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A NOTE ON SIMPSON 3/8 RULE FOR FUNCTION WHOSE MODULUS OF FIRST DERIVATIVES ARE s-CONVEX FUNCTION WITH APPLICATION

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Abstract. Researchers continue to explore and introduce new operators, methods, and applications related to fractional integrals and inequalities. In recent years, fractional integrals and inequalities have gained a lot of attention. In this paper, firstly we established the new identity for the case of differentiable function through the fractional operator (Caputo-Fabrizio). By utilizing this novel identity, the obtained results are improved for Simpson second formula-type inequality. Based on this identity the Simpson second formula-type inequality is proved for the s-convex functions. Furthermore, we also include the applications to special means.

1. Introduction

Inequalities provide a versatile tool for dealing with uncertain or variable quantities and are integral to many branches of mathematics and their applications. They allow mathematicians, scientists, and engineers to reason about relationships, make informed decisions, and solve a wide range of problems. Overall, estimation is a powerful tool that complements exact calculations and enhances your problem-solving toolkit, enabling you to make informed decisions and solve mathematical problems efficiently. Inequalities have useful and legitimate applications in the fields of probability theory, functional inequalities, interpolation spaces, sobolev spaces, and information theory.

Fractional calculus continues to be an active area of research, with ongoing studies exploring its theoretical foundations, computational methods, and diverse applications. The concept of fractional calculus dates back to the work of mathematicians like Leibniz and Liouville, but it gained renewed interest and formalization in the 20th century. Notable researchers who contributed to the field include Riemann, Liouville, Grünwald, Letnikov, and Caputo. Indeed, over the past two decades, the use of fractional calculus has experienced a notable increase in both pure and applied disciplines of science and engineering. In fractional calculus, there exist many different kinds of integral operators. These operators have applications for applied crucial

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and theory of integral inequalities. In recent years mathematicians have become more interested in fractional integral operators to developed new inequalities (see these article [1–4]). Numerous novel fractional integral inequalities, such as the Simpson's inequality, have been developed in recent years, view these paper [5, 6] for more details in this regard. Furthermore, Budak et al. [7] examined several modifications of Simpson-type inequalities within the structure of differentiable convex functions by employing generalized fractional integrals. For more on Simpson-type inequalities and other characteristics of Riemann-Liouville fractional integrals, readers see [8, 9] and its reference. The following inequality is known as Simpson's inequality the original version is defined as.

Theorem 1.1. [13] *Let $\lambda : [\pi, \vartheta] \to \mathbb{R}$ be a four times continuously differentiable mapping on (π, ϑ) and $\left\|\lambda^{(4)}\right\|_\infty = \sup_{x\in(\pi,\vartheta)} \left|\lambda^{(4)}\right| < \infty$, then following inequalities holds:*

$$\left|\left[\frac{\lambda(\pi)+\lambda(\vartheta)}{6}+\frac{2}{3}\lambda\left(\frac{\pi+\vartheta}{2}\right)\right]-\frac{1}{\vartheta-\pi}\int_\pi^\vartheta \lambda(x)\,dx\right| \le \frac{1}{2880}\left\|\lambda^{(4)}\right\|_\infty (\vartheta-\pi)^4.$$

In all of this years, Thomas Simpson established fundamental methods for numerical integration and estimate of definite integrals that are now known as Simpson's law. (1710-1761). But J. Kepler utilized an identical approximation over a century before, which is because it is often referred to as Kepler's law. Estimates based just on a three-step quadratic kernel are often referred to as Newton-type results as Simpson's method utilizes the three-point Newton-Cotes quadrature rule. Notably, the work [10] delves into fractional Newton type inequalities for functions with bounded variation, while also establishing similar inequalities for differentiable convex functions through the application of Riemann-Liouville fractional integrals. Further expanding this field, Sitthiwirattham et.al in [11] contributed additional fractional Newton- type inequalities, again focusing on bounded variation. Moreover, in the study by Gao and Shi in [12] new Newton type inequalities were formulated based on convexity principle, particulary for specific scenarios involving real functions, thereby underscoring potential applications in this domain. For those seeking a more comprehensive under standing of Newton-type inequalities, especially in the context of convex differentiable functions, the works cite in [13,14] are invaluable resources. We delineate two primary formulations as follows.

Simpson quadrature formula (Simpson's 1/3) is followed as:

$$\left|\frac{\lambda(\pi)}{6}+\frac{4}{6}\lambda\left(\frac{\pi+\vartheta}{2}\right)+\frac{\lambda(\vartheta)}{6}-\frac{1}{\vartheta-\pi}\int_\pi^\vartheta \lambda(u)\,du\right| \le \frac{(\vartheta-\pi)^4}{2880}\left\|\lambda^{(4)}\right\|_\infty. \tag{1}$$

Simpson second formula or Newton-Cotes quadrature formula (Simpson's 3/8) is followed as:

$$\left|\frac{\lambda(\pi)}{8}+\frac{3}{8}\lambda\left(\frac{2\pi+\vartheta}{2}\right)+\frac{3}{8}\lambda\left(\frac{\pi+2\vartheta}{2}\right)+\frac{\lambda(\vartheta)}{8}-\frac{1}{\vartheta-\pi}\int_\pi^\vartheta \lambda(u)\,du\right| \le \frac{(\vartheta-\pi)^4}{6480}\left\|\lambda^{(4)}\right\|_\infty. \tag{2}$$

In recently Noor et. al [15] obtained the Simpson's 3/8 type inequalities for differentiable function.

$$\left|\frac{\lambda(\pi)}{8}+\frac{3}{8}\lambda\left(\frac{2\pi+\vartheta}{2}\right)+\frac{3}{8}\lambda\left(\frac{\pi+2\vartheta}{2}\right)+\frac{\lambda(\vartheta)}{8}-\frac{1}{\vartheta-\pi}\int_\pi^\vartheta \lambda(u)\,du\right|$$

$$\leq (\vartheta-\pi)\left[\frac{17}{756}\left(\frac{973\left|\lambda'(\pi)\right|^{q}+251\left|\lambda'(\vartheta)\right|^{q}}{1224}\right)^{\frac{1}{q}}+\frac{1}{36}\left(\frac{\left|\lambda'(\pi)\right|^{q}+\left|\lambda'(\vartheta)\right|^{q}}{2}\right)^{\frac{1}{q}}\right.$$
$$\left.+\frac{17}{756}\left(\frac{251\left|\lambda'(\pi)\right|^{q}+973\left|\lambda'(\vartheta)\right|^{q}}{1224}\right)^{\frac{1}{q}}\right],$$

and

$$\left|\frac{\lambda(\pi)}{8}+\frac{3}{8}\lambda\left(\frac{2\pi+\vartheta}{2}\right)+\frac{3}{8}\lambda\left(\frac{\pi+2\vartheta}{2}\right)+\frac{\lambda(\vartheta)}{8}-\frac{1}{\vartheta-\pi}\int_{\pi}^{\vartheta}\lambda(u)\,du\right|$$
$$\leq (\vartheta-\pi)\left[\left(\frac{3^{p+1}+5^{p+1}}{24^{p+1}(p+1)}\right)^{\frac{1}{p}}\left(\frac{5\left|\lambda'(\pi)\right|^{q}+\left|\lambda'(\vartheta)\right|^{q}}{18}\right)^{\frac{1}{q}}+\left(\frac{2}{6^{p+1}(p+1)}\right)^{\frac{1}{p}}\right.$$
$$\left.\times\left(\frac{\left|\lambda'(\pi)\right|^{q}+\left|\lambda'(\vartheta)\right|^{q}}{6}\right)^{\frac{1}{q}}+\left(\frac{3^{p+1}+5^{p+1}}{24^{p+1}(p+1)}\right)^{\frac{1}{p}}\left(\frac{\left|\lambda'(\pi)\right|^{q}+5\left|\lambda'(\vartheta)\right|^{q}}{18}\right)^{\frac{1}{q}}\right].$$

In 2020 Erden et al. established the error bounds of Simpson's second formula for differentiable function [16].

THEOREM 1.2. *Let $\lambda:[\pi,\vartheta]\rightarrow\mathbb{R}$ be a differentiable mapping whose derivative is continuously on (π,ϑ) . Then for all $x\in[\pi,\vartheta]$, then following inequality holds:*

$$\left|\frac{\lambda(\pi)}{8}+\frac{3}{8}\lambda\left(\frac{2\pi+\vartheta}{2}\right)+\frac{3}{8}\lambda\left(\frac{\pi+2\vartheta}{2}\right)+\frac{\lambda(\vartheta)}{8}-\frac{1}{\vartheta-\pi}\int_{\pi}^{\vartheta}\lambda(u)\,du\right|\leq\frac{25(\vartheta-\pi)}{288}\|\lambda\|_{\infty}.$$

The concept of convexity has a strong history in ancient times. The theory of convexity has found extensive use and significance in a wide range of modern mathematical disciplines, including real analysis, functional analysis, linear algebra, and optimization. Theory of inequalities, where the idea of convexity is crucial for enhancing the estimation bounds of various kinds of integral inequalities [17, 18]. The well-known definition of convexity is followed as:

DEFINITION 1.3. [19] If $\lambda:I\rightarrow\mathbb{R}$ is called convex on I for all $(\pi,\vartheta)\in I$, and $\Pi\in[0,1]$, then following inequality holds:

$$\lambda(\Pi\pi+(1-\Pi)\vartheta)\leq\Pi\lambda(\pi)+(1-\Pi)\lambda(\vartheta). \tag{3}$$

The mapping λ is concave on I the inequality (3) holds in reversed direction for all $(\pi,\vartheta)\in I$, and $\Pi\in[0,1]$.

DEFINITION 1.4. [20] The function $\lambda:I\subseteq\mathbb{R}\rightarrow\mathbb{R}_0:=[0,\infty)$ is said to be s-convex, if

$$\lambda(\Pi\pi+(1-\Pi)\vartheta)\leq\Pi^{s}\lambda(\pi)+(1-\Pi)^{s}\lambda(\vartheta),$$

for all $\pi,\vartheta\in I$, and $\Pi\in[0,1]$.

DEFINITION 1.5. [21] Suppose $\lambda\in L[\pi,\vartheta]$. The left and right-sided Riemann-Liouville fractional integrals of order $\Delta>0$ defined by:

$$I_{\pi^{+}}^{\Delta}\lambda(x) = \frac{1}{\Gamma(\Delta)}\int_{\pi}^{x}(x-\Pi)^{\Delta-1}\lambda(\Pi)\,d\Pi,\ x>\pi$$
$$I_{\vartheta^{-}}^{\Delta}\lambda(x) = \frac{1}{\Gamma(\Delta)}\int_{x}^{\vartheta}(\Pi-x)^{\Delta-1}\lambda(\Pi)\,d\Pi,\ x<\vartheta,$$

where $\Gamma(.)$ is the gamma function and $I_{\pi^{+}}^{0}\lambda(\Pi)=I_{\vartheta^{-}}^{0}\lambda(\Pi)=\lambda(\Pi)$.

Definition 1.6. [22] Let $\lambda \in H'(\pi,\vartheta)$, $\pi<\vartheta$, for all $\Delta\in[0,1]$, where $\beta(\Delta)>0$ is a normalizer satisfying $\beta(0)=\beta(1)=1$, then the fractional integrals are defined as:

$$\begin{aligned}\left({}_{\pi}^{CF}I^{\Delta}\lambda\right)(x) &= \frac{1-\Delta}{\beta(\Delta)}\lambda(x)+\frac{\Delta}{\beta(\Delta)}\int_{\pi}^{x}\lambda(x)\,dx\\ \left({}^{CF}I_{\vartheta}^{\Delta}\lambda\right)(x) &= \frac{1-\Delta}{\beta(\Delta)}\lambda(x)+\frac{\Delta}{\beta(\Delta)}\int_{x}^{\vartheta}\lambda(x)\,dx.\end{aligned}$$

Motivated by the ongoing research, the main goal in this paper is to establish a new integral identity using the fractional operator. By using the new identity to proved the error bounds of Simpson second formula type inequalities for bounded function. Based on this identity we developed the Simpson second formula for the s-convex functions. We also include the applications to special means, and Simpson formula, taking many special cases of the main findings is discuses in literature.

2. Simpson's 3/8 Formula-Type Inequalities for Differentiable Function

In this section, we present a new identity by the mean of a fractional operator. This identity is required to prove our main results.

Lemma 2.1. *Suppose $\lambda: I\subset\mathbb{R}\to\mathbb{R}$ be a differentiable mapping on I^o where $\pi,\vartheta\in I^o$ with $\pi<\vartheta$ and $\lambda'\in L^1[\pi,\vartheta]$, then the following fractional equality is proved:*

$$\begin{aligned}&\frac{\lambda(\pi)}{8}+\frac{3}{8}\lambda\left(\frac{2\pi+\vartheta}{3}\right)+\frac{3}{8}\lambda\left(\frac{\pi+2\vartheta}{3}\right)+\frac{\lambda(\vartheta)}{8}\\&-\frac{\beta(\Delta)}{\Delta(\vartheta-\pi)}\left[\left({}_{\pi}^{CF}I^{\Delta}\lambda\right)(k)+\left({}^{CF}I_{\vartheta}^{\Delta}\lambda\right)(k)\right]+\frac{2(1-\Delta)}{\beta(\Delta)}\lambda(k)\\ =\ &\frac{(\vartheta-\pi)}{9}\left[\int_0^1\left(\Pi-\frac{3}{8}\right)\lambda'\left((1-\Pi)\pi+\Pi\frac{2\pi+\vartheta}{3}\right)d\Pi\right.\\&+\int_0^1\left(\Pi-\frac{1}{2}\right)\lambda'\left((1-\Pi)\frac{2\pi+\vartheta}{3}+\Pi\frac{\pi+2\vartheta}{3}\right)d\Pi\\&\left.+\int_0^1\left(\Pi-\frac{5}{8}\right)\lambda'\left((1-\Pi)\frac{\pi+2\vartheta}{3}+\Pi\vartheta\right)d\Pi\right],\end{aligned}$$

where $\beta(\Delta)>0$ is a normalizer function.

Proof. Let

$$\begin{aligned}&\int_0^1\left(\Pi-\frac{3}{8}\right)\lambda'\left((1-\Pi)\pi+\Pi\frac{2\pi+\vartheta}{3}\right)d\Pi\\&+\int_0^1\left(\Pi-\frac{1}{2}\right)\lambda'\left((1-\Pi)\frac{2\pi+\vartheta}{3}+\Pi\frac{\pi+2\vartheta}{3}\right)d\Pi\\&+\int_0^1\left(\Pi-\frac{5}{8}\right)\lambda'\left((1-\Pi)\frac{\pi+2\vartheta}{3}+\Pi\vartheta\right)d\Pi\\ =\ &I_1+I_2+I_3.\end{aligned}$$

By using the integration by parts, we have

$$\begin{aligned}
I_1 &= \int_0^1 \left(\Pi - \frac{3}{8}\right) \lambda' \left((1-\Pi)\pi + \Pi \frac{2\pi+\vartheta}{3}\right) d\Pi \\
&= \frac{3\left(\Pi - \frac{3}{8}\right)}{\vartheta - \pi} \lambda \left((1-\Pi)\pi + \Pi \frac{2\pi+\vartheta}{3}\right)\Bigg|_0^1 \\
&\quad - \frac{3}{\vartheta-\pi} \int_0^1 \lambda \left((1-\Pi)\pi + \Pi \frac{2\pi+\vartheta}{3}\right) d\Pi \\
&= \frac{9}{8(\vartheta-\pi)} \lambda(\pi) + \frac{15}{8(\vartheta-\pi)} \lambda\left(\frac{2\pi+\vartheta}{3}\right) - \frac{9}{(\vartheta-\pi)^2} \int_\pi^{\frac{2\pi+\vartheta}{3}} \lambda(u)\, du. \qquad (4)
\end{aligned}$$

$$\begin{aligned}
I_2 &= \int_0^1 \left(\Pi - \frac{1}{2}\right) \lambda' \left((1-\Pi)\frac{2\pi+\vartheta}{3} + \Pi \frac{\pi+2\vartheta}{3}\right) d\Pi \\
&= \frac{3\left(\Pi - \frac{1}{2}\right)}{\vartheta - \pi} \lambda \left((1-\Pi)\frac{2\pi+\vartheta}{3} + \Pi \frac{\pi+2\vartheta}{3}\right)\Bigg|_0^1 \\
&\quad - \frac{3}{\vartheta-\pi} \int_0^1 \lambda \left((1-\Pi)\frac{2\pi+\vartheta}{3} + \Pi \frac{\pi+2\vartheta}{3}\right) d\Pi \\
&= \frac{3}{2(\vartheta-\pi)} \lambda\left(\frac{2\pi+\vartheta}{3}\right) + \frac{3}{2(\vartheta-\pi)} \lambda\left(\frac{\pi+2\vartheta}{3}\right) - \frac{9}{(\vartheta-\pi)^2} \int_{\frac{2\pi+\vartheta}{3}}^{\frac{\pi+2\vartheta}{3}} \lambda(u)\, du. \qquad (5)
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
I_3 &= \int_0^1 \left(\Pi - \frac{5}{8}\right) \lambda' \left((1-\Pi)\frac{\pi+2\vartheta}{3} + \Pi\vartheta\right) d\Pi \\
&= \frac{3\left(\Pi - \frac{5}{8}\right)}{\vartheta - \pi} \lambda \left((1-\Pi)\frac{\pi+2\vartheta}{3} + \Pi\vartheta\right)\Bigg|_0^1 \\
&\quad - \frac{3}{\vartheta-\pi} \int_0^1 \lambda \left((1-\Pi)\frac{\pi+2\vartheta}{3} + \Pi\vartheta\right) d\Pi \\
&= \frac{9}{8(\vartheta-\pi)} \lambda(\vartheta) + \frac{15}{8(\vartheta-\pi)} \lambda\left(\frac{\pi+2\vartheta}{3}\right) - \frac{9}{(\vartheta-\pi)^2} \int_{\frac{\pi+2\vartheta}{3}}^{\vartheta} \lambda(u)\, du. \qquad (6)
\end{aligned}$$

Adding the equalities (4), (5) and (6), we get

$$\begin{aligned}
& I_1 + I_2 + I_3 \\
&= \frac{9}{8(\vartheta-\pi)} \lambda(\pi) + \frac{15}{8(\vartheta-\pi)} \lambda\left(\frac{2\pi+\vartheta}{3}\right) - \frac{9}{(\vartheta-\pi)^2} \int_\pi^{\frac{2\pi+\vartheta}{3}} \lambda(u)\, du \\
&\quad + \frac{3}{2(\vartheta-\pi)} \lambda\left(\frac{2\pi+\vartheta}{3}\right) + \frac{3}{2(\vartheta-\pi)} \lambda\left(\frac{\pi+2\vartheta}{3}\right) - \frac{9}{(\vartheta-\pi)^2} \int_{\frac{2\pi+\vartheta}{3}}^{\frac{\pi+2\vartheta}{3}} \lambda(u)\, du \\
&\quad + \frac{9}{8(\vartheta-\pi)} \lambda(\vartheta) + \frac{15}{8(\vartheta-\pi)} \lambda\left(\frac{\pi+2\vartheta}{3}\right) - \frac{9}{(\vartheta-\pi)^2} \int_{\frac{\pi+2\vartheta}{3}}^{\vartheta} \lambda(u)\, du \\
&= \frac{9}{8(\vartheta-\pi)} \lambda(\pi) + \frac{9}{8(\vartheta-\pi)} \lambda(\vartheta) + \frac{27}{8(\vartheta-\pi)} \left(\frac{2\pi+\vartheta}{3}\right)
\end{aligned}$$

$$+\frac{27}{8(\vartheta-\pi)}\left(\frac{\pi+2\vartheta}{3}\right)-\frac{9}{(\vartheta-\pi)^2}\int_{\pi}^{\vartheta}\lambda(u)\,du. \tag{7}$$

Multiplying the equality (7) with $\frac{(\vartheta-\pi)}{9}$ and subtracting $\frac{2(1-\Delta)}{\beta(\Delta)}\lambda(k)$, we have

$$\begin{aligned}
&(I_1+I_2+I_3)\frac{(\vartheta-\pi)}{9}-\frac{2(1-\Delta)}{\beta(\Delta)}\lambda(k)\\
=\;&\frac{\lambda(\pi)}{8}+\frac{\lambda(\vartheta)}{8}+\frac{3}{8}\lambda\left(\frac{2\pi+\vartheta}{3}\right)+\frac{3}{8}\lambda\left(\frac{\pi+2\vartheta}{3}\right)\\
&-\frac{\beta(\Delta)}{\Delta(\vartheta-\pi)}\left(\frac{\Delta}{\beta(\Delta)}\int_{\pi}^{k}\lambda(u)\,du-\frac{(1-\Delta)}{\beta(\Delta)}\lambda(k)+\frac{\Delta}{\beta(\Delta)}\int_{k}^{\vartheta}\lambda(u)\,du\right.\\
&\left.-\frac{(1-\Delta)}{\beta(\Delta)}\lambda(k)\right)\\
=\;&\frac{\lambda(\pi)}{8}+\frac{\lambda(\vartheta)}{8}+\frac{3}{8}\lambda\left(\frac{2\pi+\vartheta}{3}\right)+\frac{3}{8}\lambda\left(\frac{\pi+2\vartheta}{3}\right)\\
&-\frac{\beta(\Delta)}{\Delta(\vartheta-\pi)}\left[\left({}_{\pi}^{CF}I^{\Delta}\lambda\right)(k)+\left({}^{CF}I_{\vartheta}^{\Delta}\lambda\right)(k)\right].
\end{aligned}$$

The proof of Lemma 2.1 is completed. □

3. Simpson's 3/8 Formula-Type Inequalities for s-convex Function

In this section, we prove the error bounds of Simpson's second formula type inequalities for s-convex function.

Theorem 3.1. *Let $\lambda:[0,\infty)\to\mathbb{R}$, $\pi,\vartheta\in\mathbb{R}^+$, $\pi<\vartheta$. If $|\lambda'|$ is s-convex on $[\pi,\vartheta]$, then the following fractional inequality is established:*

$$\begin{aligned}
&\left|\left[\frac{\lambda(\pi)}{8}+\frac{3}{8}\lambda\left(\frac{2\pi+\vartheta}{3}\right)+\frac{3}{8}\lambda\left(\frac{\pi+2\vartheta}{3}\right)+\frac{\lambda(\vartheta)}{8}\right]\right.\\
&\left.-\frac{\beta(\Delta)}{\Delta(\vartheta-\pi)}\left[\left({}_{\pi}^{CF}I^{\Delta}\lambda\right)(k)+\left({}^{CF}I_{\vartheta}^{\Delta}\lambda\right)(k)\right]+\frac{2(1-\Delta)}{\beta(\Delta)}\lambda(k)\right|\\
\leq\;&\frac{(\vartheta-\pi)}{9\times 3^{-s}\times(s+1)(s+2)\times 2^{-5-3s}}\left[\left(3^{2+s}-7\times 8^{1+s}-3^{3+s}\times 8^{1+s}+12^{2+s}-17\times 16^{1+s}\right.\right.\\
&\left.\left.+21^{2+s}-2^{2+3s}\times\left(2^{1+s}-1-3^{2+s}\right)s\right)\right]\left(\left|\lambda'(\pi)\right|+\left|\lambda'(\vartheta)\right|\right).
\end{aligned}$$

Proof. By using the Lemma 2.1, since $|\lambda'|$ is s-convex, we have

$$\begin{aligned}
&\left|\left[\frac{\lambda(\pi)}{8}+\frac{3}{8}\lambda\left(\frac{2\pi+\vartheta}{3}\right)+\frac{3}{8}\lambda\left(\frac{\pi+2\vartheta}{3}\right)+\frac{\lambda(\vartheta)}{8}\right]\right.\\
&\left.-\frac{\beta(\Delta)}{\Delta(\vartheta-\pi)}\left[\left({}_{\pi}^{CF}I^{\Delta}\lambda\right)(k)+\left({}^{CF}I_{\vartheta}^{\Delta}\lambda\right)(k)\right]+\frac{2(1-\Delta)}{\beta(\Delta)}\lambda(k)\right|\\
\leq\;&\frac{(\vartheta-\pi)}{9}\left[\int_0^1\left|\Pi-\frac{3}{8}\right|\left|\lambda'\left((1-\Pi)\pi+\Pi\frac{2\pi+\vartheta}{3}\right)\right|d\Pi\right.\\
&+\int_0^1\left|\Pi-\frac{1}{2}\right|\left|\lambda'\left((1-\Pi)\frac{2\pi+\vartheta}{3}+\Pi\frac{\pi+2\vartheta}{3}\right)\right|d\Pi
\end{aligned}$$

$$
\begin{aligned}
&+\int_0^1\left|\Pi-\frac{5}{8}\right|\left|\lambda'\left((1-\Pi)\frac{\pi+2\vartheta}{3}+\Pi\vartheta\right)\right|d\Pi\Bigg]\\
=\ &\frac{(\vartheta-\pi)}{9}\left[\int_0^1\left|\Pi-\frac{3}{8}\right|\left|\lambda'\left(\frac{3-\Pi}{3}\pi+\frac{\Pi}{3}\vartheta\right)\right|d\Pi\right.\\
&+\int_0^1\left|\Pi-\frac{1}{2}\right|\left|\lambda'\left(\frac{2-\Pi}{3}\pi+\frac{1+\Pi}{3}\vartheta\right)\right|\\
&\left.+\int_0^1\left|\Pi-\frac{5}{8}\right|\left|\lambda'\left(\frac{1-\Pi}{3}\pi+\frac{2+\Pi}{3}\vartheta\right)\right|d\Pi\right]\\
=\ &\frac{(\vartheta-\pi)}{9}\left[\int_0^1\left|\Pi-\frac{3}{8}\right|\left(\left(\frac{3-\Pi}{3}\right)^s|\lambda'(\pi)|+\left(\frac{\Pi}{3}\right)^s|\lambda'(\vartheta)|\right)d\Pi\right.\\
&+\int_0^1\left|\Pi-\frac{1}{2}\right|\left(\left(\frac{2-\Pi}{3}\right)^s|\lambda'(\pi)|+\left(\frac{1+\Pi}{3}\right)^s|\lambda'(\vartheta)|\right)d\Pi\\
&\left.+\int_0^1\left|\Pi-\frac{5}{8}\right|\left(\left(\frac{1-\Pi}{3}\right)^s|\lambda'(\pi)|+\left(\frac{2+\Pi}{3}\right)^s|\lambda'(\vartheta)|\right)d\Pi\right]\\
=\ &\frac{(\vartheta-\pi)}{9}\left[\frac{1}{64^{1+s}\times(s+1)(s+2)}\left(\left(\frac{7}{3}\right)^s\left(49\times 2^{1+3s}\times 3^{2+s}-13\times 2^{5+7s}\times 7^{-s}-3^{3+s}\right.\right.\right.\\
&\left.\left.\times 4^{2+3s}\times 7^{-s}-5\times 2^{4+7s}\times 7^{-s}s+3^{2+s}\times 7^{-5}\times 8^{1+2s}s\right)\right)|\lambda'(\pi)|\\
&+\frac{3^{2+s}+8^{1+s}+5\times 2^{2+3s}s}{(s+1)(s+2)2^{5+3s}\times 3^s}|\lambda'(\vartheta)|+\frac{2^{1+2s}s+3^{2+s}-2^ss-2^{2+s}-2^{2+2s}}{(s+1)(s+2)\times 2^{1+2s}}|\lambda'(\pi)|\\
&-\frac{2^{2+s}+2^{2+2s}-3^{2+s}+2^5s-2^{1+2s}s}{(s+1)(s+2)\times 2^{1+s}\times 3^{-5}}|\lambda'(\vartheta)|+\frac{3^{2+s}+8^{1+s}+5\times 2^{2+3s}s}{(s+1)(s+2)2^{5+3s}\times 3^s}|\lambda'(\pi)|\\
&+\frac{1}{64^{1+s}\times(s+1)(s+2)}\left(\left(\frac{7}{3}\right)^s\left(49\times 2^{1+3s}\times 3^{2+s}-13\times 2^{5+7s}\right.\right.\\
&\left.\left.\left.\times 7^{-s}-3^{3+s}\times 4^{2+3s}\times 7^{-s}-5\times 2^{4+7s}\times 7^{-s}s+3^{2+s}\times 7^{-5}\times 8^{1+2s}s\right)\right)|\lambda'(\vartheta)|\right]\\
\le\ &\frac{(\vartheta-\pi)}{9\times 3^{-s}\times(s+1)(s+2)\times 2^{-5-3s}}\left[\left(3^{2+s}-7\times 8^{1+s}-3^{3+s}\times 8^{1+s}+12^{2+s}\right.\right.\\
&\left.\left.-17\times 16^{1+s}+21^{2+s}-2^{2+3s}\times\left(2^{1+s}-1-3^{2+s}\right)s\right)\right](|\lambda'(\pi)|+|\lambda'(\vartheta)|).
\end{aligned}
$$

The proof of Theorem 3.1, is completed. □

Corollary 3.2. *If we choose $s=1$ in Theorem 3.1, then we get*

$$
\begin{aligned}
&\left|\left[\frac{\lambda(\pi)}{8}+\frac{3}{8}\lambda\left(\frac{2\pi+\vartheta}{3}\right)+\frac{3}{8}\lambda\left(\frac{\pi+2\vartheta}{3}\right)+\frac{\lambda(\vartheta)}{8}\right]\right.\\
&\left.-\frac{\beta(\Delta)}{\Delta(\vartheta-\pi)}\left[\left({}_{\pi}^{CF}I^{\Delta}\lambda\right)(k)+\left({}^{CF}I_{\vartheta}^{\Delta}\lambda\right)(k)\right]+\frac{2(1-\Delta)}{\beta(\Delta)}\lambda(k)\right|\\
\le\ &\frac{25(\vartheta-\pi)}{576}(|\lambda'(\pi)|+|\lambda'(\vartheta)|).
\end{aligned}
$$

Corollary 3.3. *If we choose $|\lambda'(x)|\le M=\|\lambda'\|_\infty$, in Theorem 3.1, then we get*

$$
\left|\left[\frac{\lambda(\pi)}{8}+\frac{3}{8}\lambda\left(\frac{2\pi+\vartheta}{3}\right)+\frac{3}{8}\lambda\left(\frac{\pi+2\vartheta}{3}\right)+\frac{\lambda(\vartheta)}{8}\right]\right.
$$

$$-\frac{\beta(\Delta)}{\Delta(\vartheta-\pi)}\left[\left({}_{\pi}^{CF}I^{\Delta}\lambda\right)(k)+\left({}^{CF}I_{\vartheta}^{\Delta}\lambda\right)(k)\right]+\frac{2(1-\Delta)}{\beta(\Delta)}\lambda(k)\Bigg|$$
$$\leq \frac{25(\vartheta-\pi)}{576}\left\|\lambda'\right\|_{\infty}.$$

Remark 3.4. If we put $\Delta = 1$ and $\beta(0) = \beta(1) = 1$ in Corollary 3.3, we emphasise the fact that our obtained Corollary 3.3 improved the error bound given by Erden et. al [16, Corollary 4].

Theorem 3.5. *Let $\lambda : [0,\infty) \to \mathbb{R}$, $\pi, \vartheta \in \mathbb{R}^{+}$, $\pi < \vartheta$. If $|\lambda'|^{q}$ is s-convex on $[\pi, \vartheta]$ and $q > 1$, then the following fractional inequality is proved:*

$$\Bigg|\left[\frac{\lambda(\pi)}{8}+\frac{3}{8}\lambda\left(\frac{2\pi+\vartheta}{3}\right)+\frac{3}{8}\lambda\left(\frac{\pi+2\vartheta}{3}\right)+\frac{\lambda(\vartheta)}{8}\right]$$
$$-\frac{\beta(\Delta)}{\Delta(\vartheta-\pi)}\left[\left({}_{\pi}^{CF}I^{\Delta}\lambda\right)(k)+\left({}^{CF}I_{\vartheta}^{\Delta}\lambda\right)(k)\right]+\frac{2(1-\Delta)}{\beta(\Delta)}\lambda(k)\Bigg|$$
$$\leq \frac{(\vartheta-\pi)}{9}\left[\left(\frac{(3^{1+p}+5^{1+p})}{8^{1+p}(p+1)}\right)^{\frac{1}{p}}\left(\frac{3-2^{s+1}\left|\lambda'(\pi)\right|^{q}+\left|\lambda'(\vartheta)\right|^{q}}{3^{s}(1+s)}\right)^{\frac{1}{q}}\right.$$
$$+\left(\frac{2^{-p}}{p+1}\right)^{\frac{1}{p}}\left(\left(\frac{2^{1+s}-1}{3^{s}(s+1)}\right)\left|\lambda'(\pi)\right|^{q}+\left|\lambda'(\vartheta)\right|^{q}\right)^{\frac{1}{q}}+\left(\frac{(3^{1+p}+5^{1+p})}{8^{1+p}(p+1)}\right)^{\frac{1}{p}}$$
$$\left.\left(\frac{\left|\lambda'(\pi)\right|^{q}+3-2^{s+1}\left|\lambda'(\vartheta)\right|^{q}}{3^{s}(1+s)}\right)^{\frac{1}{q}}\right].$$

Proof. By using the Lemma 2.1, with the help of Hölder inequality and s-convexity of $|\lambda'|^{q}$, we have

$$\Bigg|\left[\frac{\lambda(\pi)}{8}+\frac{3}{8}\lambda\left(\frac{2\pi+\vartheta}{3}\right)+\frac{3}{8}\lambda\left(\frac{\pi+2\vartheta}{3}\right)+\frac{\lambda(\vartheta)}{8}\right]$$
$$-\frac{\beta(\Delta)}{\Delta(\vartheta-\pi)}\left[\left({}_{\pi}^{CF}I^{\Delta}\lambda\right)(k)+\left({}^{CF}I_{\vartheta}^{\Delta}\lambda\right)(k)\right]+\frac{2(1-\Delta)}{\beta(\Delta)}\lambda(k)\Bigg|$$
$$\leq \frac{(\vartheta-\pi)}{9}\left[\int_0^1\left|\Pi-\frac{3}{8}\right|\left|\lambda'\left((1-\Pi)\pi+\Pi\frac{2\pi+\vartheta}{3}\right)\right|d\Pi\right.$$
$$+\int_0^1\left|\Pi-\frac{1}{2}\right|\left|\lambda'\left((1-\Pi)\frac{2\pi+\vartheta}{3}+\Pi\frac{\pi+2\vartheta}{3}\right)\right|d\Pi$$
$$\left.+\int_0^1\left|\Pi-\frac{5}{8}\right|\left|\lambda'\left((1-\Pi)\frac{\pi+2\vartheta}{3}+\Pi\vartheta\right)\right|d\Pi\right]$$
$$= \frac{(\vartheta-\pi)}{9}\left[\int_0^1\left|\Pi-\frac{3}{8}\right|\left|\lambda'\left(\frac{3-\Pi}{3}\pi+\frac{\Pi}{3}\vartheta\right)\right|d\Pi\right.$$
$$+\int_0^1\left|\Pi-\frac{1}{2}\right|\left|\lambda'\left(\frac{2-\Pi}{3}\pi+\frac{1+\Pi}{3}\vartheta\right)\right|$$
$$\left.+\int_0^1\left|\Pi-\frac{5}{8}\right|\left|\lambda'\left(\frac{1-\Pi}{3}\pi+\frac{2+\Pi}{3}\vartheta\right)\right|d\Pi\right]$$
$$= \frac{(\vartheta-\pi)}{9}\left[\left(\int_0^1\left|\Pi-\frac{3}{8}\right|^{p}d\Pi\right)^{\frac{1}{p}}\left(\int_0^1\left|\lambda'\left(\left(\frac{3-\Pi}{3}\right)^{s}\pi+\left(\frac{\Pi}{3}\right)^{s}\vartheta\right)\right|^{q}d\Pi\right)^{\frac{1}{q}}\right.$$

$$+\left(\int_0^1\left|\Pi-\frac{1}{2}\right|^p d\Pi\right)^{\frac{1}{p}}\left(\int_0^1\left|\lambda'\left(\left(\frac{2-\Pi}{3}\right)^s\pi+\left(\frac{1+\Pi}{3}\right)^s\vartheta\right)\right|^q d\Pi\right)^{\frac{1}{q}}$$

$$+\left(\int_0^1\left|\Pi-\frac{5}{8}\right|^p d\Pi\right)^{\frac{1}{p}}\left(\int_0^1\left|\lambda'\left(\left(\frac{1-\Pi}{3}\right)^s\pi+\left(\frac{2+\Pi}{3}\right)^s\vartheta\right)\right|^q d\Pi\right)^{\frac{1}{q}}\Bigg]$$

$$= \frac{(\vartheta-\pi)}{9}\left[\left(\int_0^1\left|\Pi-\frac{3}{8}\right|^p d\Pi\right)^{\frac{1}{p}}\left(\int_0^1\left(\left(\frac{3-\Pi}{3}\right)^s|\lambda'(\pi)|^q+\left(\frac{\Pi}{3}\right)^s|\lambda'(\vartheta)|^q\right)d\Pi\right)^{\frac{1}{q}}\right.$$

$$+\left(\int_0^1\left|\Pi-\frac{1}{2}\right|^p d\Pi\right)^{\frac{1}{p}}\left(\int_0^1\left(\left(\frac{2-\Pi}{3}\right)^s|\lambda'(\pi)|^q+\left(\frac{1+\Pi}{3}\right)^s|\lambda'(\vartheta)|^q\right)d\Pi\right)^{\frac{1}{q}}$$

$$\left.+\left(\int_0^1\left|\Pi-\frac{5}{8}\right|^p d\Pi\right)^{\frac{1}{p}}\left(\int_0^1\left(\left(\frac{1-\Pi}{3}\right)^s|\lambda'(\pi)|^q+\left(\frac{2+\Pi}{3}\right)^s|\lambda'(\vartheta)|^q\right)d\Pi\right)^{\frac{1}{q}}\right]$$

$$\leq \frac{(\vartheta-\pi)}{9}\left[\left(\frac{(3^{1+p}+5^{1+p})}{8^{1+p}(p+1)}\right)^{\frac{1}{p}}\left(\frac{3-2^{s+1}|\lambda'(\pi)|^q+|\lambda'(\vartheta)|^q}{3^s(1+s)}\right)^{\frac{1}{q}}\right.$$

$$+\left(\frac{2^{-p}}{p+1}\right)^{\frac{1}{p}}\left(\left(\frac{2^{1+s}-1}{3^s(s+1)}\right)|\lambda'(\pi)|^q+|\lambda'(\vartheta)|^q\right)^{\frac{1}{q}}+\left(\frac{(3^{1+p}+5^{1+p})}{8^{1+p}(p+1)}\right)^{\frac{1}{p}}$$

$$\left.\left(\frac{|\lambda'(\pi)|^q+3-2^{s+1}|\lambda'(\vartheta)|^q}{3^s(1+s)}\right)^{\frac{1}{q}}\right].$$

The proof of Theorem 3.5, is finished. □

COROLLARY 3.6. *If we choose $s=1$ in Theorem 3.5, then we get*

$$\left|\left[\frac{\lambda(\pi)}{8}+\frac{3}{8}\lambda\left(\frac{2\pi+\vartheta}{3}\right)+\frac{3}{8}\lambda\left(\frac{\pi+2\vartheta}{3}\right)+\frac{\lambda(\vartheta)}{8}\right]\right.$$

$$\left.-\frac{\beta(\Delta)}{\Delta(\vartheta-\pi)}\left[\left({}^{CF}_{\pi}I^{\Delta}\lambda\right)(k)+\left({}^{CF}I^{\Delta}_{\vartheta}\lambda\right)(k)\right]+\frac{2(1-\Delta)}{\beta(\Delta)}\lambda(k)\right|$$

$$\leq \frac{(\vartheta-\pi)}{9}\left[\left(\frac{(3^{1+p}+5^{1+p})}{8^{1+p}(p+1)}\right)^{\frac{1}{p}}\left(|\lambda'(\pi)|^q+|\lambda'(\vartheta)|^q\right)^{\frac{1}{q}}+\left(\frac{2^{-p}}{p+1}\right)^{\frac{1}{p}}\left(\frac{|\lambda'(\pi)|^q+|\lambda'(\vartheta)|^q}{2}\right)^{\frac{1}{q}}\right].$$

THEOREM 3.7. *Let $\lambda:[0,\infty)\to\mathbb{R}$, $\pi,\vartheta\in\mathbb{R}^+$, $\pi<\vartheta$. If $|\lambda'|^q$ is s-convex on $[\pi,\vartheta]$ and $q\geq 1$, then the following fractional inequality is constructed:*

$$\left|\left[\frac{\lambda(\pi)}{8}+\frac{3}{8}\lambda\left(\frac{2\pi+\vartheta}{3}\right)+\frac{3}{8}\lambda\left(\frac{\pi+2\vartheta}{3}\right)+\frac{\lambda(\vartheta)}{8}\right]\right.$$

$$\left.-\frac{\beta(\Delta)}{\Delta(\vartheta-\pi)}\left[\left({}^{CF}_{\pi}I^{\Delta}\lambda\right)(k)+\left({}^{CF}I^{\Delta}_{\vartheta}\lambda\right)(k)\right]+\frac{2(1-\Delta)}{\beta(\Delta)}\lambda(k)\right|$$

$$\leq \frac{(\vartheta-\pi)}{9}\left[\left(\frac{17}{64}\right)^{1-\frac{1}{q}}\left(\frac{1}{64^{1+s}\times(s+1)(s+2)}\left(\left(\frac{7}{3}\right)^s(49\times 2^{1+3s}\times 3^{2+s}-13\times 2^{5+7s}\right.\right.\right.$$

$$\times\, 4^{2+3s}\times 7^{-s}-5\times 2^{4+7s}\times 7^{-s}s+3^{2+s}\times 7^{-5}\times 8^{1+2s}s)\big)\,|\lambda'(\pi)|^q$$

$$\left.+\frac{3^{2+s}+8^{1+s}+5\times 2^{2+3s}s}{(s+1)(s+2)\,2^{5+3s}\times 3^s}|\lambda'(\vartheta)|^q\right)^{\frac{1}{q}}$$

$$+\left(\frac{1}{4}\right)^{1-\frac{1}{q}}\frac{2^{1+2s}s+3^{2+s}-2^{s}s-2^{2+s}-2^{2+2s}}{(s+1)(s+2)\times 2^{1+2s}}\left|\lambda'(\pi)\right|^{q}$$
$$\left.-\frac{2^{2+s}+2^{2+2s}-3^{2+s}+2^{5}s-2^{1+2s}s}{(s+1)(s+2)\times 2^{1+s}\times 3^{-5}}\left|\lambda'(\vartheta)\right|^{q}\right)^{\frac{1}{q}}$$
$$+\left(\frac{17}{64}\right)^{1-\frac{1}{q}}\left(\frac{3^{2+s}+8^{1+s}+5\times 2^{2+3s}s}{(s+1)(s+2)2^{5+3s}\times 3^{s}}\left|\lambda'(\pi)\right|^{q}\right.$$
$$+\frac{1}{64^{1+s}\times(s+1)(s+2)}\left(\left(\frac{7}{3}\right)^{s}\left(49\times 2^{1+3s}\times 3^{2+s}-13\times 2^{5+7s}\times 7^{-s}\right.\right.$$
$$\left.\left.\left.-3^{3+s}\times 4^{2+3s}\times 7^{-s}-5\times 2^{4+7s}\times 7^{-s}s+3^{2+s}\times 7^{-5}\times 8^{1+2s}s\right)\right)\left|\lambda'(\vartheta)\right|^{q}\right)\Big].$$

Proof. By using the Lemma 2.1, with the help of power-mean inequality and s-convexity of $|\lambda'|^q$, we have

$$\Bigg|\left[\frac{\lambda(\pi)}{8}+\frac{3}{8}\lambda\left(\frac{2\pi+\vartheta}{3}\right)+\frac{3}{8}\lambda\left(\frac{\pi+2\vartheta}{3}\right)+\frac{\lambda(\vartheta)}{8}\right]$$
$$-\frac{\beta(\Delta)}{\Delta(\vartheta-\pi)}\left[\left({}_{\pi}^{CF}I^{\Delta}\lambda\right)(k)+\left({}^{CF}I_{\vartheta}^{\Delta}\lambda\right)(k)\right]+\frac{2(1-\Delta)}{\beta(\Delta)}\lambda(k)\Bigg|$$
$$\leq \frac{(\vartheta-\pi)}{9}\left[\int_0^1\left|\Pi-\frac{3}{8}\right|\left|\lambda'\left((1-\Pi)\pi+\Pi\frac{2\pi+\vartheta}{3}\right)\right|d\Pi\right.$$
$$+\int_0^1\left|\Pi-\frac{1}{2}\right|\left|\lambda'\left((1-\Pi)\frac{2\pi+\vartheta}{3}+\Pi\frac{\pi+2\vartheta}{3}\right)\right|d\Pi$$
$$\left.+\int_0^1\left|\Pi-\frac{5}{8}\right|\left|\lambda'\left((1-\Pi)\frac{\pi+2\vartheta}{3}+\Pi\vartheta\right)\right|d\Pi\right]$$
$$=\frac{(\vartheta-\pi)}{9}\left[\int_0^1\left|\Pi-\frac{3}{8}\right|\left|\lambda'\left(\frac{3-\Pi}{3}\pi+\frac{\Pi}{3}\vartheta\right)\right|d\Pi\right.$$
$$+\int_0^1\left|\Pi-\frac{1}{2}\right|\left|\lambda'\left(\frac{2-\Pi}{3}\pi+\frac{1+\Pi}{3}\vartheta\right)\right|$$
$$\left.+\int_0^1\left|\Pi-\frac{5}{8}\right|\left|\lambda'\left(\frac{1-\Pi}{3}\pi+\frac{2+\Pi}{3}\vartheta\right)\right|d\Pi\right]$$
$$=\frac{(\vartheta-\pi)}{9}\left[\left(\int_0^1\left|\Pi-\frac{3}{8}\right|d\Pi\right)^{1-\frac{1}{q}}\left(\int_0^1\left|\Pi-\frac{3}{8}\right|\left|\lambda'\left(\left(\frac{3-\Pi}{3}\right)^{s}\pi+\left(\frac{\Pi}{3}\right)^{s}\vartheta\right)\right|^{q}d\Pi\right)^{\frac{1}{q}}\right.$$
$$\left(\int_0^1\left|\Pi-\frac{1}{2}\right|d\Pi\right)^{1-\frac{1}{q}}\left(\int_0^1\left|\Pi-\frac{1}{2}\right|\left|\lambda'\left(\left(\frac{2-\Pi}{3}\right)^{s}\pi+\left(\frac{1+\Pi}{3}\right)^{s}\vartheta\right)\right|^{q}d\Pi\right)^{\frac{1}{q}}$$
$$\left.+\left(\int_0^1\left|\Pi-\frac{5}{8}\right|d\Pi\right)^{1-\frac{1}{q}}\left(\int_0^1\left|\Pi-\frac{5}{8}\right|\left|\lambda'\left(\left(\frac{1-\Pi}{3}\right)^{s}\pi+\left(\frac{2+\Pi}{3}\right)^{s}\vartheta\right)\right|^{q}d\Pi\right)^{\frac{1}{q}}\right]$$
$$\leq\frac{(\vartheta-\pi)}{9}\left[\left(\frac{17}{64}\right)^{1-\frac{1}{q}}\left(\frac{1}{64^{1+s}\times(s+1)(s+2)}\left(\left(\frac{7}{3}\right)^{s}\left(49\times 2^{1+3s}\times 3^{2+s}-13\times 2^{5+7s}\right.\right.\right.\right.$$
$$\left.\left.\times\, 7^{-s}-3^{3+s}\times 4^{2+3s}\times 7^{-s}-5\times 2^{4+7s}\times 7^{-s}s+3^{2+s}\times 7^{-5}\times 8^{1+2s}s\right)\right)\left|\lambda'(\pi)\right|^{q}$$

$$+\frac{3^{2+s}+8^{1+s}+5\times 2^{2+3s}s}{(s+1)(s+2)\,2^{5+3s}\times 3^{s}}\left|\lambda'(\vartheta)\right|^{q}\Bigg)^{\frac{1}{q}}$$
$$+\left(\frac{1}{4}\right)^{1-\frac{1}{q}}\Bigg(\frac{2^{1+2s}s+3^{2+s}-2^{s}s-2^{2+s}-2^{2+2s}}{(s+1)(s+2)\times 2^{1+2s}}\left|\lambda'(\pi)\right|^{q}$$
$$-\frac{2^{2+s}+2^{2+2s}-3^{2+s}+2^{5}s-2^{1+2s}s}{(s+1)(s+2)\times 2^{1+s}\times 3^{-5}}\left|\lambda'(\vartheta)\right|^{q}\Bigg)^{\frac{1}{q}}$$
$$+\left(\frac{17}{64}\right)^{1-\frac{1}{q}}\Bigg(\frac{3^{2+s}+8^{1+s}+5\times 2^{2+3s}s}{(s+1)(s+2)\,2^{5+3s}\times 3^{s}}\left|\lambda'(\pi)\right|^{q}$$
$$+\frac{1}{64^{1+s}\times(s+1)(s+2)}\left(\left(\frac{7}{3}\right)^{s}\left(49\times 2^{1+3s}\times 3^{2+s}\right.\right.$$
$$-13\times 2^{5+7s}\times 7^{-s}-3^{3+s}\times 4^{2+3s}\times 7^{-s}-5$$
$$\left.\left.\times 2^{4+7s}\times 7^{-s}s+3^{2+s}\times 7^{-5}\times 8^{1+2s}s\right)\right)\left|\lambda'(\vartheta)\right|^{q}\Bigg)\Bigg].$$

This completes the proof. □

Corollary 3.8. *If we choose $s=1$ in Theorem 3.7, then we get*

$$\left|\left[\frac{\lambda(\pi)}{8}+\frac{3}{8}\lambda\left(\frac{2\pi+\vartheta}{3}\right)+\frac{3}{8}\lambda\left(\frac{\pi+2\vartheta}{3}\right)+\frac{\lambda(\vartheta)}{8}\right]\right.$$
$$\left.-\frac{\beta(\Delta)}{\Delta(\vartheta-\pi)}\left[\left({}_{\pi}^{CF}I^{\Delta}\lambda\right)(k)+\left({}^{CF}I_{\vartheta}^{\Delta}\lambda\right)(k)\right]+\frac{2(1-\Delta)}{\beta(\Delta)}\lambda(k)\right|$$
$$\leq (\vartheta-\pi)\left[\left(\frac{17}{576}\right)\left(\frac{973\left|\lambda'(\pi)\right|^{q}+251\left|\lambda'(\vartheta)\right|^{q}}{1224}\right)^{\frac{1}{q}}+\frac{1}{36}\left(\frac{\left|\lambda'(\pi)\right|^{q}+\left|\lambda'(\vartheta)\right|^{q}}{2}\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(\frac{17}{576}\right)\left(\frac{251\left|\lambda'(\pi)\right|^{q}+973\left|\lambda'(\vartheta)\right|^{q}}{1224}\right)^{\frac{1}{q}}\right].$$

Remark 3.9. If we put $\Delta=1$ and $\beta(0)=\beta(1)=1$, in Corollary 3.8, then Corollary 3.8 reduces to [15, Corollary 3.3].

4. Simpson' s 3/8 Formula-Type Inequalities For Bounded Function

In this section, we introduce the Simpson's 3/8 formula type inequalities for the differentiable Bounded function.

Theorem 4.1. *Let $\lambda:[0,\infty)\to\mathbb{R}$, $\pi,\vartheta\in\mathbb{R}^{+}$, $\pi<\vartheta$. If there exits constants $m\leq\lambda'(x)\leq M$ for all $x\in[\pi,\vartheta]$, then the following fractional inequality holds:*

$$\left|\left[\frac{\lambda(\pi)}{8}+\frac{3}{8}\lambda\left(\frac{2\pi+\vartheta}{3}\right)+\frac{3}{8}\lambda\left(\frac{\pi+2\vartheta}{3}\right)+\frac{\lambda(\vartheta)}{8}\right]\right.$$
$$\left.-\frac{\beta(\Delta)}{\Delta(\vartheta-\pi)}\left[\left({}_{\pi}^{CF}I^{\Delta}\lambda\right)(k)+\left({}^{CF}I_{\vartheta}^{\Delta}\lambda\right)(k)\right]+\frac{2(1-\Delta)}{\beta(\Delta)}\lambda(k)\right|$$
$$\leq \frac{25(\vartheta-\pi)}{576}(M-m).$$

Proof. By using the Lemma 2.1, we have

$$\begin{aligned}
&\left|\left[\frac{\lambda(\pi)}{8}+\frac{3}{8}\lambda\left(\frac{2\pi+\vartheta}{3}\right)+\frac{3}{8}\lambda\left(\frac{\pi+2\vartheta}{3}\right)+\frac{\lambda(\vartheta)}{8}\right]\right.\\
&-\frac{\beta(\Delta)}{\Delta(\vartheta-\pi)}\left[\left({}_{\pi}^{CF}I^{\Delta}\lambda\right)(k)+\left({}^{CF}I_{\vartheta}^{\Delta}\lambda\right)(k)\right]+\frac{2(1-\Delta)}{\beta(\Delta)}\lambda(k)\\
=&\ \frac{(\vartheta-\pi)}{9}\left[\int_0^1\left(\Pi-\frac{3}{8}\right)\lambda'\left((1-\Pi)\pi+\Pi\frac{2\pi+\vartheta}{3}\right)d\Pi\right.\\
&+\int_0^1\left(\Pi-\frac{1}{2}\right)\lambda'\left((1-\Pi)\frac{2\pi+\vartheta}{3}+\Pi\frac{\pi+2\vartheta}{3}\right)d\Pi\\
&\left.+\int_0^1\left(\Pi-\frac{5}{8}\right)\lambda'\left((1-\Pi)\frac{\pi+2\vartheta}{3}+\Pi\vartheta\right)d\Pi\right]\\
=&\ \frac{(\vartheta-\pi)}{9}\left[\int_0^1\left(\Pi-\frac{3}{8}\right)\lambda'\left(\frac{3-\Pi}{3}\pi+\frac{\Pi}{3}\vartheta\right)d\Pi\right.\\
&+\int_0^1\left(\Pi-\frac{1}{2}\right)\lambda'\left(\frac{2-\Pi}{3}\pi+\frac{1+\Pi}{3}\vartheta\right)\\
&\left.+\int_0^1\left(\Pi-\frac{5}{8}\right)\lambda'\left(\frac{1-\Pi}{3}\pi+\frac{2+\Pi}{3}\vartheta\right)d\Pi\right]\\
=&\ \frac{(\vartheta-\pi)}{9}\left[\int_0^1\left(\Pi-\frac{3}{8}\right)\left(\lambda'\left(\left(\frac{3-\Pi}{3}\right)\pi+\left(\frac{\Pi}{3}\right)\vartheta\right)-\frac{m+M}{2}\right)d\Pi\right.\\
&+\int_0^1\left(\Pi-\frac{1}{2}\right)\left(\lambda'\left(\left(\frac{2-\Pi}{3}\right)\pi+\left(\frac{1+\Pi}{3}\right)\vartheta\right)-\frac{m+M}{2}\right)d\Pi\\
&\left.+\int_0^1\left(\Pi-\frac{5}{8}\right)\left(\lambda'\left(\left(\frac{1-\Pi}{3}\right)\pi+\left(\frac{2+\Pi}{3}\right)\vartheta\right)-\frac{m+M}{2}\right)d\Pi\right].
\end{aligned}\tag{8}$$

Employing the absolute value of the equality (8), we get

$$\begin{aligned}
&\left|\left[\frac{\lambda(\pi)}{8}+\frac{3}{8}\lambda\left(\frac{2\pi+\vartheta}{3}\right)+\frac{3}{8}\lambda\left(\frac{\pi+2\vartheta}{3}\right)+\frac{\lambda(\vartheta)}{8}\right]\right.\\
&\left.-\frac{\beta(\Delta)}{\Delta(\vartheta-\pi)}\left[\left({}_{\pi}^{CF}I^{\Delta}\lambda\right)(k)+\left({}^{CF}I_{\vartheta}^{\Delta}\lambda\right)(k)\right]+\frac{2(1-\Delta)}{\beta(\Delta)}\lambda(k)\right|\\
\le&\ \frac{(\vartheta-\pi)}{9}\left[\int_0^1\left|\Pi-\frac{3}{8}\right|\left|\lambda'\left(\left(\frac{3-\Pi}{3}\right)\pi+\left(\frac{\Pi}{3}\right)\vartheta\right)-\frac{m+M}{2}\right|d\Pi\right.\\
&+\int_0^1\left|\Pi-\frac{1}{2}\right|\left|\lambda'\left(\left(\frac{2-\Pi}{3}\right)\pi+\left(\frac{1+\Pi}{3}\right)\vartheta\right)-\frac{m+M}{2}\right|d\Pi\\
&\left.+\int_0^1\left|\Pi-\frac{5}{8}\right|\left|\lambda'\left(\left(\frac{1-\Pi}{3}\right)\pi+\left(\frac{2+\Pi}{3}\right)\vartheta\right)-\frac{m+M}{2}\right|d\Pi\right].
\end{aligned}\tag{9}$$

Since $m\le\lambda'(x)\le M$ for all $x\in[\pi,\vartheta]$, we have

$$\left|\lambda'\left(\left(\frac{3-\Pi}{3}\right)\pi+\left(\frac{\Pi}{3}\right)\vartheta\right)-\frac{m+M}{2}\right|\le\frac{M-m}{2},\tag{10}$$

$$\left|\lambda'\left(\left(\frac{2-\Pi}{3}\right)\pi+\left(\frac{1+\Pi}{3}\right)\vartheta\right)-\frac{m+M}{2}\right|\le\frac{M-m}{2},\tag{11}$$

similarly, we get

$$\left|\lambda'\left(\left(\frac{1-\Pi}{3}\right)\pi+\left(\frac{2+\Pi}{3}\right)\vartheta\right)-\frac{m+M}{2}\right|\leq\frac{M-m}{2}. \tag{12}$$

Using the inequalities (10)-(12) in (9), we have

$$\begin{aligned}&\left|\left[\frac{\lambda(\pi)}{8}+\frac{3}{8}\lambda\left(\frac{2\pi+\vartheta}{3}\right)+\frac{3}{8}\lambda\left(\frac{\pi+2\vartheta}{3}\right)+\frac{\lambda(\vartheta)}{8}\right]\right.\\&\left.-\frac{\beta(\Delta)}{\Delta(\vartheta-\pi)}\left[\left({}^{CF}_{\pi}I^{\Delta}\lambda\right)(k)+\left({}^{CF}I^{\Delta}_{\vartheta}\lambda\right)(k)\right]+\frac{2(1-\Delta)}{\beta(\Delta)}\lambda(k)\right|\\\leq\;&\frac{(\vartheta-\pi)(M-m)}{18}\left[\int_0^1\left|\Pi-\frac{3}{8}\right|+\int_0^1\left|\Pi-\frac{1}{2}\right|+\int_0^1\left|\Pi-\frac{5}{8}\right|\right]\\\leq\;&\frac{25(\vartheta-\pi)}{576}(M-m).\end{aligned}$$

This completes the proof. □

5. Application to special means

(a) The Arithmetic mean:

$$A=A(\pi,\vartheta):=\frac{\pi+\vartheta}{2},\ \pi,\vartheta\in\mathbb{R};$$

(b) The Logarithmic mean:

$$L=L(\pi,\vartheta):=\frac{\vartheta-\pi}{\ln\vartheta-\ln\pi},\ \pi,\vartheta\in\mathbb{R},\ \pi\neq\vartheta;$$

(c) The Generalized Logarithmic-mean:

$$L_r=L_r(\pi,\vartheta):=\left[\frac{\vartheta^{r+1}-\pi^{r+1}}{(r+1)(\vartheta-\pi)}\right]\ r\in\mathbb{R}\backslash\{-1,0\},\ \pi,\vartheta\in\mathbb{R},\ \pi\neq\vartheta;$$

Proposition 5.1. *Let $\pi,\vartheta\in\mathbb{R}$ with $0<\pi<\vartheta$, we have*

$$\begin{aligned}&\left|\frac{1}{4}A(\pi^n,\vartheta^n)+\frac{2^{n-3}}{3^{n-1}}\left(A^n(2\pi,\vartheta)+A^n(\pi,2\vartheta)\right)-L_n^n(\pi,\vartheta)\right|\\\leq\;&\frac{25n(\vartheta-\pi)}{576}\left[\pi^{n-1}+\vartheta^{n-1}\right].\end{aligned}$$

Proof. The assertion follows from Corollary 3.2 $\lambda(x)=x^n$, $\Delta=1$, and $\beta(0)=\beta(1)=1$. □

Proposition 5.2. *Let $\pi,\vartheta\in\mathbb{R}$ with $0<\pi<\vartheta$, we have*

$$\begin{aligned}&\left|A\left(\pi^3,\vartheta^3\right)+3A\left(\left(\frac{2\pi+\vartheta}{3}\right)^3,\left(\frac{\pi+2\vartheta}{3}\right)^3\right)-4L_3^3(\pi,\vartheta)\right|\\\leq\;&\frac{25}{24}\left(A\left(\pi^3,\vartheta^3\right)-G^2(\pi,\vartheta)A(\pi,\vartheta)\right).\end{aligned}$$

Proof. The assertion follows from Theorem 4.1 $\lambda(x) = x^3, \Delta = 1$, and $\beta(0) = \beta(1) = 1$. □

6. Conclusion

Fractional calculus developing efficient and accurate numerical methods for solving integral inequalities. In this article, we have established the new identity for the Caputo-Fabrizio fractional integral operator. Employing this new identity of Simpson's second formula type inequalities for s-convex functions are obtained. By using this novel identity, the present error bounds of Simpson's second formula type inequalities are improved. Moreover, we also include the applications to special means. In the future, scholars may expand this work with modified Caputo-Fabrizio fractional operators and modified A-B fractional operators.

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