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RULED SURFACES GENERATED BY SALKOWSKI CURVE AND ITS FRENET VECTORS IN EUCLIDEAN 3-SPACE

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Abstract. In present study, we introduce ruled surfaces whose base curve is the Salkowski curve in Euclidean 3-space and whose generating lines consist of the Frenet vectors of this curve (tangent, principal normal and binormal vectors). Then, we produce regular surfaces from a vector with real coefficients, which is a linear combination of these vectors, and we examine some special cases for these surfaces. Moreover, we present some geometric properties and graphics of all these surfaces.

1. Introduction

In differential geometry, surfaces have a substantial place and concepts in various disciplines such as computer graphics, physics,and engineering. Ruled surfaces, one of the most familiar examples of surfaces, were introduced by the 19^{th} century French mathematician G. Monge. A ruled surface is defined as a set of points created by continuously moving a line along a curve. This curve is called the base curve and the line is called the generating line (direction vector) of the ruled surface. For example; while a cylinder and a cone are ruled surfaces, a sphere is not a ruled surface. Ruled surfaces have applications in several disciplines such as kinematics, computer-aided geometric design, and architecture. Some studies on ruled surfaces are [1, 3–6, 12, 18, 22, 23, 25–29]. Another important area in differential geometry is the theory of curves. A smooth transformation of the form $\alpha : I \to R^3$, where I is an open interval of R, is called a curve in R^3. Frenet vectors of a differentiable curve in R^3 are tangent vector $\mathcal{T}$, principal normal vector $\mathcal{N}$, and binormal vector $\mathcal{B}$, [6]. An example of curves in R^3 are helices. An ivy wrapped around a tree or wall, a DNA model, spiral stairs, or the grooves and sets engraved on a screw are all examples of helices. Helices are called curves with constant non-zero curvature and torsion functions. The helix curve was first expressed by Lancret and proved by Sain Venant in 1845. The concept of slant helix was first defined in an article published by Izumiya and Takeuchi, [13]. Other studies on slant helices are [2, 9, 15, 19]. Salkowski curves, which are examples of the slant helices, were defined as a family of curves with constant curvature and non-constant torsion in the work of E. Leopold Salkowski's (1909), [24]. Similarly,

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curves whose curvature is not constant and whose torsion is constant are known as anti-Salkowski curves. Juan Monterde (2009) gave the Frenet vectors of Salkowski curves in his study [20]. Some other studies on Salkowski curves in Euclidean 3-space are [7,8,10,21].

In this study, we first introduce ruled surfaces whose base curve is the Salkowski curve in Euclidean 3-space and whose generating lines consists of Frenet vectors (tangent, principal normal and binormal vectors) of this curve. Then, we also generate ruled surfaces from a vector $X(t) = a\mathcal{T}(t) + b\mathcal{N}(t) + c\mathcal{B}(t)$ with real coefficients a, b, c, which consists of the linear combination of these vectors. Finally, we obtaine from vectors lying in the normal, rectifying and osculating planes of this curve. So, we calculate the equations of normal vectors, striction curves, distribution parameters, tangent and asymptotic planes for all these surfaces. Besides we examine whether the surfaces are developable or not and we provıde their graphs.

2. Preliminaries

For $m \neq \pm\dfrac{1}{\sqrt{3}}, 0 \in \mathbb{R}$ and $n = \dfrac{m}{\sqrt{m^2+1}}$, the family of curves defined by the parametric equation given below are called Salkowski curves in Euclidean 3-space [24]:

$$\begin{aligned}\Upsilon(t) = \frac{n}{4m}\Bigg[& \frac{n-1}{1+2n}\sin((1+2n)t) - \frac{n+1}{1-2n}\sin((1-2n)t) - 2\sin t, \\ & \frac{1-n}{1+2n}\cos((1+2n)t) + \frac{n+1}{1-2n}\cos((1-2n)t) + 2\cos t, \\ & \frac{1}{m}\cos(2nt)\Bigg],\end{aligned} \tag{1}$$

Figure 1.

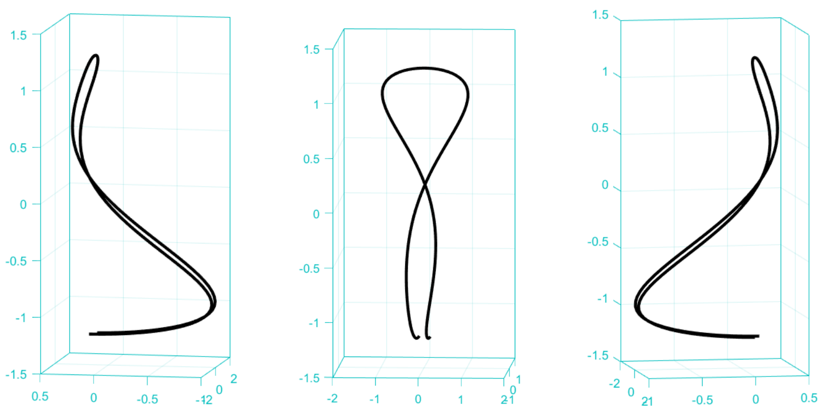

FIGURE 1. Salkowski curve for $m = \frac{1}{5}$.

The curves are regular in the interval of $\left]-\dfrac{\pi}{2n}, \dfrac{\pi}{2n}\right[$. Moreover, $\left\|\Upsilon'(t)\right\| = \dfrac{\cos(nt)}{\sqrt{m^2+1}}$.

Frenet vectors of $\Upsilon(t)$ are [20]:

$$
\begin{cases}
\mathcal{T}(t) = \left(-S(t), \; -R(t), \; -\frac{n}{m}\sin(nt)\right), \\
\mathcal{N}(t) = \left(\frac{n}{m}\sin t, \; -\frac{n}{m}\cos t, \; -n\right), \\
\mathcal{B}(t) = \left(-P(t), \; -Q(t), \; \frac{n}{m}\cos(nt)\right),
\end{cases} \tag{2}
$$

where,

$$
\begin{aligned}
P(t) &= \cos t\sin(nt) - n\sin t\cos(nt), \\
S(t) &= \cos t\cos(nt) + n\sin t\sin(nt), \\
Q(t) &= \sin t\sin(nt) + n\cos t\cos(nt), \\
R(t) &= \sin t\cos(nt) - n\cos t\sin(nt).
\end{aligned}
$$

The first derivatives of Salkowski curve and its Frenet vectors with respect to t are

$$
\Upsilon'(t) = \frac{n}{m}\cos(nt)\left(-S(t), -R(t), -\frac{n}{m}\sin(nt)\right) \tag{3}
$$

and

$$
\begin{cases}
\mathcal{T}'(t) = \frac{n^2}{m^2}\cos(nt)(\sin t, -\cos t, -m), \\
\mathcal{N}'(t) = \frac{n}{m}(\cos t, \sin t, 0), \\
\mathcal{B}'(t) = \frac{n^2}{m^2}\sin(nt)(\sin t, -\cos t, -m),
\end{cases} \tag{4}
$$

respectively.

3. Ruled Surfaces Generated by Salkowski Curve and Its Frenet Vectors in Euclidean 3-Space

In this section, we will examine ruled surfaces whose base curve is the Salkowski curve in Euclidean 3-space and whose generating lines consist of the tangent, principal normal and binormal vectors of this curve.

3.1. Ruled Surfaces Generated by Salkowski Curve and Its Tangent Vector $\mathcal{T}(t)$.

Theorem 3.1. *Let the ruled surface whose base curve is Salkowski curve $\Upsilon(t)$ in Euclidean 3-space and whose generating line is the tangent vector $\mathcal{T}(t)$ of this curve is denoted by $\varphi_{\mathcal{T}}(t, v_{\mathcal{T}})$. The parametric equation of this surface is as follows (Figure 2):*

$$
\begin{aligned}
\varphi_{\mathcal{T}}(t, v_{\mathcal{T}}) = \Bigg[& \frac{n}{4m}\left(\frac{n-1}{1+2n}(\sin(1+2n)t) - \frac{n+1}{1-2n}(\sin(1-2n)t) - 2\sin t\right) - v_{\mathcal{T}}S(t), \\
& \frac{n}{4m}\left(\frac{1-n}{1+2n}(\cos(1+2n)t) + \frac{n+1}{1-2n}(\cos(1-2n)t) + 2\cos t\right) - v_{\mathcal{T}}R(t), \\
& \frac{n}{4m^2}\cos(2nt) - \frac{v_{\mathcal{T}}n}{m}\sin(nt)\Bigg].
\end{aligned} \tag{5}
$$

Proof. The parametric equation of the ruled surface $\varphi_{\mathcal{T}}(t, v_{\mathcal{T}})$ is written as

$$\varphi_{\mathcal{T}}(t, v_{\mathcal{T}}) = \Upsilon(t) + v_{\mathcal{T}}\mathcal{T}(t). \tag{6}$$

If (1) and (2) are substituted in (6), then (5) is obtained. □

THEOREM 3.2. *The normal vector $\eta_{\mathcal{T}}(t)$ of the ruled surface $\varphi_{\mathcal{T}}(t, v_{\mathcal{T}})$ is as follows:*

$$\eta_{\mathcal{T}}(t) = \frac{v_{\mathcal{T}}n}{m}\cos(nt)\left(P(t), Q(t), -\frac{n}{m}\cos(nt)\right). \tag{7}$$

Proof. The normal vector $\eta_{\mathcal{T}}(t)$ of $\varphi_{\mathcal{T}}(t, v_{\mathcal{T}})$ is calculated with

$$\eta_{\mathcal{T}}(t) = (\varphi_{\mathcal{T}})_t(t) \wedge (\varphi_{\mathcal{T}})_{v_{\mathcal{T}}}(t), \tag{8}$$

where, the vector $(\varphi_{\mathcal{T}})_t(t)$ is derivative of $\varphi_{\mathcal{T}}(t, v_{\mathcal{T}})$ with respect to t and the vector $(\varphi_{\mathcal{T}})_{v_{\mathcal{T}}}(t)$ is derivative of $\varphi_{\mathcal{T}}(t, v_{\mathcal{T}})$ with respect to $v_{\mathcal{T}}$. From (3) and (4),

$$(\varphi_{\mathcal{T}})_t(t) = \frac{n}{m}\cos(nt)\left[S(t) - \frac{v_{\mathcal{T}}n}{m}\sin t,\ R(t) + \frac{v_{\mathcal{T}}n}{m}\cos t,\ \frac{n}{m}\sin(nt) + v_{\mathcal{T}}n\right] \tag{9}$$

and from (2),

$$(\varphi_{\mathcal{T}})_{v_{\mathcal{T}}}(t) = \mathcal{T}(t) = -\left(S(t),\ R(t),\ \frac{n}{m}\sin(nt)\right) \tag{10}$$

are obtained. Thus, if (9) and (10) are substituted in (8), then (7) is obtained. □

THEOREM 3.3. *Let the plane have a fixed point $M = (x,\ y,\ z)$ and a variable point $D = (x_0,\ y_0,\ z_0)$. The equation of the tangent plane of the ruled surface $\varphi_{\mathcal{T}}(t, v_{\mathcal{T}})$ is as follows:*

$$(x - x_0)\, mP(t) + (y - y_0)\, mQ(t) - (z - z_0)\, n\cos(nt) = 0.$$

Proof. The equation of the tangent plane of the ruled surface $\varphi_{\mathcal{T}}(t, v_{\mathcal{T}})$ is found by

$$\langle DM,\ \eta_{\mathcal{T}}(t)\rangle = 0. \tag{11}$$

From (7) and (11), the theorem is proved. □

THEOREM 3.4. *The parameter $v_{\mathcal{T}}$ of the striction curve of the ruled surface $\varphi_{\mathcal{T}}(t, v_{\mathcal{T}})$ is as follows:*

$$v_{\mathcal{T}} = 0. \tag{12}$$

Proof. The parameter $v_{\mathcal{T}}$ of the striction curve of $\varphi_{\mathcal{T}}(t, v_{\mathcal{T}})$ is calculated with

$$v_{\mathcal{T}} = -\frac{\langle \mathcal{T}(t) \wedge \mathcal{T}'(t),\ \mathcal{T}(t) \wedge \Upsilon'(t)\rangle}{\langle \mathcal{T}(t) \wedge \mathcal{T}'(t),\ \mathcal{T}(t) \wedge \mathcal{T}'(t)\rangle}. \tag{13}$$

From (2) and (4),

$$\mathcal{T}(t) \wedge \mathcal{T}'(t) = \frac{n}{m}\cos(nt)\left(-P(t),\ -\ Q(t),\ \frac{n}{m}\cos(nt)\right) \tag{14}$$

and from (2) and (3),

$$\mathcal{T}(t) \wedge \Upsilon'(t) = (0, 0, 0) \tag{15}$$

are obtained. Thus, from (14) and (15),

$$\langle \mathcal{T}(t) \wedge \mathcal{T}'(t),\ \mathcal{T}(t) \wedge \Upsilon'(t)\rangle = 0 \tag{16}$$

is obtained. If (16) is substituted in (13), then (12) is obtained. □

THEOREM 3.5. *The parametric equation of the striction curve $\psi_{\mathcal{T}}(t)$ of the ruled surface $\varphi_{\mathcal{T}}(t, v_{\mathcal{T}})$ is as follows:*

$$\begin{aligned}\psi_{\mathcal{T}}(t) \;=\; \frac{n}{4m}\Bigg[&\frac{n-1}{1+2n}(\sin(1+2n)t) - \frac{n+1}{1-2n}(\sin(1-2n)t) - 2\sin t,\\ &\frac{1-n}{1+2n}(\cos(1+2n)t) + \frac{n+1}{1-2n}(\cos(1-2n)t) + 2\cos t,\\ &\frac{1}{m}\cos(2nt)\Bigg].\end{aligned}$$

Proof. The equation of the striction curve $\psi_{\mathcal{T}}(t)$ of $\varphi_{\mathcal{T}}(t, v_{\mathcal{T}})$ is obtained by substituting the parameter $v_{\mathcal{T}}$ into the equation

(17) $$\psi_{\mathcal{T}}(t) = \Upsilon(t) + v_{\mathcal{T}}\mathcal{T}(t).$$

Thus, if (1) and (12) are substituted in (17), then the theorem is proved. □

COROLLARY 3.6. *The striction curve and the base curve (Salkowski curve) of the ruled surface $\varphi_{\mathcal{T}}(t, v_{\mathcal{T}})$ coincide.*

THEOREM 3.7. *Let the plane has a fixed point $M = (x,\ y,\ z)$ and a variable point $D = (x_0,\ y_0,\ z_0)$. The equation of the asymptotic plane of the ruled surface $\varphi_{\mathcal{T}}(t, v_{\mathcal{T}})$ is as follows:*

$$(x - x_0)\, mP(t) + (y - y_0)\, mQ(t) - (z - \ z_0)\, n\cos(nt) = 0.$$

Proof. The normal vector at infinity of $\varphi_{\mathcal{T}}(t, v_{\mathcal{T}})$ is found by $\eta_{\mathcal{T}_\infty}(t) = \mathcal{T}(t) \wedge \mathcal{T}'(t)$. From (14),

(18) $$\eta_{\mathcal{T}_\infty}(t) = -\frac{n}{m}\cos(nt)\left(P(t),\ Q(t),\ \ -\frac{n}{m}\cos(nt)\right)$$

is obtained. The equation of the asymptotic plane of the ruled surface $\varphi_{\mathcal{T}}(t, v_{\mathcal{T}})$ is found by

(19) $$\langle DM,\ \eta_{\mathcal{T}_\infty}(t)\rangle = 0.$$

From (18) and (19), the theorem is proved. □

THEOREM 3.8. *The distribution parameter $\rho_{\mathcal{T}}(t)$ of the ruled surface $\varphi_{\mathcal{T}}(t, v_{\mathcal{T}})$ is as follows:*

$$\rho_{\mathcal{T}}(t) = 0.$$

Proof. The distribution parameter $\rho_{\mathcal{T}}(t)$ of $\varphi_{\mathcal{T}}(t, v_{\mathcal{T}})$ is calculated with

(20) $$\rho_{\mathcal{T}}(t) = \frac{\det\left(\Upsilon'(t),\ \mathcal{T}(t),\ \mathcal{T}'(t)\right)}{\left\|\mathcal{T}'(t)\right\|^2}.$$

From (3) and (14),

(21) $$\det\left(\Upsilon'(t),\ \mathcal{T}(t),\ \mathcal{T}'(t)\right) = 0$$

is obtained. If (21) is substituted in (20), then the theorem is proved. □

COROLLARY 3.9. *The ruled surface $\varphi_{\mathcal{T}}(t, v_{\mathcal{T}})$ is a developable surface.*

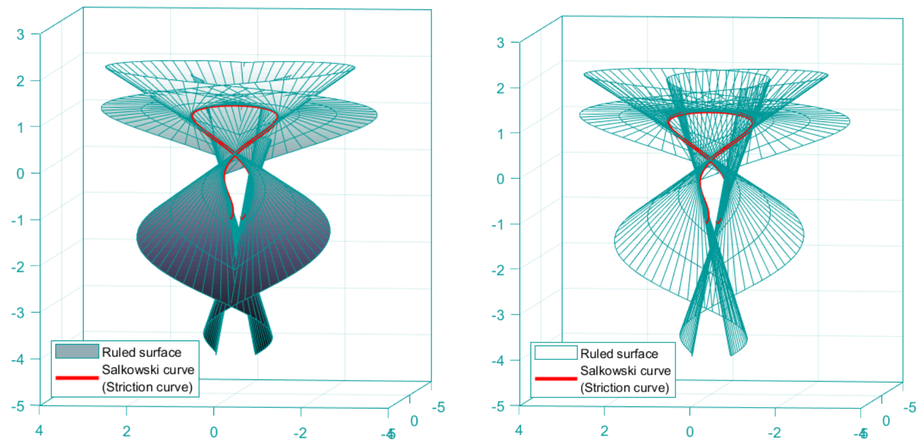

FIGURE 2. Ruled surface generated by Salkowski curve and its tangent vector $\mathcal{T}(t)$ for $m = \frac{1}{5}$. (The right image is the transparent form of the left image.)

3.2. Ruled Surfaces Generated by Salkowski Curve and Its Principal Normal Vector $\mathcal{N}(t)$.

THEOREM 3.10. *Let the ruled surface whose base curve is Salkowski curve $\Upsilon(t)$ in Euclidean 3-space and whose generating line is the principal normal vector $\mathcal{N}(t)$ of this curve is denoted by $\varphi_{\mathcal{N}}(t, v_{\mathcal{N}})$. The parametric equation of this surface is as follows (Figure 3):*

$$\varphi_{\mathcal{N}}(t, v_{\mathcal{N}}) = \frac{n}{4m}\left[\frac{n-1}{1+2n}(\sin(1+2n)t) - \frac{n+1}{1-2n}(\sin(1-2n)t) - 2\sin t + 4v_{\mathcal{N}}\sin t,\right.$$
$$\frac{1-n}{1+2n}(\cos(1+2n)t) + \frac{n+1}{1-2n}(\cos(1-2n)t) + 2\cos t - 4v_{\mathcal{N}}\cos t,$$
$$\left.\frac{1}{m}\cos(2nt) - 4v_{\mathcal{N}}m\right]. \tag{22}$$

Proof. The parametric equation of the ruled surface $\varphi_{\mathcal{N}}(t, v_{\mathcal{N}})$ is written as

$$\varphi_{\mathcal{N}}(t, v_{\mathcal{N}}) = \Upsilon(t) + v_{\mathcal{N}}\mathcal{N}(t). \tag{23}$$

If (1) and (2) are substituted in (23), then (22) is obtained. □

THEOREM 3.11. *The normal vector $\eta_{\mathcal{N}}(t)$ of the ruled surface $\varphi_{\mathcal{N}}(t, v_{\mathcal{N}})$ is as follows:*

$$\eta_{\mathcal{N}}(t) = -\frac{n}{m}\left[P(t)\cos(nt) + v_{\mathcal{N}}n\sin t,\ Q(t)\cos(nt) - v_{\mathcal{N}}n\cos t,\ \frac{n}{m}\left(v_{\mathcal{N}} - \cos^2(nt)\right)\right]. \tag{24}$$

Proof. From (3) and (4),

$$(\varphi_{\mathcal{N}})_t(t) = -\frac{n}{m}\left[S(t)\cos(nt) - v_{\mathcal{N}}\cos t,\ R(t)\cos(nt) - v_{\mathcal{N}}\sin t,\ \frac{n}{m}\cos(nt)\sin(nt)\right], \tag{25}$$

and from (2),

$$(\varphi_{\mathcal{N}})_{v_{\mathcal{N}}}(t) = \mathcal{N}(t) = \frac{n}{m}\left(\sin t,\ -\cos t,\ -\frac{m}{n}\right) \tag{26}$$

are obtained. Thus, if (25) and (26) are substituted in (8), then the theorem is proved. □

THEOREM 3.12. *Let the plane has a fixed point $M = (x,\ y,\ z)$ and a variable point $D = (x_0,\ y_0,\ z_0)$. The equation of the tangent plane of the ruled surface $\varphi_{\mathcal{N}}(t, v_{\mathcal{N}})$ is as follows:*

$$\begin{aligned} &(x - x_0)\, m\, (P(t)\cos(nt) + v_{\mathcal{N}} n \sin t) \\ &+ (y - y_0)\, m\, (Q(t)\cos(nt) - v_{\mathcal{N}} n \cos t) \\ &- (z - z_0)\, n \left(\cos^2(nt) - v_{\mathcal{N}}\right) = 0. \end{aligned}$$

Proof. The equation of the tangent plane of $\varphi_{\mathcal{N}}(t, v_{\mathcal{N}})$ is found by

$$\langle DM,\ \eta_{\mathcal{N}}(t)\rangle = 0. \tag{27}$$

From (24) and (27), the theorem is proved. □

THEOREM 3.13. *The parameter $v_{\mathcal{N}}$ of the striction curve of the ruled surface $\varphi_{\mathcal{N}}(t, v_{\mathcal{N}})$ is as follows:*

$$v_{\mathcal{N}} = \cos^2(nt). \tag{28}$$

Proof. From (2) and (4),

$$\mathcal{N}(t) \wedge \mathcal{N}'(t) = \frac{n^2}{m}\left(\sin t,\quad -\cos t,\ \frac{1}{m}\right) \tag{29}$$

and from (2) and (3),

$$\mathcal{N}(t) \wedge \Upsilon'(t) = \frac{n}{m}\cos(nt)\left(P(t),\ Q(t),\ \ -\frac{n}{m}\cos(nt)\right) \tag{30}$$

are obtained. Thus, from (29) and (30),

$$\langle \mathcal{N}(t) \wedge \mathcal{N}'(t),\ \mathcal{N}(t) \wedge \Upsilon'(t)\rangle = -\frac{n^2}{m^2}\cos^2(nt). \tag{31}$$

and

$$\langle \mathcal{N}(t) \wedge \mathcal{N}'(t),\ \mathcal{N}(t) \wedge \mathcal{N}'(t)\rangle = \frac{n^2}{m^2} \tag{32}$$

are obtained. If (31) and (32) are substituted in (13), then the theorem is proved. □

THEOREM 3.14. *The parametric equation of the striction curve $\psi_{\mathcal{N}}(t)$ of the ruled surface $\varphi_{\mathcal{N}}(t, v_{\mathcal{N}})$ is as follows:*

$$\begin{aligned} \psi_{\mathcal{N}}(t) \quad = \quad \frac{n}{4m}\Bigg[&\frac{n-1}{1+2n}(\sin(1+2n)t) - \frac{n+1}{1-2n}(\sin(1-2n)t) - 2\sin t + 4\cos^2(nt)\sin t, \\ &\frac{1-n}{1+2n}(\cos(1+2n)t) + \frac{n+1}{1-2n}(\cos(1-2n)t) + 2\cos t - 4\cos^2(nt)\cos t, \\ &\frac{1}{m}\cos(2nt) - 4m\cos^2(nt)\Bigg]. \end{aligned}$$

Proof. The equation of the striction curve $\psi_{\mathcal{N}}(t)$ of $\varphi_{\mathcal{N}}(t, v_{\mathcal{N}})$ is obtained by substituting the parameter $v_{\mathcal{N}}$ into the equation

$$\varphi_{\mathcal{N}}(t) = \Upsilon(t) + v_{\mathcal{N}}\mathcal{N}(t). \tag{33}$$

Thus, if (1), (2) and (28) are substituted in (33), then the theorem is proved. □

Corollary 3.15. *Since* $\cos(nt) \neq 0$, *the striction curve and the base curve (Salkowski curve) of the ruled surface* $\varphi_{\mathcal{N}}(t, v_{\mathcal{N}})$ *never coincide.*

Theorem 3.16. *Let the plane has a fixed point* $M = (x,\ y,\ z)$ *and a variable point* $D = (x_0,\ y_0,\ z_0)$. *The equation of the asymptotic plane of the ruled surface* $\varphi_{\mathcal{N}}(t, v_{\mathcal{N}})$ *is as follows:*

$$(x - x_0)\, m \sin t + (y - y_0)\, m \cos t + (z - z_0) = 0.$$

Proof. The normal vector at infinity of $\varphi_{\mathcal{N}}(t, v_{\mathcal{N}})$ is found by $\eta_{\mathcal{N}_\infty}(t) = \mathcal{N}(t) \wedge \mathcal{N}'(t)$. From (29),

$$\eta_{\mathcal{N}_\infty}(t) = -\frac{n}{m}\cos(nt)\left[P(t),\ Q(t),\ -\frac{n}{m}\cos(nt)\right] \tag{34}$$

is obtained. The equation of the asymptotic plane of the ruled surface $\varphi_{\mathcal{N}}(t, v_{\mathcal{N}})$ is found by

$$\langle DM,\ \eta_{\mathcal{N}_\infty}(t)\rangle = 0. \tag{35}$$

From (34) and (35), the theorem is proved. □

Theorem 3.17. *The distribution parameter* $\rho_{\mathcal{N}}(t)$ *of the ruled surface* $\varphi_{\mathcal{N}}(t, v_{\mathcal{N}})$ *is as follows:*

$$\rho_{\mathcal{N}}(t) = -\cos(nt)\sin(nt).$$

Proof. From (3) and (29),

$$\det\left(\Upsilon'(t),\ \mathcal{N}(t),\ \mathcal{N}'(t)\right) = -\frac{n^2}{m^2}\cos(nt)\sin(nt) \tag{36}$$

and from (4),

$$\left\|\mathcal{N}'(t)\right\|^2 = \frac{n^2}{m^2} \tag{37}$$

are obtained. If (36) and (37) are substituted in (70), the theorem is proved. □

Corollary 3.18. *Since* $\cos(nt) \neq 0$ *and* $\sin(nt) \neq 0$, *the ruled surface* $\varphi_{\mathcal{N}}(t, v_{\mathcal{N}})$ *is never a developable surface.*

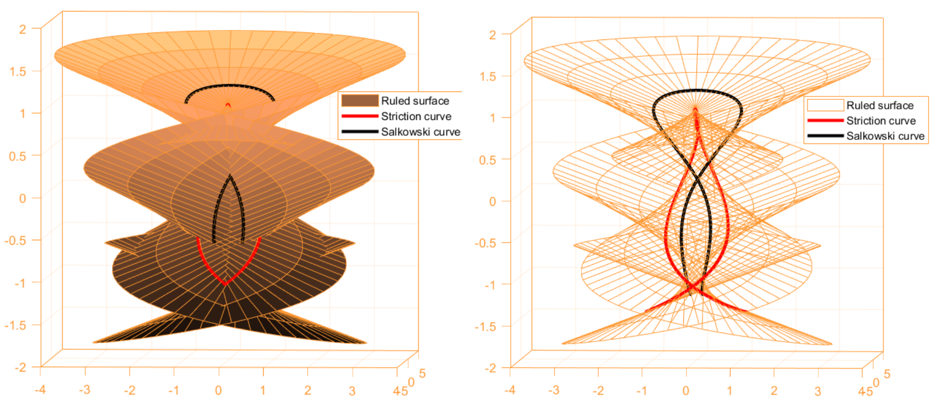

Figure 3. Ruled surface generated by Salkowski curve and its principal normal vector $\mathcal{N}(t)$ for $m = \frac{1}{5}$. (The right image is the transparent form of the left image.)

3.3. Ruled Surfaces Generated by Salkowski Curve and Its Binormal Vector $\mathcal{B}(t)$.

THEOREM 3.19. *Let the ruled surface whose base curve is Salkowski curve $\Upsilon(t)$ in Euclidean 3-space and whose generating line is the binormal vector $\mathcal{B}(t)$ of this curve is denoted by $\varphi_{\mathcal{B}}(t, v_{\mathcal{B}})$. The parametric equation of this surface is as follows (Figure 4):*

$$\begin{aligned} \varphi_{\mathcal{B}}(t, v_{\mathcal{B}}) = & \left[\frac{n}{4m}\left(\frac{n-1}{1+2n}(\sin(1+2n)t) - \frac{n+1}{1-2n}(\sin(1-2n)t) - 2\sin t\right) - v_{\mathcal{B}}P(t),\right. \\ & \frac{n}{4m}\left(\frac{1-n}{1+2n}(\cos(1+2n)t) + \frac{n+1}{1-2n}(\cos(1-2n)t) + 2\cos t\right) - v_{\mathcal{B}}Q(t), \\ & \left.\frac{n}{4m^2}\cos(2nt) + \frac{v_{\mathcal{B}}n}{m}\cos(nt)\right]. \end{aligned} \tag{38}$$

Proof. The parametric equation of the ruled surface $\varphi_{\mathcal{B}}(t, v_{\mathcal{B}})$ is written as

$$\varphi_{\mathcal{B}}(t, v_{\mathcal{B}}) = \Upsilon(t) + v_{\mathcal{B}}\mathcal{B}(t). \tag{39}$$

If (1) and (2) are substituted in (39), then (38) is obtained. □

THEOREM 3.20. *The normal vector $\eta_{\mathcal{B}}(t)$ of the ruled surface $\varphi_{\mathcal{B}}(t, v_{\mathcal{B}})$ is as follows:*

$$\begin{aligned} \eta_{\mathcal{B}}(t) = & -\frac{n}{m}\left[\frac{n}{m}\sin t\cos(nt) + v_{\mathcal{B}}S(t)\sin(nt),\right. \\ & -\frac{n}{m}\cos t\cos(nt) + v_{\mathcal{B}}R(t)\sin(nt), \\ & \left.\frac{n}{m}\left(m\cos(nt) - v_{\mathcal{B}}\sin^2(nt)\right)\right]. \end{aligned} \tag{40}$$

Proof. From (3) and (4),

$$\begin{aligned} (\varphi_{\mathcal{B}})_t(t) = & -\frac{n}{m}\left[S(t)\cos(nt) - \frac{v_{\mathcal{B}}n}{m}\sin t\sin(nt),\right. \\ & R(t)\cos(nt) + \frac{v_{\mathcal{B}}n}{m}\cos t\sin(nt), \\ & \left.\frac{n}{m}\sin(nt)\left(\cos(nt) + v_{\mathcal{B}}m\right)\right] \end{aligned} \tag{41}$$

and from (2),

$$(\varphi_{\mathcal{B}})_{v_{\mathcal{B}}}(t) = \mathcal{B}(t) = \left(-P(t),\ -Q(t),\ \frac{n}{m}\cos(nt)\right) \tag{42}$$

are obtained. Thus, if (41) and (42) are substituted in (8), then the theorem is proved. □

THEOREM 3.21. *Let the plane has a fixed point $M = (x,\ y,\ z)$ and a variable point $D = (x_0,\ y_0,\ z_0)$. The equation of the tangent plane of the ruled surface $\varphi_{\mathcal{B}}(t, v_{\mathcal{B}})$ is as follows:*

$$\begin{aligned} & (x - x_0)\left(n\sin t\cos(nt) + v_{\mathcal{B}}mS(t)\sin(nt)\right) \\ & -(y - y_0)\left(n\cos t\cos(nt) - v_{\mathcal{B}}mR(t)\sin(nt)\right) \\ & -(z - z_0)\left(nm\cos(nt) - v_{\mathcal{B}}n\sin^2(nt)\right) = 0. \end{aligned}$$

Proof. The equation of the tangent plane of $\varphi_{\mathcal{B}}(t, v_{\mathcal{B}})$ is found by

$$\langle DM,\ \eta_{\mathcal{B}}(t)\rangle = 0. \tag{43}$$

From (40) and (43), the theorem is proved. □

Theorem 3.22. *The parameter $v_{\mathcal{B}}$ of the striction curve of the ruled surface $\varphi_{\mathcal{B}}(t, v_{\mathcal{B}})$ is as follows:*

$$v_{\mathcal{B}} = 0. \tag{44}$$

Proof. From (2) and (4),

$$\mathcal{B}(t) \wedge \mathcal{B}'(t) = \frac{n}{m}\sin(nt)\left(S(t),\ R(t),\ \frac{n}{m}\sin(nt)\right) \tag{45}$$

and from (2) and (3),

$$\mathcal{B}(t) \wedge \Upsilon'(t) = \frac{n^2}{m^2}\cos(nt)\,(\sin t,\ \ -\cos t,\ \ -m) \tag{46}$$

are obtained. Thus, from (45) and (46),

$$\langle \mathcal{B}(t) \wedge \mathcal{B}'(t),\ \mathcal{B}(t) \wedge \Upsilon'(t)\rangle = 0 \tag{47}$$

is obtained. If (47) is substituted in (13), then the theorem is proved. □

Theorem 3.23. *The parametric equation of the striction curve $\psi_{\mathcal{B}}(t)$ of the ruled surface $\varphi_{\mathcal{B}}(t, v_{\mathcal{B}})$ is as follows:*

$$\begin{aligned} \psi_{\mathcal{B}}(t) \;=\; \frac{n}{4m}\Bigg[& \frac{n-1}{1+2n}(\sin(1+2n)t) - \frac{n+1}{1-2n}(\sin(1-2n)t) - 2\sin t, \\ & \frac{1-n}{1+2n}(\cos(1+2n)t) + \frac{n+1}{1-2n}(\cos(1-2n)t) + 2\cos t, \\ & \frac{1}{m}\cos(2nt)\Bigg]. \end{aligned}$$

Proof. The equation of the striction curve $\psi_{\mathcal{B}}(t)$ of $\varphi_{\mathcal{B}}(t, v_{\mathcal{B}})$ is obtained by substituting the parameter $v_{\mathcal{B}}$ into the equation

$$\varphi_{\mathcal{B}}(t) = \Upsilon(t) + v_{\mathcal{B}}\mathcal{B}(t). \tag{48}$$

Thus, if (1), (2) and (44) are substituted in (48), then the theorem is proved. □

Corollary 3.24. *The striction curve and the base curve (Salkowski curve) of the ruled surface $\varphi_{\mathcal{B}}(t, v_{\mathcal{B}})$ coincide.*

Theorem 3.25. *Let the plane has a fixed point $M = (x,\ y,\ z)$ and a variable point $D = (x_0,\ y_0,\ z_0)$. The equation of the asymptotic plane of the ruled surface $\varphi_{\mathcal{B}}(t, v_{\mathcal{B}})$ is as follows:*

$$(x - x_0)\, mS(t) + (y - y_0)\, mR(t) + (z - z_0)\, n\sin(nt) = 0.$$

Proof. The normal vector at infinity of $\varphi_{\mathcal{B}}(t, v_{\mathcal{B}})$ is found by $\eta_{\mathcal{B}_\infty}(t) = \mathcal{B}(t) \wedge \mathcal{B}'(t)$. From (45),

$$\eta_{\mathcal{B}_\infty}(t) = \frac{n}{m}\sin(nt)\left(S(t),\ R(t),\ \frac{n}{m}\sin(nt)\right) \tag{49}$$

is obtained. The equation of the asymptotic plane of the ruled surface $\varphi_{\mathcal{B}}(t, v_{\mathcal{B}})$ is found by

$$\langle DM,\ \eta_{\mathcal{B}_\infty}(t)\rangle = 0. \tag{50}$$

From (49) and (50), the theorem is proved. □

THEOREM 3.26. *The distribution parameter* $\rho_{\mathcal{B}}(t)$ *of the ruled surface* $\varphi_{\mathcal{B}}(t, v_{\mathcal{B}})$ *is as follows:*

$$\rho_{\mathcal{B}}(t) = -\frac{\cos(nt)}{\sin(nt)}.$$

Proof. From (3) and (45),

$$\det\left(\Upsilon'(t), \ \mathcal{B}(t), \ \mathcal{B}'(t)\right) = -\frac{n^2}{m^2}\cos(nt)\sin(nt) \tag{51}$$

and from (4),

$$\|\mathcal{B}'(t)\|^2 = \frac{n^2}{m^2}\sin^2(nt) \tag{52}$$

are obtained. If (51) and (52) are substituted in (70), then the theorem is proved. □

COROLLARY 3.27. *Since* $\cos(nt) \neq 0$*, the ruled surface* $\varphi_{\mathcal{B}}(t, v_{\mathcal{B}})$ *is never a developable surface.*

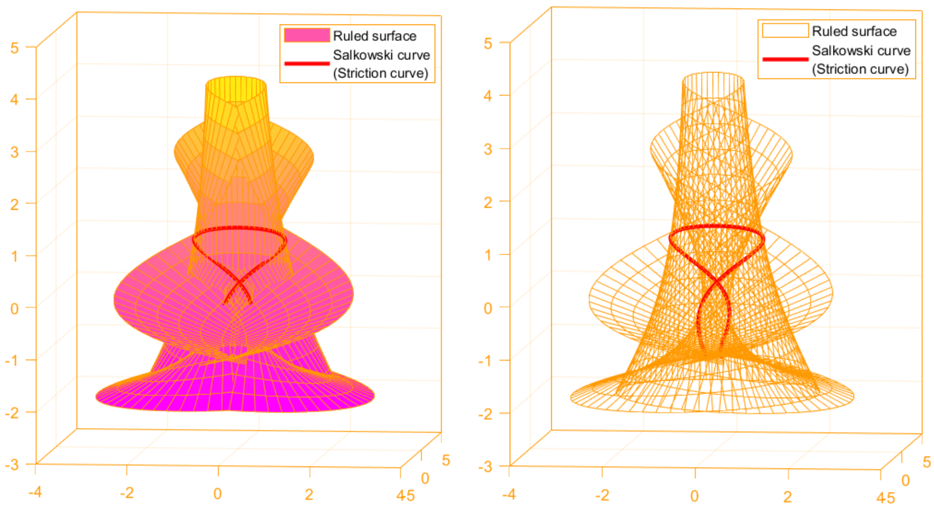

FIGURE 4. Ruled surface generated by Salkowski curve and its binormal vector $\mathcal{B}(t)$ for $m = \frac{1}{5}$. (The right image is the transparent form of the left image.)

4. Ruled Surfaces Generated by Salkowski Curve and Linear Combinations of Its Frenet Vectors in Euclidean 3-Space

In this section, firstly ruled surfaces whose base curve is the Salkowski curve and direction vector is the vector

$$X(t) = a\mathcal{T}(t) + b\mathcal{N}(t) + c\mathcal{B}(t), \quad a, b, c \in \mathbb{R}, \tag{53}$$

are obtained, where $X(t)$ represents the vectors with real coefficients obtained from the linear combinations of the Frenet vectors of the Salkowski curve. Now, let's compute the vector $X(t)$. If the vectors in (2) is substituted in (53),

$$X(t) = -\left(aS(t) + cP(t) - \frac{bn}{m}\sin t, aR(t) + cQ(t) + \frac{bn}{m}\cos t, \frac{n}{m}C(t) + bn\right) \tag{54}$$

is obtained, where

$$C(t) = a\sin(nt) - c\cos(nt).$$

The first derivative of $X(t)$ with respect to t is

$$X'(t) = \frac{n^2}{m^2}\left(A(t)\sin t + b\cos t, -A(t)\cos t + b\sin t, -mA(t)\right), \tag{55}$$

where

$$A(t) = a\cos(nt) + c\sin(nt).$$

THEOREM 4.1. *Let the ruled surface whose base curve is Salkowski curve $\Upsilon(t)$ in Euclidean 3-space and whose generating line is the vector $X(t) = a\mathcal{T}(t) + b\mathcal{N}(t) + c\mathcal{B}(t)$ is denoted by $\varphi_X(t, v_X)$. The parametric equation of this surface is as follows (Figure 5):*

$$\begin{aligned}\varphi_X(t, v_X) \;=\; & \left[\frac{n}{4m}\left(\frac{n-1}{1+2n}\sin((1+2n)t) - \frac{n+1}{1-2n}\sin((1-2n)t) - 2\sin t\right)\right. \\ & -v_X\left(aS(t) + cP(t) - \frac{bn}{m}\sin t\right), \\ & \frac{n}{4m}\left(\frac{1-n}{1+2n}\cos((1+2n)t) + \frac{n+1}{1-2n}\cos((1-2n)t) + 2\cos t\right) \\ & -v_X\left(aR(t) + cQ(t) + \frac{bn}{m}\cos t\right), \\ & \left.\frac{n}{4m^2}\cos(2nt) - v_X n\left(\frac{1}{m}C(t) + b\right)\right].\end{aligned} \tag{56}$$

Proof. The parametric equation of the ruled surface $\varphi_X(t, v_X)$ is written as

$$\varphi_X(t, v_X) = \Upsilon(t) + v_X X(t). \tag{57}$$

If (1) and (54) are substituted in (57), then (56) is obtained. □

THEOREM 4.2. *The normal vector* $\eta_X(t)$ *of the ruled surface* $\varphi_X(t, v_X)$ *is as follows:*

$$\begin{aligned} \eta_X(t) \quad = \quad & -\frac{n}{m}\Bigg(\cos(nt)\left(bP(t)+\frac{cn}{m}\sin t\right) \\ & + v_X\left(bn\sin t\left(\frac{1}{m}C(t)+b\right) - A(t)\left(aP(t)-cS(t)\right)\right), \\ & \cos(nt)\left(bQ(t)-\frac{cn}{m}\cos t\right) \\ & - v_X\left(bn\cos t\left(\frac{1}{m}C(t)+b\right) + A(t)\left(aQ(t)-cR(t)\right)\right), \\ & - n\cos(nt)\left(\frac{b}{m}\cos(nt)+c\right) \\ & - v_X n\left(b\left(C(t)-\frac{b}{m}\right)-\frac{1}{m}A^2(t)\right)\Bigg). \end{aligned} \tag{58}$$

Proof. From (3) and (55),

$$\begin{aligned} (\varphi_X)_t(t) \quad = \quad & -\frac{n}{m}\Bigg(S(t)\cos(nt) - v_X\left(\frac{n}{m}A(t)\sin t + b\cos t\right), \\ & R(t)\cos(nt) + v_X\left(\frac{n}{m}A(t)\cos t - b\sin t\right), \\ & \frac{n}{m}\cos(nt)\sin(nt) + v_X nA(t)\Bigg) \end{aligned} \tag{59}$$

and from (54),

$$(\varphi_X)_{v_X}(t) = -\left(aS(t)+cP(t)-\frac{bn}{m}\sin t,\, aR(t)+cQ(t)+\frac{bn}{m}\cos t,\, \frac{n}{m}C(t)+bn\right) \tag{60}$$

are obtained. Thus, if (59) and (60) are substituted in (8), then (58) is obtained. □

THEOREM 4.3. *Let the plane have a fixed point* $M = (x,\ y,\ z)$ *and a variable point* $D = (x_0,\ y_0,\ z_0)$*. The equation of the tangent plane of the ruled surface* $\varphi_X(t, v_X)$ *is as follows:*

$$\begin{aligned} & (x-x_0)\left[\cos(nt)\left(bP(t)+\frac{cn}{m}\sin t\right) - v_X\left(bn\sin t\left(\frac{1}{m}C(t)+b\right) - A(t)\left(aP(t)-cS(t)\right)\right)\right] \\ & -(y-y_0)\left[\cos(nt)\left(bQ(t)-\frac{cn}{m}\cos t\right) + v_X\left(bn\cos t\left(\frac{1}{m}C(t)+b\right) + A\left(aQ(t)-cR(t)\right)\right)\right] \\ & -(z-z_0)\left[n\cos(nt)\left(c+\frac{b}{m}\cos(nt)\right) + v_X n\left(b\left(C(t)-\frac{b}{m}\right)-\frac{1}{m}A^2(t)\right)\right] = 0. \end{aligned}$$

Proof. The equation of the tangent plane of the ruled surface $\varphi_X(t, v_X)$ is found by

$$\langle DM,\ \eta_X(t)\rangle = 0. \tag{61}$$

From (58) and (61), the theorem is proved. □

THEOREM 4.4. *The parameter* v_X *of the striction curve of the ruled surface* $\varphi_X(t, v_X)$ *is as follows:*

$$v_X = \frac{b\cos^2(nt)}{A^2(t)+b^2}. \tag{62}$$

Proof. From (54) and (55),

$$\begin{aligned} X(t)\wedge X'(t) &= \frac{n}{m}\left(A(t)\left(cS(t)-aP(t)\right)+bn\sin t\left(\frac{1}{m}C(t)+b\right),\right.\\ &\qquad A(t)\left(cR(t)-aQ(t)\right)-bn\cos t\left(\frac{1}{m}C(t)+b\right),\\ &\qquad \left.\frac{n}{m}A^2(t)-bn\left(C(t)-\frac{b}{m}\right)\right) \end{aligned} \tag{63}$$

and from (3) and (54),

$$X(t)\wedge\Upsilon'(t)=\frac{n}{m}\cos(nt)\left(bP(t)+\frac{cn}{m}\sin t,\ bQ(t)-\frac{cn}{m}\cos t,\ -cn-\frac{bn}{m}\cos(nt)\right) \tag{64}$$

are obtained. Thus, from (63) and (64),

$$\langle X(t)\wedge X'(t),\ X(t)\wedge\Upsilon'(t)\rangle=-\frac{bn^2}{m^2}\cos^2(nt)\left(a^2+b^2+c^2\right) \tag{65}$$

and

$$\langle X(t)\wedge X'(t),\ X(t)\wedge X'(t)\rangle=\frac{n^2}{m^2}\left(A^2(t)+b^2\right)\left(a^2+b^2+c^2\right) \tag{66}$$

are obtained. If (65) and (66) are substituted in (13), then (62) is obtained. □

THEOREM 4.5. *The parametric equation of the striction curve $\psi_X(t)$ of the ruled surface $\varphi_X(t,v_X)$ is as follows:*

$$\begin{aligned} \psi_X(t) &= \left[\frac{n}{4m}\left(\frac{n-1}{1+2n}\sin((1+2n)t)-\frac{n+1}{1-2n}\sin((1-2n)t)-2\sin t\right)\right.\\ &\qquad -\frac{b\cos^2(nt)}{A^2(t)+b^2}\left(aS(t)+cP(t)-\frac{bn}{m}\sin t\right),\\ &\qquad \frac{n}{4m}\left(\frac{1-n}{1+2n}\cos((1+2n)t)+\frac{n+1}{1-2n}\cos((1-2n)t)+2\cos t\right)\\ &\qquad -\frac{b\cos^2(nt)}{A^2(t)+b^2}\left(aR(t)+cQ(t)+\frac{bn}{m}\cos t\right),\\ &\qquad \left.\frac{n}{4m^2}\cos(2nt)-\frac{bn\cos^2(nt)}{A^2(t)+b^2}\left(\frac{1}{m}C(t)+b\right)\right]. \end{aligned}$$

Proof. The equation of the striction curve $\psi_X(t)$ of $\varphi_X(t,v_X)$ is obtained by substituting the parameter v_X into the equation

$$\psi_X(t)=\Upsilon(t)+v_X X(t). \tag{67}$$

Thus, if (1), (54) and (62) are substituted in (67), then the theorem is proved. □

COROLLARY 4.6. *If $b=0$, then the striction curve and the base curve (Salkowski curve) of the ruled surface $\varphi_X(t,v_X)$ coincide.*

THEOREM 4.7. *Let the plane has a fixed point $M=(x,\ y,\ z)$ and a variable point $D=(x_0,\ y_0,\ z_0)$. The equation of the asymptotic plane of the ruled surface $\varphi_X(t,v_X)$*

is as follows:

$$\begin{aligned}&(x-x_0)\left[A(t)\left(C(t)\cos t-nA(t)\sin t\right)-bn\sin t\left(\frac{1}{m}C(t)+b\right)\right]\\&+(y-y_0)\left[A(t)\left(C(t)\sin t+nA(t)\cos t\right)-bn\cos t\left(\frac{1}{m}C(t)+b\right)\right]\\&-(z-z_0)\left[n\left(\frac{1}{m}A^2(t)-b\left(C(t)-\frac{b}{m}\right)\right)\right]=0.\end{aligned}$$

Proof. The normal vector at infinity of $\varphi_X(t,v_X)$ is found by $\eta_{X_\infty}(t)=X(t)\wedge X'(t)$. From (63),

$$\begin{aligned}\eta_{X_\infty}(t)&=\frac{n}{m}\Bigg(A(t)\left(cS(t)-aP(t)\right)+bn\sin t\left(\frac{1}{m}C(t)+b\right),\\&\qquad A(t)\left(cR(t)-aQ(t)\right)-bn\cos t\left(\frac{1}{m}C(t)+b\right),\\&\qquad \frac{n}{m}A^2(t)-bn\left(C(t)-\frac{b}{m}\right)\Bigg)\end{aligned}\tag{68}$$

is obtained. The equation of the asymptotic plane of the ruled surface $\varphi_X(t,v_X)$ is found by

$$\langle DM,\ \eta_{X_\infty}(t)\rangle=0.\tag{69}$$

From (68) and (69), the theorem is proved. □

THEOREM 4.8. *The distribution parameter $\rho_X(t)$ of the ruled surface $\varphi_X(t,v_X)$ is as follows:*

$$\rho_X(t)=-\frac{\cos(nt)\left(ac\cos(nt)+(b^2+c^2)\sin(nt)\right)}{A^2(t)+b^2}.$$

Proof. From (3) and (63),

$$\det\left(\Upsilon'(t),\ X(t),\ X'(t)\right)=-\frac{n^2}{m^2}\cos(nt)\left(ac\cos(nt)+\left(b^2+c^2\right)\sin(nt)\right)\tag{70}$$

and

$$\left\|X'(t)\right\|^2=\frac{n^2}{m^2}\left(A^2(t)+b^2\right)\tag{71}$$

are obtained. If (70) and (71) are substituted in (20), then the theorem is proved. □

COROLLARY 4.9. *If $b=c=0$, the ruled surface $\varphi_X(t,v_X)$ is a developable surface.*

Now let's examine some special cases for the vector $X(t)$:

- Let's give the propositions for the ruled surfaces generated by Salkowski curve and the vector

$$X_{\mathcal{NB}}(t)=\left(-cP(t)+\frac{bn}{m}\sin t,\ \ -cQ(t)-\frac{bn}{m}\cos t,\ \frac{cn}{m}\cos(nt)-bn\right)$$

  lying on the normal plane.

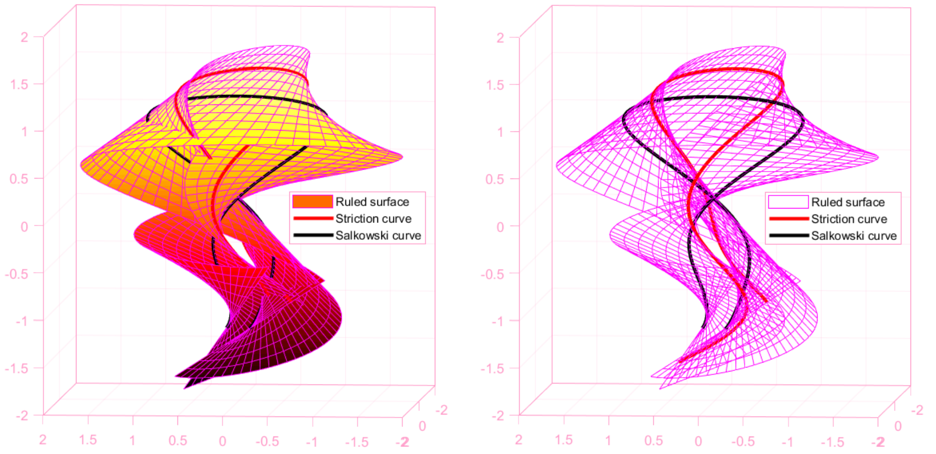

FIGURE 5. Ruled surface generated by Salkowski curve and the vector $X(t)$ for $a = b = c = m = \frac{1}{5}$. (The right image is the transparent form of the left image.)

PROPOSITION 4.1. *Let the ruled surface whose base curve is Salkowski curve $\Upsilon(t)$ in Euclidean 3-space and generating line is the vector $X_{\mathcal{NB}}(t)$ is denoted by $\varphi_{\mathcal{NB}}(t, v_{\mathcal{TB}})$. The parametric equation of this surface is as follows (Figure 6):*

$$\begin{aligned}\varphi_{\mathcal{NB}}(t, v_{\mathcal{NB}}) = \Bigg[& \frac{n}{4m}\left(\frac{n-1}{1+2n}(\sin(1+2n)t) - \frac{n+1}{1-2n}(\sin(1-2n)t) - 2\sin t\right) \\ & + v_{\mathcal{NB}}\left(-cP(t) + \frac{bn}{m}\sin t\right), \\ & \frac{n}{4m}\left(\frac{1-n}{1+2n}(\cos(1+2n)t) + \frac{n+1}{1-2n}(\cos(1-2n)t) + 2\cos t\right) \\ & - v_{\mathcal{NB}}\left(cQ(t) + \frac{bn}{m}\cos t\right), \\ & \frac{n}{4m^2}\cos(2nt) + v_{\mathcal{NB}} n\left(\frac{c}{m}\cos(nt) - b\right)\Bigg].\end{aligned}$$

PROPOSITION 4.2. *The normal vector $\eta_{\mathcal{NB}}(t)$ of the ruled surface $\varphi_{\mathcal{NB}}(t, v_{\mathcal{NB}})$ is as follows:*

$$\begin{aligned}\eta_{\mathcal{NB}}(t) = -\frac{n}{m}\Bigg[& \cos(nt)\left(bP + \frac{cn}{m}\sin t\right) + v_{\mathcal{NB}}\left(bn\sin t\left(b - \frac{c}{m}\cos(nt)\right) + c^2 S\sin(nt)\right), \\ & \cos(nt)\left(bQ - \frac{cn}{m}\cos t\right) - v_{\mathcal{NB}}\left(bn\cos t\left(b - \frac{c}{m}\cos(nt)\right) - c^2 R\sin(nt)\right), \\ & -\cos(nt)\left(\frac{bn}{m}\cos(nt) + cn\right) + v_{\mathcal{NB}} n\left(b\left(c\cos(nt) + \frac{b}{m}\right) + \frac{c^2}{m}\sin^2(nt)\right)\Bigg].\end{aligned}$$

PROPOSITION 4.3. *Let the plane has a fixed point $M = (x,\ y,\ z)$ and a variable point $D = (x_0,\ y_0,\ z_0)$. The equation of the tangent plane of the ruled surface $\varphi_{\mathcal{NB}}(t, v_{\mathcal{NB}})$ is as follows:*

$$\begin{aligned}& (x - x_0)\left[\cos(nt)\left(bmP(t) + cn\sin t\right) + v_{NB}\left(bn\sin t\left(bm - c\cos(nt)\right) + c^2 mS(t)\sin(nt)\right)\right] \\ & + (y - y_0)\left[\cos(nt)\left(bmQ(t) - cn\cos t\right) - v_{NB}\left(bn\cos t\left(bm - c\cos(nt)\right) - c^2 mR\sin(nt)\right)\right] \\ & - (z - z_0)\left[n\cos(nt)\left(b\cos(nt) + cm\right) - v_{NB}\left(bn\left(cm\cos(nt) + b\right) + c^2 n\sin^2(nt)\right)\right] = 0.\end{aligned}$$

PROPOSITION 4.4. *The parameter $v_{\mathcal{NB}}$ of the striction curve of the ruled surface $\varphi_{\mathcal{NB}}(t, v_{\mathcal{NB}})$ is as follows:*

$$v_{\mathcal{NB}} = \frac{b\cos^2(nt)}{b^2 + c^2\sin^2(nt)}.$$

PROPOSITION 4.5. *The parametric equation of the striction curve $\psi_{\mathcal{NB}}(t)$ of the ruled surface $\varphi_{\mathcal{NB}}(t, v_{\mathcal{NB}})$ is as follows:*

$$\begin{aligned}
\psi_{\mathcal{NB}}(t) = & \left[\frac{n}{4m}\left(\frac{n-1}{1+2n}\sin((1+2n)t) - \frac{n+1}{1-2n}\sin((1-2n)t) - 2\sin t\right)\right. \\
& -\frac{b\cos^2(nt)}{b^2+c^2\sin^2(nt)}\left(cP(t) - \frac{bn}{m}\sin t\right), \\
& \frac{n}{4m}\left(\frac{1-n}{1+2n}\cos((1+2n)t) + \frac{n+1}{1-2n}\cos((1-2n)t) + 2\cos t\right) \\
& -\frac{b\cos^2(nt)}{b^2+c^2\sin^2(nt)}\left(cQ(t) + \frac{bn}{m}\cos t\right), \\
& \left.\frac{n}{4m^2}\cos(2nt) - \frac{bn\cos^2(nt)}{b^2+c^2\sin^2(nt)}\left(b - \frac{c}{m}\cos(nt)\right)\right].
\end{aligned}$$

PROPOSITION 4.6. *Let the plane has a fixed point $M = (x,\ y,\ z)$ and a variable point $D = (x_0,\ y_0,\ z_0)$. The equation of the asymptotic plane of the ruled surface $\varphi_{\mathcal{NB}}(t, v_{\mathcal{NB}})$ is as follows:*

$$\begin{aligned}
&(x - x_0)\left[c^2mS(t)\sin(nt) + bn\sin t\,(bm - c\cos(nt))\right] \\
&+ (y - y_0)\left[c^2mR(t)\sin(nt) - bn\cos t\,(bm - c\cos(nt))\right] \\
&+ (z - z_0)\left[n\left(c^2\sin^2(nt) + b\,(b + cm\cos(nt))\right)\right] = 0.
\end{aligned}$$

PROPOSITION 4.7. *The distribution parameter $\rho_{\mathcal{NB}}(t)$ of the ruled surface $\varphi_{\mathcal{NB}}(t, v_{\mathcal{NB}})$ is as follows:*

$$\rho_{\mathcal{NB}}(t) = -\frac{(b^2+c^2)\cos(nt)\sin(nt)}{c^2\sin^2(nt) + b^2}.$$

- Let's give the propositions for the ruled surfaces generated by Salkowski curve and the vector
$$X_{\mathcal{TB}}(t) = -\left(aS(t) + cP(t),\, aR(t) + cQ(t),\ \frac{n}{m}C(t)\right)$$
lying on the rectifying plane.

PROPOSITION 4.8. *Let the ruled surface whose base curve is Salkowski curve $\Upsilon(t)$ in Euclidean 3-space and whose generating line is the vector $X_{\mathcal{TB}}(t)$ is denoted by $\varphi_{\mathcal{TB}}(t, v_{\mathcal{TB}})$. The parametric equation of this surface is as follows (Figure 7):*

$$\begin{aligned}
\varphi_{\mathcal{TB}}(t, v_{\mathcal{TB}}) = & \left[\frac{n}{4m}\left(\frac{n-1}{1+2n)}(\sin(1+2n)t) - \frac{n+1}{1-2n)}(\sin(1-2n)t) - 2\sin t\right)\right. \\
& -v_{\mathcal{TB}}(aS(t) + cP(t)), \\
& \frac{n}{4m}\left(\frac{1-n}{1+2n)}(\cos(1+2n)t) + \frac{n+1}{1-2n)}(\cos(1-2n)t) + 2\cos t\right) \\
& -v_{\mathcal{TB}}(aR(t) + cQ(t)), \\
& \left.\frac{n}{4m^2}\cos(2nt) - \frac{v_{\mathcal{TB}}n}{m}C(t)\right].
\end{aligned}$$

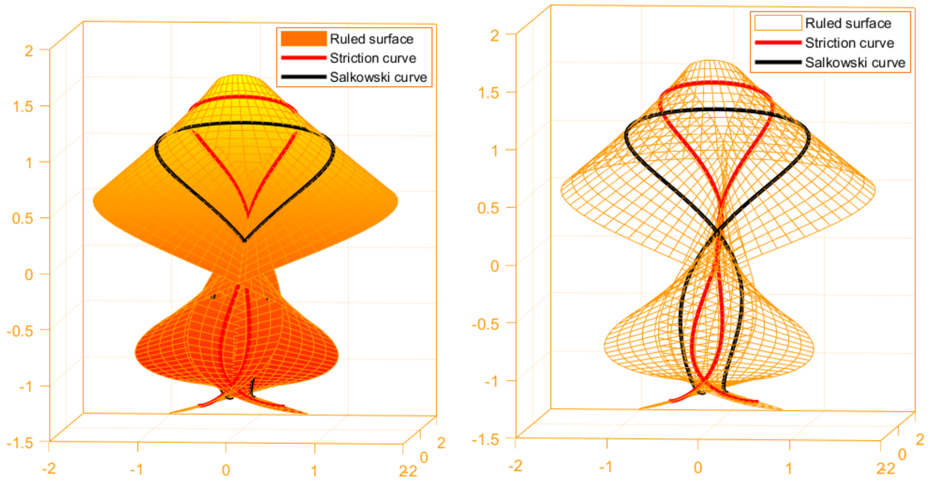

FIGURE 6. Ruled surface generated by Salkowski curve and the vector $X_{\mathcal{NB}}(t)$ for $a=0,\ b=c=m=\frac{1}{5}$. (The right image is the transparent form of the left image.)

PROPOSITION 4.9. *The normal vector $\eta_{\mathcal{TB}}(t)$ of the ruled surface $\varphi_{\mathcal{TB}}(t, v_{\mathcal{TB}})$ is as follows:*

$$\eta_{\mathcal{TB}}(t) = \frac{n}{m}\Big[-\frac{cn}{m}\sin t\cos(nt) + v_{\mathcal{TB}}\left(A(t)\left(aP(t) - cS(t)\right)\right), \frac{cn}{m}\cos t\cos(nt) + v_{\mathcal{TB}}\left(A(t)\left(aQ(t) - cR(t)\right)\right), cn\cos(nt) - \frac{v_{\mathcal{TB}}n}{m}A^2(t)\Big].$$

PROPOSITION 4.10. *Let the plane has a fixed point $M=(x,\ y,\ z)$ and a variable point $D=(x_0,\ y_0,\ z_0)$. The equation of the tangent plane of the ruled surface $\varphi_{\mathcal{TB}}(t, v_{\mathcal{TB}})$ is as follows:*

$$(x-x_0)\left[-cn\sin t\cos(nt) + v_{\mathcal{TB}}m\left(A(t)\left(aP(t)-cS(t)\right)\right)\right] + (y-y_0)\left[cn\cos t\cos(nt) + v_{\mathcal{TB}}m\left(A(t)\left(aQ(t)-cR(t)\right)\right)\right] + (z-z_0)\left[cnm\cos(nt) - v_{\mathcal{TB}}nA^2(t)\right] = 0.$$

PROPOSITION 4.11. *The parameter $v_{\mathcal{TB}}$ of the striction curve of the ruled surface $\varphi_{\mathcal{TB}}(t, v_{\mathcal{TB}})$ is as follows:*

$$v_{\mathcal{TB}} = 0.$$

PROPOSITION 4.12. *The parametric equation of the striction curve $\psi_{\mathcal{TB}}(t)$ of the ruled surface $\varphi_{\mathcal{TB}}(t, v_{\mathcal{TB}})$ is as follows:*

$$\psi_{\mathcal{TB}}(t) = \frac{n}{4m}\Big[\frac{n-1}{1+2n}\sin((1+2n)t) - \frac{n+1}{1-2n}\sin((1-2n)t) - 2\sin t, \frac{1-n}{1+2n}\cos((1+2n)t) + \frac{n+1}{1-2n}\cos((1-2n)t) + 2\cos t, \frac{1}{m}\cos(2nt)\Big].$$

Proposition 4.13. *Let the plane has a fixed point $M = (x,\ y,\ z)$ and a variable point $D = (x_0,\ y_0,\ z_0)$. The equation of the asymptotic plane of the ruled surface $\varphi_{\mathcal{TB}}(t, \upsilon_{\mathcal{TB}})$ is as follows:*

$$\begin{aligned} &(x - x_0)\, m\,(C(t)\cos t - nA(t)\sin t) \\ &+ (y - y_0)\, m\,(C(t)\sin t + nA(t)\cos t) \\ &- (z - z_0)\, nA(t) = 0 \end{aligned}$$

Proposition 4.14. *The distribution parameter $\rho_{\mathcal{TB}}(t)$ of the ruled surface $\varphi_{\mathcal{TB}}(t, \upsilon_{\mathcal{TB}})$ is as follows:*

$$\rho_{\mathcal{TB}}(t) = -\frac{c\cos(nt)}{A(t)}.$$

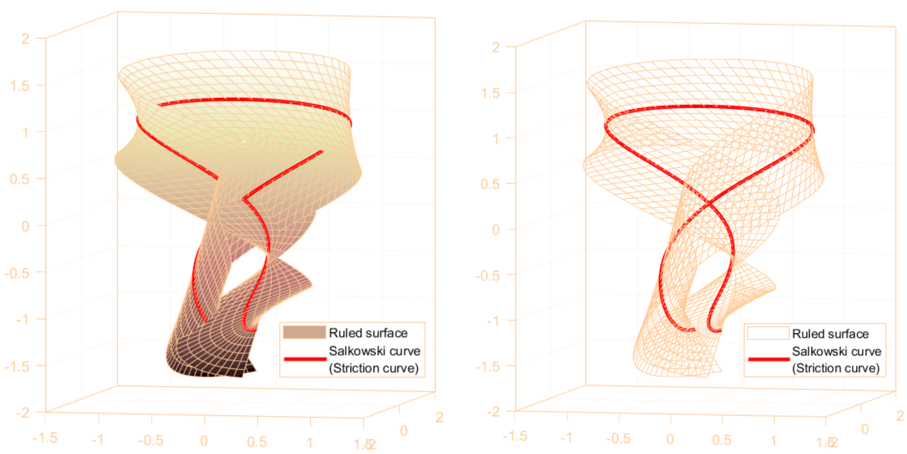

Figure 7. Ruled surface generated by Salkowski curve and the vector $X_{\mathcal{TB}}(t)$ for $b = 0,\ a = c = m = \frac{1}{5}$. (The right image is the transparent form of the left image.)

- Let's give the propositions for the ruled surfaces generated by Salkowski curve and the vector

$$X_{\mathcal{TN}}(t) = -\left(aS(t) - \frac{bn}{m}\sin t,\ aR(t) + \frac{bn}{m}\cos t,\ n\left(b + \frac{a}{m}\sin(nt)\right)\right)$$

lying on the osculator plane.

Proposition 4.15. *Let the ruled surface whose base curve is Salkowski curve $\Upsilon(t)$ in Euclidean 3-space and whose generating line is the vector $X_{\mathcal{TN}}(t)$ is denoted by*

$\varphi_{\mathcal{TN}}(t, v_{\mathcal{TN}})$. *The parametric equation of this surface is as follows (Figure 8):*

$$\begin{aligned}
\varphi_{\mathcal{TN}}(t, v_{\mathcal{TN}}) &= \Bigg[\frac{n}{4m}\left(\frac{n-1}{1+2n}(\sin(1+2n)t) - \frac{n+1}{1-2n}(\sin(1-2n)t) - 2\sin t\right) \\
&\quad -v_{\mathcal{TN}}\left(aS(t) - \frac{bn}{m}\sin t\right), \\
&\quad \frac{n}{4m}\left(\frac{1-n}{1+2n}(\cos(1+2n)t) + \frac{n+1}{1-2n}(\cos(1-2n)t) + 2\cos t\right) \\
&\quad -v_{\mathcal{TN}}\left(aR(t) + \frac{bn}{m}\cos t\right) \\
&\quad \frac{n}{4m^2}\cos(2nt) - v_{\mathcal{TN}}\, n\left(b + \frac{a}{m}\sin(nt)\right)\Bigg].
\end{aligned}$$

Proposition 4.16. *The normal vector $\eta_{\mathcal{TN}}(t)$ of the ruled surface $\varphi_{\mathcal{TN}}(t, v_{\mathcal{TN}})$ is as follows:*

$$\begin{aligned}
\eta_{\mathcal{TN}}(t) &= -\frac{n}{m}\Bigg[bP(t)\cos(nt) + v_{\mathcal{TN}}\left(bn\sin t\left(b + \frac{a}{m}\sin(nt)\right) - a^2P(t)\cos(nt)\right), \\
&\quad bQ(t)\cos(nt) - v_{\mathcal{TN}}\left(bn\cos t\left(b + \frac{a}{m}\sin(nt)\right) + a^2Q(t)\cos(nt)\right), \\
&\quad -\frac{bn}{m}\cos^2(nt) + v_{\mathcal{TN}}n\left(b\left(\frac{b}{m} - a\sin(nt)\right) + \frac{a^2}{m}\cos^2(nt)\right)\Bigg].
\end{aligned}$$

Proposition 4.17. *Let the plane has a fixed point $M = (x,\ y,\ z)$ and a variable point $D = (x_0,\ y_0,\ z_0)$. The equation of the tangent plane of the ruled surface $\varphi_{\mathcal{TN}}(t, v_{\mathcal{TN}})$ is as follows:*

$$\begin{aligned}
&(x - x_0)\left[bP(t)\cos(nt) + v_{\mathcal{TN}}\left(bn\sin t\left(b + \frac{a}{m}\sin(nt)\right) - a^2P(t)\cos(nt)\right)\right] \\
&+ (y - y_0)\left[bQ(t)\cos(nt) - v_{\mathcal{TN}}\left(bn\cos t\left(b + \frac{a}{m}\sin(nt)\right) + a^2Q(t)\cos(nt)\right)\right] \\
&+ (z - z_0)\left[-\frac{bn}{m}\cos^2(nt) + v_{\mathcal{TN}}n\left(b\left(\frac{b}{m} - a\sin(nt)\right) + \frac{a^2}{m}\cos^2(nt)\right)\right] = 0.
\end{aligned}$$

Proposition 4.18. *The parameter $v_{\mathcal{TN}}$ of the striction curve of the ruled surface $\varphi_{\mathcal{TN}}(t, v_{\mathcal{TN}})$ is as follows:*

$$v_{\mathcal{TN}} = \frac{b\cos^2(nt)}{a^2\cos^2(nt) + b^2}.$$

Proposition 4.19. *The parametric equation of the striction curve $\psi_{\mathcal{TN}}(t)$ of the ruled surface $\varphi_{\mathcal{TN}}(t, v_{\mathcal{TN}})$ is as follows:*

$$\begin{aligned}\psi_{\mathcal{TN}}(t) \;=\; &\left[\frac{n}{4m}\left(\frac{n-1}{1+2n}\sin((1+2n)t)-\frac{n+1}{1-2n}\sin((1-2n)t)-2\sin t\right)\right.\\
&-\frac{b\cos^2(nt)}{a^2\cos^2(nt)+b^2}\left(aS(t)-\frac{bn}{m}\sin t\right),\\
&\frac{n}{4m}\left(\frac{1-n}{1+2n}\cos((1+2n)t)+\frac{n+1}{1-2n}\cos((1-2n)t)+2\cos t\right)\\
&-\frac{b\cos^2(nt)}{a^2\cos^2(nt)+b^2}\left(aR(t)+\frac{bn}{m}\cos t\right),\\
&\left.\frac{n}{4m^2}\cos(2nt)-\frac{bn\cos^2(nt)}{a^2\cos^2(nt)+b^2}\left(b+\frac{a}{m}\sin(nt)\right)\right].\end{aligned}$$

Proposition 4.20. *Let the plane have a fixed point* $M=(x,\ y,\ z)$ *and a variable point* $D=(x_0,\ y_0,\ z_0)$. *The equation of the asymptotic plane of the ruled surface* $\varphi_{\mathcal{TN}}(t,v_{\mathcal{TN}})$ *is as follows:*

$$\begin{aligned}&(x-x_0)\left[a^2P(t)\cos(nt)-bn\sin t\left(b+\frac{a}{m}\sin(nt)\right)\right]\\
&+(y-y_0)\left[a^2Q(t)\cos(nt)+bn\cos t\left(b+\frac{a}{m}\sin(nt)\right)\right]\\
&-(z-z_0)\left[\frac{a^2n}{m}\cos^2(nt)+bn\left(\frac{b}{m}-a\sin(nt)\right)\right]=0.\end{aligned}$$

Proposition 4.21. *The distribution parameter* $\rho_{\mathcal{TN}}(t)$ *of the ruled surface* $\varphi_{\mathcal{TN}}(t,v_{\mathcal{TN}})$ *is as follows:*

$$\rho_{\mathcal{TN}}(t)=-\frac{b^2\cos(nt)\sin(nt)}{a^2\cos^2(nt)+b^2}.$$

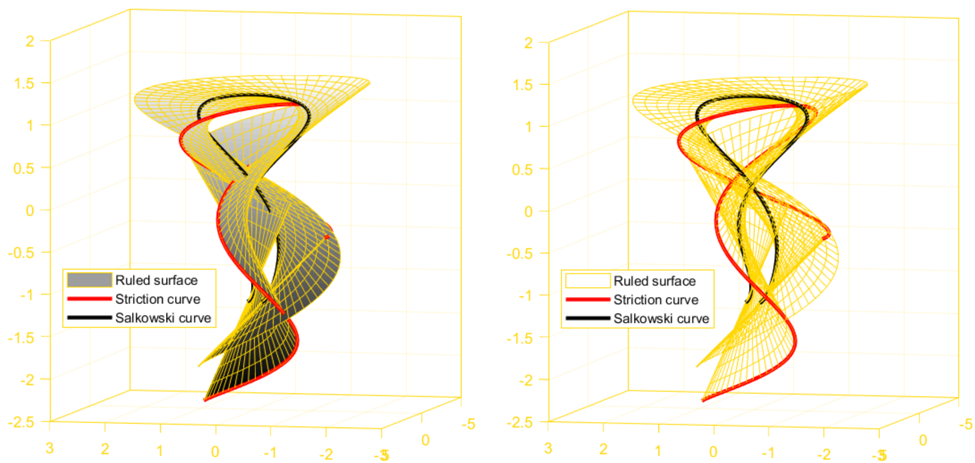

Figure 8. Ruled surface generated by Salkowski curve and the vector $X_{\mathcal{TN}}(t)$ for $c=0,\ a=b=m=\frac{1}{5}$. (The right image is the transparent form of the left image.)

- Let's give the propositions for the ruled surfaces generated by Salkowski curve and the vector

$$X_{\mathcal{TNB}}(t) = \mathcal{T}(t) + \mathcal{N}(t) + \mathcal{B}(t)$$
$$= -\left(S(t) + P(t) - \frac{n}{m}\sin t,\ R(t) + Q(t) + \frac{n}{m}\cos t,\ \frac{n}{m}(m - \cos(nt) + \sin(nt))\right).$$

Proposition 4.22. *Let the ruled surface whose base curve is Salkowski curve $\Upsilon(t)$ in Euclidean 3-space and whose generating line is the vector $X_{\mathcal{TNB}}(t)$ is denoted by $\varphi_{\mathcal{TNB}}(t, v_{\mathcal{TNB}})$. The parametric equation of this surface is as follows (Figure 9):*

$$\begin{aligned}\varphi_{\mathcal{TNB}}(t, v_{\mathcal{TNB}}) \quad = \quad & \Bigg[\frac{n}{4m}\left(\frac{n-1}{1+2n}(\sin(1+2n)t) - \frac{n+1}{1-2n}(\sin(1-2n)t) - 2\sin t\right) \\ & -v_{\mathcal{TNB}}\left(S(t) + P(t) - \frac{n}{m}\sin t\right), \\ & \frac{n}{4m}\left(\frac{1-n}{1+2n)}(\cos(1+2n)t) + \frac{n+1}{1-2n)}(\cos(1-2n)t) + 2\cos t\right) \\ & - v_{\mathcal{TNB}}\left(R(t) + Q(t) + \frac{n}{m}\cos t\right), \\ & \frac{n}{4m^2}\cos(2nt) - \frac{v_{\mathcal{TNB}}n}{m}(m - \cos(nt) + \sin(nt))\Bigg].\end{aligned}$$

Proposition 4.23. *The normal vector $\eta_{\mathcal{TNB}}(t)$ of the ruled surface $\varphi_{\mathcal{TNB}}(t, v_{\mathcal{TNB}})$ is as follows:*

$$\begin{aligned}\eta_{\mathcal{TNB}}(t) \quad = \quad & -\frac{n}{m}\Bigg[\cos(nt)\left(P(t) + \frac{n}{m}\sin t\right) \\ & +v_{\mathcal{TNB}}\left(S(t)\cos(nt) - P(t)\sin(nt) + \frac{n}{m}\sin t\,(2m - \cos(nt) + \sin(nt))\right), \\ & \cos(nt)\left(Q(t) - \frac{n}{m}\cos t\right) \\ & +v_{\mathcal{TNB}}\left(R(t)\cos(nt) - Q(t)\sin(nt) - \frac{n}{m}\cos t\,(2m - \cos(nt) + \sin(nt))\right), \\ & -n\cos(nt)\left(1 + \frac{1}{m}\cos(nt)\right) \\ & +v_{\mathcal{TNB}}n\left(\frac{2}{m}(1 + \cos(nt)\sin(nt)) + \cos(nt) - \sin(nt)\right)\Bigg].\end{aligned}$$

Proposition 4.24. *Let the plane has a fixed point $M = (x,\ y,\ z)$ and a variable point $D = (x_0,\ y_0,\ z_0)$. The equation of the tangent plane of the ruled surface $\varphi_{\mathcal{TNB}}(t, v_{\mathcal{TNB}})$ is as follows:*

$$\begin{aligned}& (x - x_0)\Bigg[\cos(nt)\left(P(t) + \frac{n}{m}\sin t\right) \\ & \qquad +v_{\mathcal{TNB}}\left(S(t)\cos(nt) - P(t)\sin(nt) + \frac{n}{m}\sin t\,(2m - \cos(nt) + \sin(nt))\right)\Bigg] \\ & + (y - y_0)\Bigg[\cos(nt)\left(Q(t) - \frac{n}{m}\cos t\right) \\ & \qquad +v_{\mathcal{TNB}}\left(R(t)\cos(nt) - Q(t)\sin(nt) - \frac{n}{m}\cos t\,(2m - \cos(nt) + \sin(nt))\right)\Bigg] \\ & + (z - z_0)\Bigg[-n\cos(nt)\left(1 + \frac{1}{m}\cos(nt)\right) \\ & \qquad +v_{\mathcal{TNB}}n\left(\frac{2}{m}(1 + \cos(nt)\sin(nt)) + \cos(nt) - \sin(nt)\right)\Bigg] = 0.\end{aligned}$$

PROPOSITION 4.25. *The parameter* $v_{\mathcal{TNB}}$ *of the striction curve of the ruled surface* $\varphi_{\mathcal{TNB}}(t, v_{\mathcal{TNB}})$ *is as follows:*

$$v_{\mathcal{TNB}} = \frac{\cos^2(nt)}{2(1+\cos(nt)\sin(nt))}.$$

PROPOSITION 4.26. *The parametric equation of the striction curve* $\psi_{\mathcal{TNB}}(t)$ *of the ruled surface* $\varphi_{\mathcal{TNB}}(t, v_{\mathcal{TNB}})$ *is as follows:*

$$\begin{aligned}\psi_{\mathcal{TNB}}(t) = & \Bigg[\frac{n}{4m}\left(\frac{n-1}{1+2n}\sin((1+2n)t) - \frac{n+1}{1-2n}\sin((1-2n)t) - 2\sin t\right) \\ & -\frac{\cos^2(nt)}{2(1+\cos(nt)\sin(nt)}\left(S(t)+P(t)-\frac{n}{m}\sin t\right), \\ & \frac{n}{4m}\left(\frac{1-n}{1+2n}\cos((1+2n)t) + \frac{n+1}{1-2n}\cos((1-2n)t) + 2\cos t\right) \\ & -\frac{\cos^2(nt)}{2(1+\cos(nt)\sin(nt)}\left(R(t)+Q(t)+\frac{n}{m}\cos t\right), \\ & \frac{n}{4m^2}\cos(2nt) - \frac{n\cos^2(nt)}{2m(1+\cos(nt)\sin(nt)}(m-\cos(nt)+\sin(nt))\Bigg].\end{aligned}$$

PROPOSITION 4.27. *Let the plane has a fixed point* $M = (x,\ y,\ z)$ *and a variable point* $D = (x_0,\ y_0,\ z_0)$*. The equation of the asymptotic plane of the ruled surface* $\varphi_{\mathcal{TNB}}(t, v_{\mathcal{TNB}})$ *is as follows:*

$$\begin{aligned}&(x-x_0)\left[P(t)\sin(nt) - S(t)\cos(nt) - 2n\sin t - \frac{n}{m}\sin t(\sin(nt)-\cos(nt))\right] \\ &+(y-y_0)\left[Q(t)\sin(nt) - R(t)\cos(nt) + 2n\cos t + \frac{n}{m}\cos t(\sin(nt)-\cos(nt))\right] \\ &-(z-z_0)\left[n\left(\frac{2}{m}(1+\cos(nt)\sin(nt)) + \cos(nt) - \sin(nt)\right)\right] = 0.\end{aligned}$$

PROPOSITION 4.28. *The distribution parameter* $\rho_{\mathcal{TNB}}(t)$ *of the ruled surface* $\varphi_{\mathcal{TNB}}(t, v_{\mathcal{TNB}})$ *is as follows:*

$$\rho_{\mathcal{TNB}}(t) = -\frac{\cos(nt)(\cos(nt)+2\sin(nt))}{2(1+\cos(nt)\sin(nt))}.$$

We have only examined four special cases here, but it is clear that countless ruled surfaces can be obtained for different values of the real coefficients a, b, c.

5. Conclusions

In this study, ruled surfaces $\varphi_{\mathcal{T}}(t, v_{\mathcal{T}})$, $\varphi_{\mathcal{N}}(t, v_{\mathcal{N}})$, $\varphi_{\mathcal{B}}(t, v_{\mathcal{B}})$, $\varphi_{X}(t, v_{X})$, $\varphi_{\mathcal{NB}}(t, v_{\mathcal{NB}})$, $\varphi_{\mathcal{TB}}(t, v_{\mathcal{TB}})$, $\varphi_{\mathcal{TN}}(t, v_{\mathcal{TN}})$ and $\varphi_{\mathcal{TNB}}(t, v_{\mathcal{TNB}})$ are generated, respectively. The equations of normal vectors, striction curves, distribution parameters, tangent and asymptotic planes of these surfaces are calculated. It is concluded that, the striction curve and the base curve of the ruled surface $\varphi_{\mathcal{T}}(t, v_{\mathcal{T}})$ coincide and $\varphi_{\mathcal{T}}(t, v_{\mathcal{T}})$ is developable; the base curve and the striction curve of the ruled surface $\varphi_{\mathcal{N}}(t, v_{\mathcal{N}})$ never coincide and $\varphi_{\mathcal{N}}(t, v_{\mathcal{N}})$ is never developable; the base curve and the striction curve of the ruled surface $\varphi_{\mathcal{B}}(t, v_{\mathcal{B}})$ are coincide and $\varphi_{\mathcal{B}}(t, v_{\mathcal{B}})$ is never develobaple.

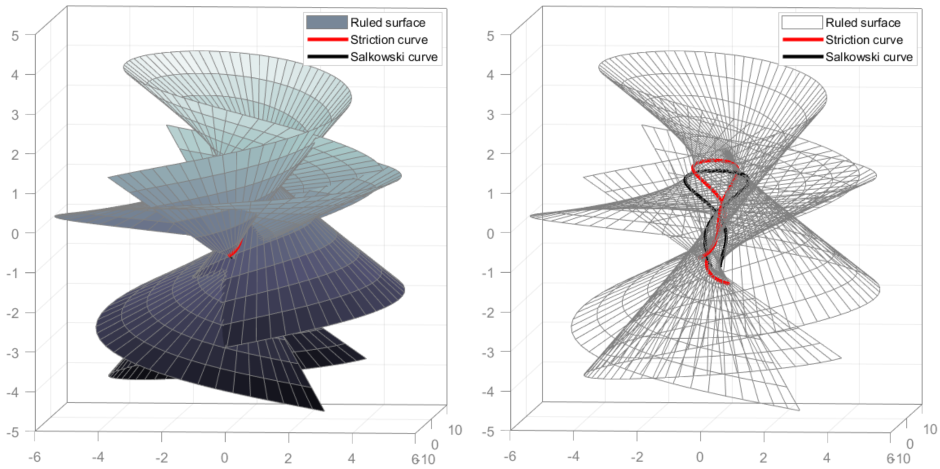

FIGURE 9. Ruled surface generated by Salkowski curve and the vector $X_{\mathcal{TNB}}(t)$ for $a = b = c = 1,\ m = \frac{1}{5}$. (The right image is the transparent form of the left image.)

Moreover, the striction curve and the base curve of the ruled surface $\varphi_X\,(t, v_X)$ coincide, if $b = 0$ and $\varphi_X\,(t, v_X)$ is developable, if $b = c = 0$; the base curve and the striction curve of the ruled surface the striction curve and the base curve of the ruled surface $\varphi_{\mathcal{NB}}\,(t, v_{\mathcal{NB}})$ coincide, if $b = 0$ and $\varphi_{\mathcal{NB}}\,(t, v_{\mathcal{NB}})$ is developable, if $b = c = 0$; the striction curve and the base curve of the ruled surface $\varphi_{\mathcal{TB}}\,(t, v_{\mathcal{TB}})$ coincide and $\varphi_{\mathcal{TB}}\,(t, v_{\mathcal{TB}})$ is developable; if $c = 0$; the striction curve and the base curve of the ruled surface $\varphi_{\mathcal{TN}}\,(t, v_{\mathcal{TN}})$ coincide, if $b = 0$ and $\varphi_{\mathcal{TN}}\,(t, v_{\mathcal{TN}})$ is developable, if $b = 0$; the striction curve and the base curve of the ruled surface $\varphi_{\mathcal{TNB}}\,(t, v_{\mathcal{TNB}})$ never coincide and $\varphi_{\mathcal{TNB}}\,(t, v_{\mathcal{TNB}})$ is never developable. Other geometric properties of these surfaces, such as their fundamental forms, Gaussian and mean curvatures, singularities, can also be examined by considering studies [11,14,16–18,26]. Similar studies can also be done on different curves (for example anti-Salkowski curves) or in various spaces.

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