\varphi\psi^2([\beta^2(x),[\beta(y),\alpha(z)]]))=\circlearrowleft_{x,y,z}(\varphi\psi^2([x,y,z])-\varphi\psi^2([y,x,z]))\\&=\circlearrowleft_{x,y,z}([x,y,z]_{\varphi,\psi}-[y,x,z]_{\varphi,\psi}).\end{aligned}$$

The second assertion is proved as follows:

$$\begin{aligned}&\varphi\alpha([x,y]_{\varphi,\psi})=\varphi\alpha([\varphi(x),\psi(y)])=[\varphi\alpha\varphi(x),\varphi\alpha\psi(y)]\\&=[\varphi(\varphi\alpha(x)),\psi(\varphi\alpha(y))]=[\varphi\alpha(x),\varphi\alpha(y)]_{\varphi,\psi},\\&\psi\beta([x,y]_{\varphi,\psi})=\psi\beta([\varphi(x),\psi(y)])=[\psi\beta\varphi(x),\psi\beta\psi(y)]\\&=[\varphi(\psi\beta(x)),\psi(\psi\beta(y))]=[\psi\beta(x),\psi\beta(y)]_{\varphi,\psi},\end{aligned}$$

and

$$\varphi\alpha([x,y,z]_{\varphi,\psi}) = \varphi\alpha\varphi\psi^2([x,y,z]) = \varphi\psi^2([\varphi\alpha(x),\varphi\alpha(y),\varphi\alpha(z)]) = [\varphi\alpha(x),\varphi\alpha(y),\varphi\alpha(z)]_{\varphi,\psi}$$
$$\psi\beta([x,y,z]_{\varphi,\psi}) = \psi\beta\varphi\psi^2([x,y,z]) = \varphi\psi^2([\psi\beta(x),\psi\beta(y),\psi\beta(z)]) = [\psi\beta(x),\psi\beta(y),\psi\beta(z)]_{\varphi,\psi}.$$

This completes the proof. □

Corollary 3.7. *If $(A,[.,.],[.,.,.],\alpha,\beta)$ is a multiplicative BiHom-Akivis algebra, then so is A_{α^n,β^m} for all $n,m\in\mathbb{N}$.*

Proof. This follows from Theorem 3.6 if take $\varphi=\alpha^n$ and $\psi=\beta^m$. □

The following result gives a method for constructing BiHom-Akivis algebras from Akivis algebra and their self-morphisms. It is an extension in binary-ternary algebras case of the well-known construction of a class of BiHom-algebras from the corresponding class of untwisted algebras.

Corollary 3.8. *Let $(A,[.,.],[.,.,.])$ be an Akivis algebra and α,β self-morphisms of $(A,[.,.],[.,.,.])$. Define on A a bilinear operation $[.,.]_{\alpha,\beta}$ and a trilinear operation $[.,.,.]_{\alpha,\beta}$ by*
$[x,y]_{\alpha,\beta} := [\alpha(x),\beta(y)]$,
$[x,y,z]_{\alpha,\beta} := \alpha\beta^2([x,y,z]))$,
for all $x,y,z\in A$. Then $A_{\alpha,\beta} = (A,[.,.]_{,\alpha,\beta},[,,]_{\alpha,\beta},\alpha,\beta)$ is a multiplicative BiHom-Akivis algebra.
Moreover, suppose that $(B,\{.,.\},\{.,.,.\})$ is another Akivis algebra and that φ,ψ are endomorphisms of B. If $f: A\to B$ is an Akivis algebra morphism satisfying $f\circ\alpha = \varphi\circ f$ and $f\circ\beta=\psi\circ f$, then $f:(A,[.,.]_{\alpha,\beta},[,,]_{\alpha,\beta},\alpha,\beta)\to(B,\{.,.\}_{\varphi,\psi},\{.,.,.\}_{\varphi,\psi},\varphi,\psi)$ is a morphism of multiplicative BiHom-Akivis algebras.

Proof. The first of this theorem is a special case of Theorem3.6 above when $\alpha=\beta=id$. The second part is proved Similarly as in Theorem 3.6. Clearly, we prove as follows:

$$f([x,y]_{\alpha,\beta}) = f([\alpha(x),\beta(y)]) = \{f\alpha(x),f\beta(y)\} = \{\varphi f(x),\psi f(y)\} = \{f(x),f(y)\}_{\varphi,\psi}$$
$$f([x,y,z]_{\alpha,\beta}) = f\alpha\beta^2([x,y,z]) = \varphi\psi^2(\{f(x),f(y),f(z)\}) = \{f(x),f(y),f(z)\}_{\varphi,\psi}.$$

This ends the proof. □

Example 3.9. Consider the two-dimensional Akivis algebra $(A,[.,.],[.,.,.])$ with basis (e_1,e_2) given by

$$[e_1,e_2] = [e_1,e_2,e_2] = [e_2,e_2,e_2] = e_1$$

and all missing products are 0 (see Example 4.7 in [12]). For any $r,s\in\mathbb{R}$, the maps α_r and β_s defined by $\alpha_r(e_1)=(r+1)e_1$, $\alpha_r(e_2)=re_1+e_2$ and $\beta_s(e_1)=(s+1)e_1$, $\beta_s(e_2)=se_1+e_2$ are commuting morphisms of A. Note that $\alpha_r\neq\beta_s$ if and only if $r\neq s$. Next, if we define the operations $[.,.]_{\alpha_r,\beta_s}$ and $[.,.,.]_{\alpha_r,\beta_s}$ with non-zero products by

$$[e_1,e_2]_{\alpha_r,\beta_s} = (r+1)e_1 \text{ and } [e_1,e_2,e_2]_{\alpha_r,\beta_s} = [e_2,e_2,e_2]_{\alpha_r,\beta_s} = (r+1)(s+1)^2e_1,$$

we get, by Corollary 3.8, that $A_{\alpha_r,\beta_s} = (A,[.,.]_{\alpha_r,\beta_s},[.,.,.]_{\alpha_r,\beta_s}\alpha_r,\beta_s)$ are BiHom-Akivis algebras.

4. BiHom-Malcev structure on BiHom-Akivis algebras

In this section we define and study BiHom-alternative and BiHom-flexible BiHom-Akivis algebras. Next, we provide a characterization of BiHom-alternative algebras through associated BiHom-Akivis algebras. The relevant result here is to prove another version of Theorem 2.2 in [7] i.e. the BiHom-Akivis algebra associated with a regular BiHom-alternative algebra is BiHom-Malcev algebra.

Definition 4.1. A BiHom-Akivis algebra $\mathcal{A} := (A, [.,.], [.,.,.], \alpha, \beta)$ is said:
(i) *BiHom-flexible*, if

$$[\alpha(x), \alpha(y), \alpha(z)] + [\alpha(z), \alpha(y), \alpha(x)] = 0 \text{ for all } x, y \in A. \tag{11}$$

(ii) *BiHom-alternative*, if

$$[\alpha^2(x), \alpha^2(y), \beta(z)] + [\alpha^2(y), \alpha^2(x), \beta(z)] = 0 \text{ for all } x, y \in A, \tag{12}$$

$$[\alpha(x), \beta^2(y), \beta^2(z)] + [\alpha(x), \beta^2(z), \beta^2(y)] = 0 \text{ for all } x, y \in A. \tag{13}$$

Remark 4.2. 1. The BiHom-flexible law (11) in $\mathcal{A}$ is equivalent to $[\alpha(x), \alpha(y), \alpha(x)] = 0$ for all $x, y \in A$.
2. The identities (12) and (13) are respectively called the left BiHom-alternativity and the right BiHom-alternativity. They are respectively equivalent to $[\alpha^2(x), \alpha^2(x), \beta(y)] = 0$ *and* $[\alpha(x), \beta^2(y), \beta^2(y)] = 0$ *for all* $x, y \in \mathcal{A}$.

The next result follows from Theorem 3.3 and Definition 4.1.

Proposition 4.3. *Let $\mathcal{A} = (A, \mu, \alpha, \beta)$ be a multiplicative regular BiHom-algebra and*
$\mathcal{A}_{\mathcal{K}} = (A, [.,.] = \mu - \mu \circ (\alpha^{-1}\beta \otimes \alpha\beta^{-1}) \circ \tau, [.,.,.] = as_{\alpha,\beta} \circ (\alpha^{-1}\beta^2 \otimes \beta \otimes \alpha), \alpha, \beta)$ its associated BiHom-Akivis algebra.

1. *If (A, μ, α, β) is BiHom-flexible, then $\mathcal{A}_{\mathcal{K}}$ is BiHom-flexible.*
2. *If (A, μ, α, β) is BiHom-alternative, then so is $\mathcal{A}_{\mathcal{K}}$.*

Now, we obtain the following characterization of BiHom-Lie algebras in terms of BiHom-Akivis algebras.

Proposition 4.4. *Let $\mathcal{A} := (A, [.,.], [.,.,.], \alpha, \beta)$ be a BiHom-flexible BiHom-Akivis algebra such that α is surjective. Then $\mathcal{A}_L = (A, [.,.], \alpha, \beta)$ is a BiHom-Lie algebra if, and only if $\circlearrowleft_{x,y,z} [x, y, z] = 0$, for all $x, y, z \in A$.*

Proof. Pick x, y, z in A. Then there exists $a, b, c \in A$ such that $x = \alpha(a), y = \alpha(b), z = \alpha(c)$ since α is surjective. Therefore, the BiHom-Akivis identity (10) and the BiHom-flexibility (11) in $\mathcal{A}$ imply

$$\circlearrowleft_{x,y,z} [\beta^2(x), [\beta(y), \alpha(z)]] = \circlearrowleft_{x,y,z} [x, y, z] - \circlearrowleft_{x,y,z} [y, x, z] = \circlearrowleft_{a,b,c} [\alpha(a), \alpha(b), \alpha(c)] - \circlearrowleft_{a,b,c} [\alpha(b), \alpha(a), \alpha(c)] = 2 \circlearrowleft_{a,b,c} [\alpha(a), \alpha(b), \alpha(c)] = 2 \circlearrowleft_{x,y,z} [x, y, z].$$

Hence, $\circlearrowleft_{x,y,z} [\beta^2(x), [\beta(y), \alpha(z)]] = 0$ if and only if $\circlearrowleft_{x,y,z} [x, y, z] = 0$ since the characteristic of the ground field $\mathbb{K}$ is 0. □

The following result generalizes Proposition 2.1 in [7], which in turn is a generalization of a similar well-known result in alternative rings. The reader can also see Proposition 3.17 in [19] for the Hom-version of this result.

Proposition 4.5. *Let $\mathcal{A} := (A, [.,.], [.,.,.], \alpha, \beta)$ be a BiHom-alternative BiHom-Akivis algebra such that α and β are surjective. Then*

$$\circlearrowleft_{x,y,z} [\beta^2(x), [\beta(y), \alpha(z)]] = 6[x, y, z] \tag{14}$$

for all $x, y, z \in A$.

Proof. Pick $x, y, z \in A$. Then, there exists $a, b, c \in A$ such that $x = \alpha^2(a), y = \alpha^2(b), z = \alpha(c)$. Hence, the application to (10) of the BiHom-alternativity in $\mathcal{A}$ gives :

$$\circlearrowleft_{x,y,z} [\beta^2(x), [\beta(y), \alpha(z)]] = \circlearrowleft_{x,y,z} [x, y, z] - \sigma[y, x, z] = \circlearrowleft_{x,y,z} [x, y, z] - \circlearrowleft_{x,y,z} [\alpha^2(b), \alpha^2(a), \beta(c)]$$
$$\circlearrowleft_{x,y,z} [x, y, z] + \circlearrowleft_{a,b,c} [\alpha^2(a), \alpha^2(b), \beta(c)] = 2 \circlearrowleft_{x,y,z} [x, y, z]. \tag{15}$$

Next, again by the BiHom-alternativity in $\mathcal{A}$ and surjectivity of α and β, we prove that $\sigma[x, y, z] = 3[x, y, z]$. Therefore

$$\circlearrowleft_{x,y,z} [\beta^2(x), [\beta(y), \alpha(z)]] = 6[x, y, z].$$

□

First, let recall the following.

Definition 4.6. [7] Let (A, μ, α, β) be a regular BiHom-algebra. Define the BiHom-Bruck-Kleinfeld function $f : A^{\otimes 4} \longrightarrow A$ as the multilinear map

$$f(w, x, y, z) = as_{\alpha,\beta}(\beta^2(w)\alpha\beta(x), \alpha^2\beta(y), \alpha^3(z)) - as_{\alpha,\beta}(\beta^2(x), \alpha\beta(y), \alpha^2(z))\alpha^3\beta(w)$$
$$-\alpha^2\beta^2(x)as_{\alpha,\beta}(\alpha\beta(w), \alpha^2(y), \alpha^3\beta^{-1}(z)). \tag{16}$$

The following result is very useful.

Lemma 4.7. *[7] Let (A, μ, α, β) be a regular BiHom-alternative algebra. Then the BiHom-Bruck-Kleinfeld function f is alternating.*

Proposition 4.8. *Let (A, μ, α, β) be a regular BiHom-alternative algebra. Then*

$$as_{\alpha,\beta}(\beta^3(x), \alpha\beta^2(y), \alpha\beta(x)\alpha^2(z)) = as_{\alpha,\beta}(\alpha^{-1}\beta^3(x), \beta^2(y), \alpha\beta(z))\alpha^2\beta^2(x) \tag{17}$$
$$as_{\alpha,\beta}(\beta^3(x), \alpha\beta^2(y), \alpha\beta(z)\alpha^2(x)) = \alpha\beta^3(x)as_{\alpha,\beta}(\beta^2(x), \alpha\beta(y), \alpha^2(z)) \tag{18}$$

for all $x, y, z \in A$.

Proof. For (17), we compute as follows,

$$\begin{aligned}
&as_{\alpha,\beta}(\beta^3(x), \alpha\beta^2(y), \alpha\beta(x)\alpha^2(z)) = as_{\alpha,\beta}(\beta^2(\beta(x)), \alpha\beta(\beta(y)), \alpha^2(\alpha^{-1}\beta(x)z))\\
&= as_{\alpha,\beta}(\beta^2(\alpha^{-1}\beta(x)z), \alpha\beta(\beta(x)), \alpha^2(\beta(y))) \ \text{(by alternativity of } as_{\alpha,\beta}(\beta^2 \otimes \alpha\beta \otimes \alpha^2) \ \text{)}\\
&= as_{\alpha,\beta}(\beta^2(\alpha^{-1}\beta(x))\alpha\beta(\alpha^{-1}\beta(z)), \alpha^2\beta(\alpha^{-1}\beta(x)), \alpha^3(\alpha^{-1}\beta(y)))\\
&= f(\alpha^{-1}\beta(x), \alpha^{-1}\beta(z), \alpha^{-1}\beta(x), \alpha^{-1}\beta(y))\\
&+as_{\alpha,\beta}(\beta^2(\alpha^{-1}\beta(z)), \alpha\beta(\alpha^{-1}\beta(x)), \alpha^2(\alpha^{-1}\beta(y)))\alpha^3\beta(\alpha^{-1}\beta(x))\\
&+\alpha^2\beta^2(\alpha^{-1}\beta(z))as_{\alpha,\beta}(\alpha\beta(\alpha^{-1}\beta(x)), \alpha^2(\alpha^{-1}\beta(x)), \alpha^3\beta^{-1}(\alpha^{-1}\beta(y)))\\
&= as_{\alpha,\beta}(\alpha^{-1}\beta^3(x), \beta^2(y), \alpha\beta(z))\alpha^2\beta^2(x) \ \text{(by alternativity of } f \text{ and } as_{\alpha,\beta}(\beta^2 \otimes \alpha\beta \otimes \alpha^2) \ \text{)}.
\end{aligned}$$

This finishes the proof of (17).

For (18), we compute as follows

$$\begin{aligned}
&as_{\alpha,\beta}(\beta^3(x), \alpha\beta^2(y), \alpha\beta(z)\alpha^2(x)) = as_{\alpha,\beta}(\beta^2(\beta(x)), \alpha\beta(\beta(y)), \alpha^2(\alpha^{-1}\beta(z)x))\\
&= as_{\alpha,\beta}(\beta^2(\alpha^{-1}\beta(z)x), \alpha\beta(\beta(x)), \alpha^2(\beta(y)))\ \ (\text{ by alternativity of } as_{\alpha,\beta}(\beta^2\otimes\alpha\beta\otimes\alpha^2)\ \ )\\
&= as_{\alpha,\beta}(\beta^2(\alpha^{-1}\beta(z))\alpha\beta(\alpha^{-1}\beta(x)), \alpha^2\beta(\alpha^{-1}\beta(x)), \alpha^3(\alpha^{-1}\beta(y)))\\
&= f(\alpha^{-1}\beta(z), \alpha^{-1}\beta(x), \alpha^{-1}\beta(x), \alpha^{-1}\beta(y))\\
&+as_{\alpha,\beta}(\beta^2(\alpha^{-1}\beta(x)), \alpha\beta(\alpha^{-1}\beta(x)), \alpha^2(\alpha^{-1}\beta(y)))\alpha^3\beta(\alpha^{-1}\beta(z))\\
&+\alpha^2\beta^2(\alpha^{-1}\beta(x))as_{\alpha,\beta}(\alpha\beta(\alpha^{-1}\beta(z)), \alpha^2(\alpha^{-1}\beta(x)), \alpha^3\beta^{-1}(\alpha^{-1}\beta(y)))\\
&= \alpha\beta^3(x)as_{\alpha,\beta}(\beta^2(x), \alpha\beta(y), \alpha^2(z))\ \ (\text{ by alternativity of } f \text{ and } as_{\alpha,\beta}(\beta^2\otimes\alpha\beta\otimes\alpha^2)\ \ ).
\end{aligned}$$

This finishes the proof of (18). □

Corollary 4.9. *Let (A, μ, α, β) be a regular BiHom-alternative algebra. Then*

$$as_{\alpha,\beta}(\beta^3(x), \alpha\beta^2(y), [\alpha\beta(x), \alpha^2(z)]) = [as_{\alpha,\beta}(\alpha^{-1}\beta^3(x), \beta^2(y), \alpha\beta(z)), \alpha^2\beta^2(x)] \tag{19}$$

for all $x, y, z \in A$ where $[.,.] = \mu - \mu \circ (\alpha^{-1}\beta \otimes \alpha\beta^{-1}) \circ \tau$ is the BiHom-commutator bracket.

Proof. Indeed, we have

$$\begin{aligned}
&as_{\alpha,\beta}(\beta^3(x), \alpha\beta^2(y), [\alpha\beta(x), \alpha^2(z)])\\
&= as_{\alpha,\beta}(\beta^3(x), \alpha\beta^2(y), \alpha\beta(x)\alpha^2(z)) - as_{\alpha,\beta}(\beta^3(x), \alpha\beta^2(y), \alpha\beta(z)\alpha^2(x))\\
&= as_{\alpha,\beta}(\alpha^{-1}\beta^3(x), \beta^2(y), \alpha\beta(z))\alpha^2\beta^2(x) - \alpha\beta^3(x)as_{\alpha,\beta}(\beta^2(x), \alpha\beta(y), \alpha^2(z))\\
&(\text{ by (17) and (18) })\\
&= [as_{\alpha,\beta}(\alpha^{-1}\beta^3(x), \beta^2(y), \alpha\beta(z)), \alpha^2\beta^2(x)]
\end{aligned}$$

as desired. □

We now come to the main result of this section, which is Theorem 2.2 in [7] but from a point of view of BiHom-Akivis algebras.

Theorem 4.10. *Let (A, μ, α, β) be a multiplicative regular BiHom-alternative BiHom-algebra and $\mathcal{A}_{\mathcal{K}} = (A, [.,.] = \mu - \mu \circ (\alpha^{-1}\beta \otimes \alpha\beta^{-1}) \circ \tau, [.,.,.] = as_{\alpha,\beta} \circ (\alpha^{-1}\beta^2 \otimes \beta \otimes \alpha), \alpha, \beta)$ its associated BiHom-Akivis algebra. Then $(A, [.,.], \alpha, \beta)$ is a BiHom-Malcev algebra.*

Proof. From Proposition 4.3 we get that $\mathcal{A}_{\mathcal{K}}$ is BiHom-alternative so that (14) implies

$$\begin{aligned}
&J_{\alpha,\beta}(\alpha\beta(x), \alpha\beta(y), [\beta(x), \alpha(y)]) = 6[\alpha\beta(x), \alpha\beta(y), [\beta(x), \alpha(z)]]\\
&= 6as_{\alpha,\beta}(\beta^3(x), \alpha\beta^2(y), [\alpha\beta(x), \alpha^2(z)])\\
&= [6as_{\alpha,\beta}(\alpha^{-1}\beta^3(x), \beta^2(y), \alpha\beta(z)), \alpha^2\beta^2(x)]\ (\text{ by (19) })\\
&= [[\beta(x), \beta(y), \beta(z)], \alpha^2\beta^2(x)] = [J_{\alpha,\beta}(\beta(x), \beta(y), \beta(z)), \alpha^2\beta^2(x)]
\end{aligned}$$

and so, we obtain the BiHom-Malcev identity (9). Hence, $(A, [.,.], \alpha, \beta)$ is a BiHom-Malcev algebra. □

5. Some characterizations of BiHom-Leibniz algebras

This section is devoted to BiHom-version of some known properties of left Leibniz algebras. Using the associated BiHom-Akivis algebra to a given regular BiHom-Leibniz algebra, we give a characteristic property of BiHom-Leibniz algebras (Proposition 5.5). Basing on this property a necessary and sufficient condition for the BiHom-Lie admissibility of these BiHom-algebras is obtained (Corollary 5.7). In the sequel, for a given bilinear map μ on A, the terms $\mu(x,y)$, $\mu(\mu(x,y),z)$ and $\mu(x,\mu(y,z))$ will be often denoted by xy, $xy\cdot z$ and $x\cdot yz$ respectively for all $x,y,z\in A$ to reduce the number of braces. First, let recall the following:

Definition 5.1. A (left) BiHom-Leibniz algebra is a multiplicative BiHom-algebra (A,μ,α,β) such that the following identity

$$\mu(\alpha\beta(x),\mu(y,z)) = \mu(\mu(\beta(x),y),\beta(z)) + \mu(\beta(y),\mu(\alpha(x),z)) \tag{20}$$

holds for all $x,y,z\in A$. In terms of BiHom-associators, the identity

(20) called the BiHom-Leibniz identity, is written as

$$as_{\alpha,\beta}(\beta(x),y,z) = -\beta(x)\cdot(\alpha(x)\cdot z). \tag{21}$$

Therefore, from the definition above and Remark 2.3, we observe that BiHom-Leibniz algebras are examples of non-BiHom-associative algebras. Hence, thanks to Theorem 3.3, we obtain.

Corollary 5.2. *Let $\mathcal{A}=(A,\mu,\alpha,\beta)$ be a multiplicative regular BiHom-Leibniz algebra. Then $\mathcal{A}_{\mathcal{K}}=(A,[.,.]=\mu-\mu\circ(\alpha^{-1}\beta\otimes\alpha\beta^{-1})\circ\tau,[.,.,.]=as_{\alpha,\beta}\circ(\alpha^{-1}\beta^2\otimes\beta\otimes\alpha),\alpha,\beta)$ is a BiHom-Akivis algebra.*

We have the following result:

Proposition 5.3. *Let (A,μ,α,β) be a multiplicative BiHom-Leibniz algebra. Then*

$$(\beta(x)\alpha(y)+\beta(y)\alpha(x))\cdot\beta(z) = 0 \tag{22}$$

$$\alpha\beta(x)[\beta(y),\alpha^2(z)] = [\beta(x)\beta(y),\alpha^2\beta(z)] + [\beta^2(y),\alpha(x)\alpha^2(z)] \tag{23}$$

for all $x,y,z\in A$.

Proof. The BiHom-Leibniz identity (20) implies $(\beta(x)\alpha(y))\cdot\beta(z)=\alpha\beta(x)\cdot\alpha(y)z-\alpha\beta(y)\cdot\alpha(x)z$. Similarly, interchanging x and y, we get $(\beta(y)\alpha(x))\cdot\beta(z)=\alpha\beta(y)\cdot\alpha(x)z-\alpha\beta(x)\cdot\alpha(y)z$. Then adding memberwise these equalities, we obtain (22).

Next, we get

$$\begin{aligned}
&[\beta(x)\beta(y),\alpha^2\beta(z)]+[\beta^2(y),\alpha(x)\alpha^2(z)]\\
&=\beta(x)\beta(y)\cdot\alpha^2\beta(z)-\beta\alpha^{-1}(\alpha^2\beta(z))\cdot\alpha\beta^{-1}(\beta(x)\beta(y))+\beta^2(y)\cdot\alpha(x)\alpha^2(z)\\
&-\beta\alpha^{-1}(\alpha(x)\alpha^2(z))\cdot\alpha\beta^{-1}(\beta^2(y))\\
&=\beta(x)\beta(y)\cdot\alpha^2\beta(z)-\alpha\beta^2(z)\cdot\alpha(x)\alpha(y)+\beta^2(y)\cdot\alpha(x)\alpha^2(z)\\
&-\beta(x)\alpha\beta(z))\cdot\alpha\beta(y)\\
&=\alpha\beta(x)\cdot\beta(y)\alpha^2(z)-\alpha\beta(\beta(z))\cdot\alpha(x)\alpha(y)-\beta(x)\alpha\beta(z)\cdot\alpha\beta(y)\text{ (by (20))}\\
&=\alpha\beta(x)\cdot\beta(y)\alpha^2(z)-\beta^2(z)\alpha(x)\cdot\beta\alpha(y)-\beta\alpha(x)\cdot\alpha\beta(z)\alpha(y)\\
&-\beta(x)\alpha\beta(z)\cdot\alpha\beta(y)\text{ (again by (20))}\\
&=\alpha\beta(x)[\beta(y),\alpha^2(z)]-(\beta(\beta(z))\alpha(x)+\beta(x)\alpha(\beta(z)))\cdot\beta(\alpha(y))\\
&=\alpha\beta(x)[\beta(y),\alpha^2(z)]\text{ (by (22))}
\end{aligned}$$

and then, we obtain (23). □

Remark 5.4. If $\alpha=\beta=Id$ in Proposition , then one gets the well-known identities of Leibniz algebras: $(xy+yx)\cdot z=0$ and $x[y,z]=[xy,z]+[y,xz]$(see [5], [15]). The readers can also see the Hom-versions of these properties in [13].

Proposition 5.5. *Let (A,μ,α,β) be a regular BiHom-Leibniz algebra. Then*

$$J_{\alpha,\beta}(x,y,z)=\circlearrowleft_{(x,y,z)}\big(\alpha^{-1}\beta^2(x)\beta(y)\cdot\alpha\beta(z)\big). \tag{24}$$

Proof. Note that by (21), the ternary operation $[.,.,.]$ of the associated BiHom-Akivis algebra to the considered regular BiHom-Leibniz algebra (see Corollary5.2) is

$$[x,y,z]=-\beta^2(y)\cdot\beta(x)\alpha(z)$$

and then by (10), we obtain

$$\begin{aligned}
J_{\alpha,\beta}(x,y,z)\quad&=\circlearrowleft_{(x,y,z)}\big(-\beta^2(y)\cdot\beta(x)\alpha(z)\big)-\circlearrowleft_{(x,y,z)}\big(-\beta^2(x)\cdot\beta(y)\alpha(z)\big)\\
&=\circlearrowleft_{(x,y,z)}\big(\beta^2(x)\cdot\beta(y)\alpha(z)-\beta^2(y)\cdot\beta(x)\alpha(z)\big)\\
&=\circlearrowleft_{(x,y,z)}\big(\alpha\beta(\alpha^{-1}\beta(x))\cdot\beta(y)\alpha(z)-\beta(\beta(y))\cdot\alpha(\alpha^{-1}\beta(x))\alpha(z)\big)\\
&=\circlearrowleft_{(x,y,z)}\big(\alpha^{-1}\beta^2(x)\beta(y)\cdot\alpha\beta(z)\big)\text{(by (20)).}
\end{aligned}\tag{25}$$

Hence, we get the desired identity. □

Note that in BiHom-Leibniz algebras case, equation (24) is the specific form of the BiHom-Akivis identity (10).

Definition 5.6. [9] A multiplicative regular BiHom-algebra (A,μ,α,β) is said BiHom-Lie admissible if $(A,[.,.],\alpha,\beta)$ is a BiHom-Lie algebra where $[.,.]=\mu-\mu\circ(\alpha^{-1}\beta\otimes\alpha\beta^{-1})\circ\tau$.

From Proposition 5.5 one obtains the following necessary and sufficient condition for the BiHom-Lie admissibility of a given regular Hom-Leibniz algebra.

Corollary 5.7. *A regular BiHom-Leibniz algebra (A,μ,α,β) is BiHom-Lie admissible if and only if*

$$\circlearrowleft_{(x,y,z)}\big(\alpha^{-1}\beta^2(x)\beta(y)\cdot\alpha\beta(z)\big)=0$$

for all $x,y,z\in A$.

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