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A NEW STUDY ON SIMPSON'S TYPE INEQUALITIES VIA GENERALIZED CONVEXITY WITH APPLICATION

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Abstract. Convexity plays a crucial role in the development of fractional integral inequalities. A large number of fractional integral inequalities are obtained by use of convexity methods and attributes. In this paper, we use generalize convex functions to derive new Simpson's type inequalities. Additionally, several novel connected findings of Simpson's inequality for concave functions are generated. Also, some new applications to specific means are given.

1. Introduction

Research on Simpson's type inequalities and their variants for various convexities has been conducted in the last few years. Convexity is useful in many different areas of study such as data science, machine learning, and coding theory. The theory of mathematical inequality relies heavily on convex mapping because to its extensive use in fields as diverse as mechanics [1], statistics [2], pure and applied mathematics, [3], and economics [4]. Effective and powerful techniques for solving the various issues that emerge in many fields of pure and practical mathematics are provided by convex analysis. For example, in [5] Dragomir demonstrated several inequalities of the Simpson type and provided their applications using quadrature rules. Many authors have used convex functions to prove various kinds of Newton's and Simpson's inequalities due to their extensive applicability. Also for s-convex functions Alomari et al. proved inequalities of the Simpson type in [6]. In subsequent work, Sarikaya et al. demonstrated a convexity-based variant of Simpson's type inequality in [7]. A large number of mathematicians have conducted research in mathematical analysis [8]- [11].

Definition 1.1. A function $\xi : \Phi \neq I \subseteq \mathbb{R} \to \mathbb{R}$ is convex on I, then the following inequality holds:

$$\xi\left(\chi \Delta_1 + (1-\chi)\Delta_2\right) \leq \chi \xi\left(\Delta_1\right) + (1-\chi)\xi\left(\Delta_2\right), \text{ for all } \Delta_1, \Delta_2 \in I, \chi \in [0,1].$$

Definition 1.2. [12] A function $\xi : [0,d] \to \mathbb{R}$, $d > 0$ is called (s,m)-convex functions, where $(s,m) \in [0,1]^2$, if for every $\Delta_1, \Delta_2 \in [0,\infty]$ and $\chi \in [0,1]$, then:

$$\xi\left(\chi \Delta_2 + m(1-\chi)\Delta_1\right) \leq \chi^s \xi\left(\Delta_2\right) + m\left(1-\chi^s\right)\xi\left(\Delta_1\right),$$

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denote by $z_{2m}^{s}(d)$ the class of all (s,m)-convex functions on $[0,d]$ for which $\xi(0)\leq 0$.

Simpson's inequality stated that if $\xi^{(4)}$ exist and is bounded on (Δ_1,Δ_2) as [6]:

$$\left|\int_{\Delta_1}^{\Delta_2}\xi(u)\,du-\frac{(\Delta_2-\Delta_1)}{3}\left[\frac{1}{2}\xi(\Delta_1)+\frac{1}{2}\xi(\Delta_2)+2\xi\left(\frac{\Delta_1+\Delta_2}{2}\right)\right]\right| \leq \frac{(\Delta_2-\Delta_1)^5}{2880}\left\|\xi^{(4)}\right\|_\infty . \tag{1}$$

THEOREM 1.3. [5] *Let $\xi:[\Delta_1,\Delta_2]\to\mathbb{R}$ is a differentiable mapping whose derivative is continuous on (Δ_1,Δ_2) and $\xi'\in L_1[\Delta_1,\Delta_2]$, then the following inequality holds:*

$$\left|\int_{\Delta_1}^{\Delta_2}\xi(u)\,du-\frac{(\Delta_2-\Delta_1)}{3}\left[\frac{1}{2}\xi(\Delta_1)+\frac{1}{2}\xi(\Delta_2)+2\xi\left(\frac{\Delta_1+\Delta_2}{2}\right)\right]\right| \leq \frac{(\Delta_2-\Delta_1)}{3}\|\xi\|_1 . \tag{2}$$

Bound of (2) for L-Lipschitzian was given in [7] by $\frac{5}{36}(\Delta_2-\Delta_1)$.

THEOREM 1.4. *Suppose $\xi:[\Delta_1,\Delta_2]\to\mathbb{R}$ is an absolutely continuous mapping on $[\Delta_1,\Delta_2]$ whose derivative belongs to $L_p[\Delta_1,\Delta_2]$, then the following inequality holds:*

$$\left|\frac{1}{3}\left[\frac{\xi(\Delta_1)+\xi(\Delta_2)}{2}+2\xi\left(\frac{\Delta_1+\Delta_2}{2}\right)\right]-\frac{1}{\Delta_2-\Delta_1}\int_{\Delta_1}^{\Delta_2}\xi(u)du\right| \leq \frac{1}{6}\left[\frac{2^{q+1}+1}{3(q+1)}\right]^{\frac{1}{q}}(\Delta_2-\Delta_1)^{\frac{1}{q}}\left\|\xi'\right\|_p .$$

THEOREM 1.5. [13] *Let $\xi: I\subset\mathbb{R}\to\mathbb{R}$ be a differentiable function on I^0, where $\Delta_1,\Delta_2\in I$ with $\Delta_1<\Delta_2$. If the mapping $|\xi'|$ is convex on $[\Delta_1,\Delta_2]$, then the following inequality holds:*

$$\left|\frac{1}{\Delta_2-\Delta_1}\int_{\Delta_1}^{\Delta_2}\xi(u)du-\xi\left(\frac{\Delta_1+\Delta_2}{2}\right)\right|\leq\frac{\Delta_2-\Delta_1}{8}\left[\left|\xi'(\Delta_1)\right|+\left|\xi'(\Delta_2)\right|\right].$$

In [14], Dragomir et. al presented a result which is given below:

THEOREM 1.6. *Consider the function $\xi:[0,\infty)\to[0,\infty)$ is convex in the 2nd-sense, with $s\in(0,1)$ and let $\Delta_1,\Delta_2\in[0,\infty)$, $\Delta_1<\Delta_2$ and $m\in(0,1)$. If $\xi\in L_1[m\Delta_1,\Delta_2]$, then the following inequality holds:*

$$2^{s-1}\xi\left(\frac{m\Delta_1+\Delta_2}{2}\right)\leq\frac{1}{m\Delta_2-\Delta_1}\int_{m\Delta_1}^{\Delta_2}\xi(x)dx\leq\frac{\xi(m\Delta_1)+\xi(\Delta_2)}{s+1}.$$

In [15], Sarikaya presented the following inequality as follows.

THEOREM 1.7. *Let $I\subset\mathbb{R}$ be an open interval, $\Delta_1,\Delta_2\in I$ with $\Delta_1<\Delta_2$ and $\xi:[\Delta_1,\Delta_2]\to\mathbb{R}$ be a differentiable function such that ξ' is integrable and $0<s\leq 1$ on*

(Δ_1, Δ_2) *with* $\Delta_1 < \Delta_2$. *If* $|\xi'|$ *is s-convex on* $[\Delta_1, \Delta_2]$, *then the following inequality holds:*

$$\left|\frac{1}{6}\left[\xi(\Delta_1) + 4\xi\left(\frac{\Delta_1+\Delta_2}{2}\right) + \xi(\Delta_2)\right] - \frac{1}{\Delta_2-\Delta_1}\int_{\Delta_1}^{\Delta_2}\xi(x)dx\right|$$

$$\leq (\Delta_2-\Delta_1)\frac{2(5)^{s+2} + (s-4)6^{s+1} - 2(3)^{s+2} + 2}{6^{s+2}(s+1)(s+2)}\left[|\xi'(\Delta_2)| + |\xi'(\Delta_1)|\right]. \tag{3}$$

THEOREM 1.8. *Let* $f : I \subset [0, \infty) \to \mathbb{R}$ *be a differentiable mapping on* I^0 *such that* $\xi' \in L_1[\Delta_1, \Delta_2]$ *where* $\Delta_1, \Delta_2 \in I$ *with* $\Delta_1 < \Delta_2$. *If* $|\xi'|^q$ *is s-convex on* $[\Delta_1, \Delta_2]$, *for some fixed* $s \in (0, 1]$ *and* $q \geq 1$, *then the following inequality holds:*

$$\left|\frac{1}{6}\left[\xi(\Delta_1) + 4\xi\left(\frac{\Delta_1+\Delta_2}{2}\right) + \xi(\Delta_2)\right] - \frac{1}{\Delta_2-\Delta_1}\int_{\Delta_1}^{\Delta_2}\xi(x)dx\right|$$

$$\leq \frac{(\Delta_2-\Delta_1)}{2}\left(\frac{5}{36}\right)^{1-1/q}$$

$$\times\left\{\left(\frac{2(5)^{s+2} + (s-4)6^{s+1} - (2s+7)3^{s+1}}{3\times(6)^{s+1}(s+1)(s+2)}u_1|\xi'(\Delta_2)|^q + \frac{2+(2s+1)3^{s+1}}{3\times(6)^{s+1}(s+1)(s+2)}u_2|\xi'(\Delta_1)|^q\right)^{1/q}\right.$$

$$\left.+\left(\frac{2+(2s+1)3^{s+1}}{3\times(6)^{s+1}(s+1)(s+2)}u_3|\xi'(\Delta_2)|^q + \frac{2(5)^{s+2} + (s-4)6^{s+1} - (2s+7)3^{s+1}}{3\times(6)^{s+1}(s+1)(s+2)}u_4|\xi'(\Delta_1)|^q\right)^{1/q}\right\},$$

where

$$u_1 = \frac{2(5)^{s+2} + (s-4)6^{s+1} - (2s+7)3^{s+1}}{3\times(6)^{s+1}(s+1)(s+2)},$$
$$u_2 = \frac{2+(2s+1)3^{s+1}}{3\times(6)^{s+1}(s+1)(s+2)},$$

$$u_3 = \frac{2+(2s+1)3^{s+1}}{3\times(6)^{s+1}(s+1)(s+2)},$$
$$u_4 = \frac{2(5)^{s+2} + (s-4)6^{s+1} - (2s+7)3^{s+1}}{3\times(6)^{s+1}(s+1)(s+2)}.$$

For some inequalities related to Simpson's type are given in ([16]- [20] and [21]- [24]).

The main purpose of this article is to develop new inequalities of Simpson's type for (s, m)-convexity. Some new refined results related to concave functions are also derived.

2. Main Results

In order to prove our main results, we need the following Lemma.

LEMMA 2.1. *Let* $\xi : [0, \infty) \to \mathbb{R}$ *be differentiable mapping on* I^0 *such that* $\xi' \in L_1[\Delta_1, \Delta_2]$ *where* $\Delta_1, \Delta_2 \in I$ *with* $\Delta_1 < \Delta_2$ *and* $z_2, z_1 \in \mathbb{R}$, *then we have following* *equality:*

$$\left[z_1\xi(m\Delta_1)+(1-z_2)\xi(\Delta_2)+(z_2-z_1)\,\xi\left(\frac{\Delta_2+m\Delta_1}{2}\right)\right]-\frac{1}{\Delta_2-m\Delta_1}\int_{m\Delta_1}^{\Delta_2}\xi\,(x)\,dx$$
$$=(\Delta_2-m\Delta_1)\left[\int_0^{1/2}(\chi-z_1)\xi'(\chi\Delta_2+m(1-\chi)\Delta_1)d\chi\right.$$
$$\left.+\int_{1/2}^1(\chi-z_2)\,\xi'(\chi\Delta_2+m(1-\chi)\Delta_1)d\chi\right].$$

Proof. By using an integration by parts and changing the variables, we can obtain our required result. □

THEOREM 2.2. *Under the assumption of Lemma 2.1 holds. If the mapping $|\xi'|$ is (s,m)-convex on $[\Delta_1,\Delta_2]$ for some $(s,m)\in[0,1]^2$, then we have following inequality:*

$$\left|\left[z_1\xi(m\Delta_1)+(1-z_2)\xi(\Delta_2)+(z_2-z_1)\,\xi\left(\frac{\Delta_2+m\Delta_1}{2}\right)\right]-\frac{1}{\Delta_2-m\Delta_1}\int_{m\Delta_1}^{\Delta_2}\xi\,(x)\,dx\right|$$
(4)
$$\leq(\Delta_2-m\Delta_1)\left[v_1\left|\xi'\right|(\Delta_2)+mv_2\left|\xi'\Delta_1\right|\right],$$

where

$$v_1=\frac{s-sz_2+2z_2^{s+2}+2^{-s-1}(s-(s+2)(z_2+z_1)+1)-2z_2+2z_1^{s+2}+1}{(s+1)(s+2)}$$
$$v_2=\frac{4z_1(2z_1-1)+4z_2(2z_2-3)+6}{8}$$
$$-\frac{(s-sz_2+2z_1{}^{s+2}+2^{-s-1}(s-(s+2)(z_1+z_2)+1)+2z_2{}^{s+2}-2z_2+1)}{(s+1)(s+2)}.$$

Proof. Taking modulus of Lemma 2.1, and by using (s,m) convexity we get

$$\left|\left[z_1\xi(m\Delta_1)+(1-z_2)\xi(\Delta_2)+(z_2-z_1)\,\xi\left(\frac{\Delta_2+m\Delta_1}{2}\right)\right]-\frac{1}{\Delta_2-m\Delta_1}\int_{m\Delta_1}^{\Delta_2}\xi\,(x)\,dx\right|$$
$$\leq(\Delta_2-m\Delta_1)\int_0^{1/2}|\chi-z_1|\,\xi'(\chi\Delta_2+m(1-\chi)\Delta_1)d\chi$$
$$+(\Delta_2-m\Delta_1)\int_{1/2}^1|\chi-z_2|\,\xi'(\chi\Delta_2+m(1-\chi)\Delta_1)d\chi$$
$$\leq(\Delta_2-m\Delta_1)\int_0^{1/2}|\chi-z_1|\left[\chi^s\left|\xi'(\Delta_2)\right|+m(1-\chi^s)\left|\xi'(\Delta_1)\right|\right]d\chi$$
$$+(\Delta_2-m\Delta_1)\int_{1/2}^1|\chi-z_2|\left[\chi^s\left|\xi'(\Delta_2)\right|+m(1-\chi^s)\left|\xi'(\Delta_1)\right|\right]d\chi.$$

By simple calculations, we have

$$\int_0^{1/2}|\chi-z_1|\,\chi^s d\chi+\int_{1/2}^1|\chi-z_2|\,\chi^s d\chi$$
$$=\frac{s-sz_2+2z_2^{s+2}+2^{-s-1}(s-(s+2)(z_2+z_1)+1)-2z_2+2z_1^{s+2}+1}{(s+1)(s+2)},$$

and

$$\begin{aligned}&\int_0^{1/2} |\chi - z_1| (1-\chi^s) d\chi + \int_{1/2}^1 |\chi - z_2| (1-\chi^s) d\chi \\&= \frac{4z_1(2z_1-1)+4z_2(2z_2-3)+6}{8} \\&- \frac{s - sz_2 + 2z_2^{s+2} + 2^{-s-1}(s-(s+2)(z_2+z_1)+1) - 2z_2 + 2z_1^{s+2}+1}{(s+1)(s+2)}.\end{aligned}$$

Hence proved. □

Corollary 2.3. *Under the assumption of inequality (4), Putting* $z_2 = 5/6$ *and* $z_1 = 1/6$, *we get*

$$\begin{aligned}&\left|\left[\frac{1}{6}\xi(m\Delta_1) + (1-\frac{5}{6})\xi(\Delta_2) + \left(\frac{5}{6}-\frac{1}{6}\right)\xi\left(\frac{\Delta_2+m\Delta_1}{2}\right)\right] - \frac{1}{\Delta_2 - m\Delta_1}\int_{m\Delta_1}^{\Delta_2} \xi(x)dx\right| \\&\leq (\Delta_2 - m\Delta_1)\left[v_3 |\xi'| (\Delta_2) + m v_4 |\xi' \Delta_1|\right],\end{aligned}$$

where

$$\begin{aligned}v_3 &= \frac{6^{-s} + (5)^{s+2}6^{-s} - 9(2)^{-s} + 3s - 12}{18(s+1)(s+2)}\ , \\v_4 &= \left(\frac{5}{36} - v_3\right).\end{aligned}$$

Remark 2.4. Putting $s = m = 1$, in the above Corollary 2.3, we obtain

$$\begin{aligned}&\left|\left[\frac{1}{6}\xi(\Delta_1) + (1-\frac{5}{6})\xi(\Delta_2) + \left(\frac{5}{6}-\frac{1}{6}\right)\xi\left(\frac{\Delta_2+\Delta_1}{2}\right)\right] - \frac{1}{\Delta_2 - \Delta_1}\int_{\Delta_1}^{\Delta_2} \xi(x)dx\right| \\&\leq \frac{5(\Delta_2-\Delta_1)}{72}\left[|\xi'| (\Delta_2) + |\xi'| (\Delta_1)\right].\end{aligned}$$

Remark 2.5. Which is proved in [15]. Hence Sarikaya's result [15] is the special case of Theorem 2.2.

Corollary 2.6. *If we taking* $\xi(m\Delta_1) = \xi\left(\frac{\Delta_2+m\Delta_1}{2}\right) = \xi(b)$ *in inequality (4), then we have*

$$\left|\frac{1}{\Delta_2 - m\Delta_1}\int_{m\Delta_1}^{\Delta_2} \xi(x)dx - \xi\left(\frac{\Delta_2+m\Delta_1}{2}\right)\right| \leq (\Delta_2 - m\Delta_1)\left[v_3 |\xi'| (\Delta_2) + m v_4 |\xi'| (\Delta_1)\right].$$

Remark 2.7. If we consider $s = m = 1$, in the above Corollary 2.6, which has proved by Kiramic [25].

Theorem 2.8. *Under the assumption of Lemma 2.1 holds. If the mapping* $|\xi'|^q$ *is* (s,m)*-convex on* $[\Delta_1, \Delta_2]$ *for some fixed* $(s,m) \in [0,1]^2$ *and* $q > 1$ *with* $\frac{1}{p} + \frac{1}{q} = 1$, *then we have following inequality:*

$$\left|\left[z_1\xi(m\Delta_1)+(1-z_2)\xi(\Delta_2)+(z_2-z_1)\,\xi\left(\frac{\Delta_2+m\Delta_1}{2}\right)\right]-\frac{1}{\Delta_2-m\Delta_1}\int_{m\Delta_1}^{\Delta_2}\xi\,(x)\,dx\right|$$
$$\leq(\Delta_2-m\Delta_1)\left(\frac{1}{s+1}\right)^{\frac{1}{q}}\left(z_1^{p+1}+\left(\frac{1-2z_1}{2}\right)^{p+1}\right)^{\frac{1}{p}}\times\left\{\left|\xi'\left(\frac{\Delta_2+m\Delta_1}{2}\right)\right|^q+|\xi'(\Delta_1)|^q\right\}^{\frac{1}{q}}$$
(5)
$$+(\Delta_2-m\Delta_1)\left(\frac{1}{s+1}\right)^{\frac{1}{q}}\left((1-z_2)^{p+1}+\left(\frac{2z_2-1}{2}\right)^{p+1}\right)^{\frac{1}{p}}\times\left\{\left|\xi'\left(\frac{\Delta_2+m\Delta_1}{2}\right)\right|^q+|\xi'(\Delta_2)|^q\right\}^{\frac{1}{q}}.$$

Proof. Using Hölders inequality and Lemma 2.1, we get

$$\left|\left[z_1\xi(m\Delta_1)+(1-z_2)\xi(\Delta_2)+(z_2-z_1)\,\xi\left(\frac{\Delta_2+m\Delta_1}{2}\right)\right]-\frac{1}{\Delta_2-m\Delta_1}\int_{m\Delta_1}^{\Delta_2}\xi\,(x)\,dx\right|$$
$$\leq(\Delta_2-m\Delta_1)\left(\int_0^{1/2}|(\chi-z_1)|^p\,d\chi\right)^{\frac{1}{p}}\left(\int_0^{1/2}|\xi'\left(\chi\Delta_2+m(1-\chi)\,\Delta_1\right|^q d\chi\right)^{\frac{1}{q}}$$
$$+(\Delta_2-m\Delta_1)\left(\int_{1/2}^{1}|(\chi-z_2)|^p\,d\chi\right)^{\frac{1}{p}}\left(\int_{1/2}^{1}|\xi'\left(\chi\Delta_2+m(1-\chi)\,\Delta_1\right|^q d\chi\right)^{\frac{1}{q}}.$$

Also the (s,m)-convexity of $|\xi'|^q$, implies that

$$\int_0^{1/2}|\xi'(\chi\Delta_2+m(1-\chi))\Delta_1|^q\,d\chi \leq \left(\frac{1}{s+1}\right)^{\frac{1}{q}}\left\{\left|\xi'(\frac{\Delta_2+m\Delta_1}{2})\right|^q+|\xi'(\Delta_1)|^q\right\},$$
$$\int_{1/2}^{1}|\xi'(\chi\Delta_2+m(1-\chi))\Delta_1|^q\,d\chi \leq \left(\frac{1}{s+1}\right)^{\frac{1}{q}}\left\{\left|\xi'(\frac{\Delta_2+m\Delta_1}{2})\right|^q+|\xi'(\Delta_2)|^q\right\}.$$

Therefore, by using above results, we get our required inequality. □

Corollary 2.9. *Using inequality (5) putting* $z_2=5/6$ *and* $z_1=1/6$ *and* $s=m=1$*, then we have*

$$\left|[\frac{1}{6}\xi(\Delta_1)+(1-\frac{5}{6})\xi(\Delta_2)+\left(\frac{5}{6}-\frac{1}{6}\right)\xi(\frac{\Delta_2+\Delta_1}{2})]-\frac{1}{\Delta_2-\Delta_1}\int_{\Delta_1}^{\Delta_2}\xi(x)dx\right|$$
$$\leq 2^{\frac{-1}{q}}(\Delta_2-\Delta_1)\left(\frac{1+2^{p+1}}{6^{p+1}(p+1)}\right)^{\frac{1}{p}}$$
$$\times\left[\left(|\xi'(\Delta_1)|^q+\left|\xi'\left(\frac{\Delta_1+\Delta_2}{2}\right)\right|^q\right)^{\frac{1}{q}}+\left(\left|\xi'\left(\frac{\Delta_1+\Delta_2}{2}\right)\right|^q+|\xi'(\Delta_2)|^q\right)^{\frac{1}{q}}\right].$$

THEOREM 2.10. *Let ξ be defined as in Theorem 2.8. If $|\xi'|^q$ is (s,m)-convex on $[\Delta_1, \Delta_2]$, for some fixed $(s,m) \in [0,1]^2$ and $q \geq 1$, then we have following inequality:*

$$\left|\left[z_1\xi(m\Delta_1) + (1-z_2)\xi(\Delta_2) + (z_2 - z_1)\xi\left(\frac{\Delta_2 + m\Delta_1}{2}\right)\right] - \frac{1}{\Delta_2 - m\Delta_1}\int_{m\Delta_1}^{\Delta_2} \xi(x)\,dx\right|$$

$$\leq (\Delta_2 - m\Delta_1)\left(\frac{4z_1(2z_1-1)+1}{8}\right)^{1-\frac{1}{q}}\left(v_5|\xi'(\Delta_2)|^q + mv_6|\xi'(\Delta_1)|^q\right)^{\frac{1}{q}}$$

(6)

$$+ (\Delta_2 - m\Delta_1)\left(\frac{4z_2(2z_2-3)+5}{8}\right)^{1-\frac{1}{q}}\left(v_7|\xi'(\Delta_2)|^q + v_8 m|\xi'(\Delta_1)|^q\right)^{\frac{1}{q}},$$

where

$$v_5 = \frac{2z_1^{s+2}}{(s+1)(s+2)} + \frac{\frac{1}{2^{s+2}}\left[(s+1) - 2z_1(s+2)\right]}{(s+1)(s+2)},$$

$$v_6 = \frac{4z_1(2z_1-1)+1}{8} - v_5,$$

$$v_7 = \frac{2z_2^{s+2}}{(s+1)(s+2)} + \frac{\frac{1}{2^{s+2}}\left[(s+1) - 2z_2(s+2)\right]}{(s+1)(s+2)} + \frac{\left[(s+1) - z_2(s+2)\right]}{(s+1)(s+2)},$$

$$v_8 = \frac{4z_2(2z_2-3)+5}{8} - v_7.$$

Proof. By Lemma 2.1 and using power mean inequality, we get

$$\left|\left[z_1\xi(m\Delta_1) + (1-z_2)\xi(\Delta_2) + (z_2 - z_1)\xi\left(\frac{\Delta_2 + m\Delta_1}{2}\right)\right] - \frac{1}{\Delta_2 - m\Delta_1}\int_{m\Delta_1}^{\Delta_2} \xi(x)\,dx\right|$$

$$\leq (\Delta_2 - m\Delta_1)\left(\int_0^{1/2} |(\chi - z_1)|\,d\chi\right)^{1-\frac{1}{q}}\left(\int_0^{1/2} (\chi - z_1)\left|\xi'\left(\chi\Delta_2 + m(1-\chi)\Delta_1\right|^q d\chi\right)^{\frac{1}{q}}$$

$$+ (\Delta_2 - m\Delta_1)\left(\int_{1/2}^{1} |(\chi - z_2)|\,d\chi\right)^{1-\frac{1}{q}}\left(\int_{1/2}^{1} (\chi - z_2)\left|\xi'\left(\chi\Delta_2 + m(1-\chi)\Delta_1\right|^q d\chi\right)^{\frac{1}{q}}.$$

The (s,m)-convexity of $|\xi'|^q$, gives that

$$\int_0^{1/2} |(\chi - z_1)|\,|f'(\chi\Delta_2 + m(1-\chi)\Delta_1|^q\,d\chi \leq v_5|\xi'(\Delta_2)|^q + mv_6|\xi'(\Delta_1)|^q,$$

$$\int_{1/2}^{1} |(\chi - z_2)|\,|f'(\chi\Delta_2 + m(1-\chi)\Delta_1|^q\,d\chi \leq v_7|\xi'(\Delta_2)|^q + mv_8|\xi'(\Delta_1)|^q.$$

By combining above inequalities, we get required inequality. □

Corollary 2.11. *Let ξ be defined as in inequality (4) and $z_2 = 5/6$ and $z_1 = 1/6$ and $s = m = 1$, then for convex functions:*

$$\left|\left[\frac{1}{6}\xi(\Delta_1) + (1-\frac{5}{6})\xi(\Delta_2) + \left(\frac{5}{6}-\frac{1}{6}\right)\xi\left(\frac{\Delta_2+\Delta_1}{2}\right)\right] - \frac{1}{\Delta_2-\Delta_1}\int_{\Delta_1}^{\Delta_2}\xi(x)dx\right|$$
$$\leq (\Delta_2-\Delta_1)\left(\frac{5}{72}\right)^{\frac{1}{p}}$$
$$\times\left[\left(\frac{29}{1296}\left|\xi'\left(\Delta_2\right)\right|^q + \frac{61}{1296}\left|\xi'(\Delta_1)\right|^q + \frac{61}{1296}\left|\xi'\left(\Delta_2\right)\right|^q + \frac{29}{1296}\left|\xi'\left(\Delta_1\right)\right|^q\right)^{\frac{1}{q}}\right].$$

Theorem 2.12. *Let $\xi : I \subset 0,\infty) \to \mathbb{R}$ be a differentiable function on I^0, $\Delta_1, \Delta_2 \in I$ with mapping $|\xi'|$ which is concave on $[\Delta_1, \Delta_2]$, then we have following inequality:*

$$\left|\left[z_1\xi(m\Delta_1) + (1-z_2)\xi(\Delta_2) + (z_2-z_1)\xi\left(\frac{\Delta_2+m\Delta_1}{2}\right)\right] - \frac{1}{\Delta_2-m\Delta_1}\int_{m\Delta_1}^{\Delta_2}\xi\left(x\right)dx\right|$$
$$\leq (\Delta_2-m\Delta_1)\times\left\{\left[\frac{4z_1\left(2z_1-1\right)+1}{8}\right]\left|\xi'\left(\frac{\frac{1-3z_1+8z_1^3}{24}\Delta_2+\frac{2-9z_1+24z_1^2-8z_1^3}{24}\Delta_1}{\frac{4z_1(2z_1-1)+1}{8}}\right)\right|\right.$$
$$(7)\qquad \left.+\left[\frac{4z_2\left(2z_2-3\right)+5}{8}\right]\times\left|\xi'\left(\frac{\frac{9-15z_2+8z_2^3}{24}\Delta_2+\frac{6-21z_2+24z_2^2-8z_2^3}{24}\Delta_1}{\frac{4z_2(2z_2-3)+5}{8}}\right)\right|\right\}.$$

Proof. By concavity of $|\xi'|$, we have

$$|\xi'(\chi\Delta_1 + (1-\chi)\Delta_2| \geq \chi\left|\xi'(\Delta_1)\right| + (1-\chi)\left|\xi'(\Delta_2)\right|.$$

According to Jensens inequality, we have

$$\int_0^{1/2}|(\chi-z_1)|\,\xi'\,|(\chi\Delta_2+(1-\chi)\Delta_1|\,d\chi$$
$$\leq\left(\int_0^{1/2}|\chi-z_1|\,d\chi\right)\left|\xi'\left(\frac{\int_0^{1/2}|\chi-z_1|\,|\chi\Delta_2+(1-\chi)\,\Delta_1|\,d\chi}{\int_0^{1/2}|\chi-z_1|\,d\lambda}\right)\right|$$
$$(8)\qquad =\left[\frac{4z_1\left(2z_1-1\right)+1}{8}\right]\left|\xi'\left(\frac{\frac{1-3z_1+8z_1^3}{24}\Delta_2+\frac{2-9z_1+24z_1^2-8z_1^3}{24}\Delta_1}{\frac{4z_1(2z_1-1)+1}{8}}\right)\right|,$$

and

$$\int_{1/2}^{1}|(\chi-z_2)|\,\xi'(\chi\Delta_2+(1-\chi)\Delta_1)d\chi$$
$$\leq\left(\int_{1/2}^{1}|\chi-z_2|\,d\chi\right)\left|\xi'\left(\frac{\int_{1/2}^{1}|\chi-z_2|\,|\chi\Delta_2+(1-\chi)\,\Delta_1|\,d\chi}{\int_{1/2}^{1}|\chi-z_2|\,d\chi}\right)\right|$$
$$(9)\qquad =\left[\frac{4z_2\left(2z_2-3\right)+5}{8}\right]\left|\xi'\left(\frac{\frac{9-15z_2+8z_2^3}{24}\Delta_2+\frac{6-21z_2+24z_2^2-8z_2^3}{24}\Delta_1}{\frac{4z_2(2z_2-3)+5}{8}}\right)\right|.$$

By combining the above equalities (8) and (9), we have

$$\left|\left[z_1\xi(m\Delta_1)+(1-z_2)\xi(\Delta_2)+(z_2-z_1)\,\xi\left(\frac{\Delta_2+m\Delta_1}{2}\right)\right]-\frac{1}{\Delta_2-m\Delta_1}\int_{m\Delta_1}^{\Delta_2}\xi\left(x\right)dx\right|$$
$$\leq\left(\Delta_2-m\Delta_1\right)$$
$$\times\left\{\left[\frac{4z_1\left(2z_1-1\right)+1}{8}\right]\left|\xi'\left(\frac{\frac{1-3z_1+8z_1{}^3}{24}\Delta_2+\frac{2-9z_1+24z_1{}^2-8z_1{}^3}{24}\Delta_1}{\frac{4z_1(2z_1-1)+1}{8}}\right)\right|\right.$$
$$\left.+\left[\frac{4z_2\left(2z_2-3\right)+5}{8}\right]\times\left|\xi'\left(\frac{\frac{9-15z_2+8z_2{}^3}{24}\Delta_2+\frac{6-21z_2+24z_2{}^2-8z_2{}^3}{24}\Delta_1}{\frac{4z_2(2z_2-3)+5}{8}}\right)\right|\right\}.$$

This completes the proof. □

Corollary 2.13. *By putting $z_2=5/6$, $z_1=1/6$ and $m=1$, in inequality (7), then we have:*

$$\left|\left[\frac{1}{6}f(\Delta_1)+(1-\frac{5}{6})f(\Delta_2)+\left(\frac{5}{6}-\frac{1}{6}\right)f\left(\frac{\Delta_2+\Delta_1}{2}\right)\right]-\frac{1}{\Delta_2-\Delta_1}\int_{\Delta_1}^{\Delta_2}f(x)dx\right|$$
$$\leq\frac{5(\Delta_2-\Delta_1)}{72}\left[\left|\xi'\left(\frac{29\Delta_1+61\Delta_2}{90}\right)\right|+\left|\xi'\left(\frac{61\Delta_1+29\Delta_2}{90}\right)\right|\right]. \tag{10}$$

Inequality (10) is a generalization of obtained inequality as in [6, Theorem 8] .

3. Application to Some Special Means

Now using the inequalities of section 2, some new results are designed for the Arithmetic, Generalized-logarithmic, and Logarithmic means.

Consider

$$\xi:[\Delta_1,\Delta_2]\to\mathbb{R},\quad\left(0<\Delta_1<\Delta_2\right),\,\xi\left(x\right)=x^s,\ s\in\left(0,1\right],$$

then

$$\frac{1}{\Delta_2-\Delta_1}\int_{\Delta_1}^{\Delta_2}\xi\left(x\right)dx=L_s^s\left(\Delta_1,\Delta_2\right),$$
$$\frac{\xi\left(\Delta_1\right)+\xi\left(\Delta_2\right)}{2}=A\left(\Delta_1^s,\Delta_2^s\right),$$
$$\xi\left(\frac{\Delta_1+\Delta_2}{2}\right)=A^s\left(\Delta_1,\Delta_2\right).$$

Under the assumption of Corollary 2.3, we have

$$\left|\frac{1}{3}A\left(\Delta_1^s,\Delta_2^s\right)+\frac{2}{3}A^s\left(\Delta_1,\Delta_2\right)-L_s^s\left(\Delta_1,\Delta_2\right)\right|$$
$$\leq s\left(\Delta_2-m\Delta_1\right)\left[v_1\Delta_2^{(s-1)}+mv_2\Delta_1^{(s-1)}\right],$$

where

$$v_1 = \frac{s - sz_2 + 2z_2^{s+2} + 2^{-s-1}(s-(s+2)(z_2+z_1)+1) - 2z_2 + 2z_1^{s+2} + 1}{(s+1)(s+2)}$$

$$v_2 = \frac{4z_1(2z_1-1)+4z_2(2z_2-3)+6}{8} - \frac{(s - sz_2 + 2{z_1}^{s+2} + 2^{-s-1}(s-(s+2)(z_1+z_2)+1) + 2{z_2}^{s+2} - 2z_2 + 1)}{(s+1)(s+2)}.$$

If $z_2 = 5/6$ and $z_1 = 1/6$, $s = m = 1$, then we have

$$\left|A(\Delta_1, \Delta_2) - L(\Delta_1, \Delta_2)\right| \leq \frac{5}{36}(\Delta_2 - \Delta_1).$$

Under the assumption of Corollary 2.9, we have

$$\left|\frac{1}{3}A(\Delta_1, \Delta_2) + \frac{2}{3}A^s(\Delta_1, \Delta_2) - L_s^s(\Delta_1, \Delta_2)\right|$$
$$\leq 2^{\frac{-1}{q}}(\Delta_2 - \Delta_1)\left(\frac{1+2^{p+1}}{6^{p+1}(p+1)}\right)^{\frac{1}{p}}\left[\left\{\Delta_1^{(s-1)q} + [A(\Delta_1, \Delta_2)]^{(s-1)q}\right\}^{\frac{1}{q}} + \left\{\Delta_2^{(s-1)q} + [A(\Delta_1, \Delta_2)]^{(s-1)q}\right\}^{\frac{1}{q}}\right],$$

where $p > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$. For instance, if $s = 1$ then we have

$$\left|A(\Delta_1, \Delta_2) - L(\Delta_1, \Delta_2)\right| \leq 2(\Delta_2 - \Delta_1)\left(\frac{1+2^{p+1}}{6^{p+1}(p+1)}\right)^{\frac{1}{p}}.$$

(2) Let $f : [a, b] \subseteq [0, \infty) \to \mathbb{R}$, $(0 < a < b)$, $f(x) = \frac{1}{x} \in K_s^2$, $s \in (0, 1]$. Then,

$$\frac{1}{\Delta_2 - \Delta_1}\int_{\Delta_1}^{\Delta_2} \xi(x)\, dx = L_{-s}^{-s}(\Delta_1, \Delta_2),$$
$$\frac{\xi(\Delta_1) + \xi(\Delta_2)}{2} = A\left(\Delta_1^{-s}, \Delta_2^{-s}\right),$$
$$\xi\left(\frac{\Delta_1 + \Delta_2}{2}\right) = A^{-s}(\Delta_1, \Delta_2).$$

Under the assumption of Corollary 2.3, we have

$$\left|\frac{1}{3}A\left(\Delta_1^{-s}, \Delta_2^{-s}\right) + \frac{2}{3}A^{-s}(\Delta_1, \Delta_2) - L_{-s}^{-s}(\Delta_1, \Delta_2)\right|$$
$$\leq s(\Delta_2 - m\Delta_1)\left[v_1 \Delta_2^{(-s-1)} + m v_2 \Delta_1^{(-s-1)}\right],$$

where

$$v_1 = \frac{s - sz_2 + 2z_2^{s+2} + 2^{-s-1}(s - (s+2)(z_2 + z_1) + 1) - 2z_2 + 2z_1^{s+2} + 1}{(s+1)(s+2)}$$

$$v_2 = \frac{4z_1(2z_1 - 1) + 4z_2(2z_2 - 3) + 6}{8} - \frac{(s - sz_2 + 2z_1{}^{s+2} + 2^{-s-1}(s - (s+2)(z_1 + z_2) + 1) + 2z_2{}^{s+2} - 2z_2 + 1)}{(s+1)(s+2)}.$$

If $z_2 = 5/6$ and $z_1 = 1/6$, $s = m = 1$, then we have

$$\left| \frac{1}{3} A\left(\Delta_1^{-1}, \Delta_2^{-1}\right) + \frac{2}{3} A^{-1}\left(\Delta_1, \Delta_2\right) - L_{-1}^{-1}\left(\Delta_1, \Delta_2\right) \right| \leq \frac{5}{36}\left(\Delta_2 - \Delta_1\right) A\left(\Delta_1^{-1}, \Delta_2^{-1}\right).$$

4. Conclusion

As convexity becomes more important in contemporary science and engineering, the study of convexity becomes an increasingly fruitful and dynamic field of study. This work presented new Simpson's inequalities that are derived from generalized convex functions. Additionally, several novel connected findings of Simpson's inequality for concave functions are generated. There are also some novel uses for exceptional real numbers that are covered. The authors might use the novel methods and practical concepts of convexity presented in this study for various fractional integral operators in their future works. Furthermore, numerical integration, optimization, and related fields may find special use for our results.

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