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A CANONICAL CHRISTOFFEL TRANSFORMATION OF THE STRICT THIRD DEGREE CLASSICAL LINEAR FORMS

Mohamed Khalfallah

Abstract. The aim of this paper is to study several characterizations of a large family of semiclassical linear forms of class one, which are of strict third degree and are not included in either the family of symmetric forms or the quasi-symmetric family. In fact, using the Stieltjes function and the moments, we describe a canonical Christoffel transformation w of the strict third degree classical linear form $\mathcal{V}_q^{k,l} := \mathcal{J}(k+q/3, l-q/3), k+l \geq -1, k, l \in \mathbb{Z}, q \in \{1,2\}$, meaning $w = (x-c)\mathcal{V}_q^{k,l}, |c| > 1$.

1. Introduction

The present paper concerns the theory of semiclassical orthogonal polynomials (OPs). Since the seminal paper on semiclassical orthogonal polynomials ([22]), many authors have dealt with this subject (see [1–8, 11–17, 19, 20, 22]). They arise as a natural extension of the well-known classical OPs of Hermite, Laguerre, Jacobi and Bessel, having been the focus of great research activity since the 1980's. Specifically, this theory has been developed from an algebraic aspect and a distributional one by P. Maroni and extensively studied during the last three decades (see [19] for a nice survey on this topic, as well as [11] with the applications in the framework of Sobolev inner products).

The study of the semiclassical forms of classes greater than or equal to one is a hard problem. Taking into account the difficulties of solving the Laguerre-Freud equations, it is more important to use other tools for the construction and characterizations of some semiclassical forms ([2–5]) based either on the moments, the corresponding Stieltjes function or their integral representation. For instance, the study of the third degree regular forms (TDRFs, in short) aims to provide a detailed description and characterization of specific semiclassical forms [1–8, 12–17] based either on the moments, the corresponding Stieltjes function. These forms are characterized by the fact that their formal Stieltjes function $S(w)(z) := -\sum_{n\geq 0} \langle w, x^n\rangle / z^{n+1}$ satisfies a cubic equation with polynomial coefficients

$$AS^3(w) + BS^2(w) + CS(w) + D = 0.$$

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A regular form w is called a strict third degree form (STDRF, in short) if it is a TDRF and not a second degree regular form (i.e., does not fulfill an analogous relation of second order [21]). Some TDRF properties are given in [4, 9]. In [3, 4], the authors determine all classical forms which are TDRFs. It is worth mentioning that the unique classical strict third degree forms are the Jacobi forms $\mathcal{J}(k+q/3, l-q/3)$, where $q \in \{1,2\}$ and k, l are integers with $k+l \geq -1$ (see [4]).

In general, STDRF is a Laguerre-Hahn form (see [4]). Taking into account that rational spectral transformations of STDRF preserve such a family [7] as well as according to [23], the linear spectral transformations are generated by Christoffel and Geronimus transformations which yield linear forms v, w, defined by $v = p(x)u$, and $p(x)w = u$, respectively, where $p(x)$ is a polynomial and u is a linear form.

This paper focuses on the analysis of semiclassical forms of class $s = 1$ which are STDRFs. In particular, the utmost interest of this paper is the description, using the third degree character, of a large family of forms obtained through a canonical Christoffel transformation of the strict third degree classical linear forms, i.e., the linear forms $(x-c)\mathcal{J}(k+q/3, l-q/3)$, where $q \in \{1,2\}$, k, l are integers with $k+l \geq -1$, and c is a complex number such that $|c| > 1$. Our main tool is the representation of the corresponding Stieltjes functions and, as a consequence, the moments of the forms are deduced in a straightforward way.

A brief description of the paper organization is given as follows. In Section 2 we review some basic notations, definitions and results used in the forthcoming sections. In Section 3, we first recall the definitions as well as the main proprieties of third degree forms. Second, we give some results concerning strict third degree classical forms, denoted by $\mathcal{V}_q^{k,l} := \mathcal{J}(k+q/3, l-q/3)$, $k+l \geq -1,\ k, l \in \mathbb{Z},\ q \in \{1,2\}$, which are needed in the sequel. In Section 4, we state our main result. Through the third degree forms, and by using a canonical Christoffel transformation of classical linear forms of strict third degree, we provide an identification for a large family of strict third degree linear forms which are semiclassical of class one and, as a consequence, do not belong to either the family of symmetric forms or the quasi-symmetric family.

2. Notation and basic background

In this section, we present some basic definitions, notation, and results which are used throughout this paper. Let $\mathcal{P}$ be the vector space of polynomials of one real variable and with complex coefficients and let $\mathcal{P}'$ be its algebraic dual. The elements of $\mathcal{P}'$ will be called linear forms (linear functionals). By $\langle \cdot, \cdot \rangle$, we denote the duality bracket between $\mathcal{P}$ and $\mathcal{P}'$.
Given a form $w \in \mathcal{P}'$. The sequence of complex numbers $(w)_n,\ n \geq 0$, denotes the moments of w with respect to the sequence $\{x^n\}_{n\geq 0}$, namely, the moment of order n for the form w is denoted by $(w)_n =: \langle w, x^n \rangle$. Thus, the form w is completely determined by its moments.

We define the following operations on $\mathcal{P}'$. For any $a \in \mathbb{C} - \{0\}, b, c \in \mathbb{C},\ p, q \in \mathcal{P}$, and $w \in \mathcal{P}'$

$$\langle pw, q\rangle = \langle w, pq\rangle,\ \langle w', p\rangle = -\langle w, p'\rangle,$$
$$\langle h_a w, p\rangle = \langle w, h_a p\rangle = \langle w, p(ax)\rangle,\ \langle \tau_b w, p\rangle = \langle w, \tau_{-b} p\rangle = \langle w, p(x+b)\rangle,$$
$$\langle (x-c)^{-1} w, p\rangle = \langle w, \theta_c p\rangle = \left\langle w, \frac{p(x)-p(c)}{x-c}\right\rangle.$$

For $p \in \mathcal{P}$ and $w \in \mathcal{P}'$, the product wp is the polynomial $(wp)(x) := \left\langle w, \frac{xp(x)-\zeta p(\zeta)}{x-\zeta}\right\rangle$ [19]. This allows us to define the Cauchy product of two linear forms

$$\langle vw, p\rangle := \langle v, wp\rangle, \quad v, w \in \mathcal{P}', \quad p \in \mathcal{P}.$$

The above product is commutative, associative and distributive with respect to the sum of forms.

The linear form $w \in \mathcal{P}'$ is said to be a rational perturbation of $v \in \mathcal{P}'$, if there exist polynomials p and q, such that

$$q(x)w = p(x)v.$$

In particular, we say that v is a Christoffel transformation of w if $p(x) = 1$. On the other hand, we say that v is a Geronimus transformation of w if $q(x) = 1$.

We will also use the so-called formal Stieltjes function associated with $w \in \mathcal{P}'$ that is defined as ([10, 19])

$$S(w)(z) = -\sum_{n\geq 0} \frac{(w)_n}{z^{n+1}}.$$

Remark 2.1. For any $p \in \mathcal{P}$ and $w \in \mathcal{P}'$, $S(w)(z) = p(z)$ if and only if $w = 0$ and $p = 0$.

For any $p \in \mathcal{P}$ and $w \in \mathcal{P}'$, the following property holds ([19])

$$S(pw)(z) = p(z)S(w)(z) + (w\theta_0 p)(z). \tag{2.1}$$

Let us recall that a form w is called regular (quasi-definite) if there exists a monic polynomial sequence $\{W_n\}_{n\geq 0}$ with $\deg W_n = n$ such that [10]

$$\langle w, W_n W_m\rangle = r_n \delta_{n,m},\ n, m \geq 0,$$

where $\{r_n\}_{n\geq 0}$ is a sequence of nonzero complex numbers and $\delta_{n,m}$ is the Kronecker symbol.

$\{W_n\}_{n\geq 0}$ is called a monic orthogonal polynomial sequence (MOPS, in short) with respect to the form w. It is characterized by the following three-term recurrence relation

$$\begin{aligned} &W_0(x) = 1,\ W_1(x) = x - \beta_0,\\ &W_{n+2}(x) = (x - \beta_{n+1})W_{n+1}(x) - \gamma_{n+1}W_n(x),\ n \geq 0. \end{aligned} \tag{2.2}$$

Here $\{\beta_n\}_{n\geq 0}$ and $\{\gamma_{n+1}\}_{n\geq 0}$ are sequences of complex numbers such that $\gamma_{n+1} \neq 0$ for all n. This is the so called Favard's theorem (see [10, 19, 20]). The form w is said to be normalized if $(w)_0 = 1$. In the sequel, we only consider normalized forms.

A form w is called semiclassical when it is regular and there exist non zero polynomials ϕ and ψ, ϕ monic, $\deg\phi \geq 0$, $\deg\psi \geq 1$, such that w satisfies a Pearson's equation

$$(\phi w)' + \psi w = 0. \tag{2.3}$$

Equivalently, the formal Stieltjes function of w satisfies a nonhomogeneous first order linear differential equation with polynomial coefficients

$$A_0(z)S'(w)(z) = C_0(z)S(w)(z) + D_0(z), \tag{2.4}$$

where

$$A_0 = \phi,\ C_0 = -\phi' - \psi,\ D_0 = -(w\theta_0\phi)' - (w\theta_0\psi). \tag{2.5}$$

Furthermore, if the polynomials A_0, C_0, and D_0 appearing in (2.5) are coprime, then the class of w is defined by

$$s = \max\{\deg C_0 - 1, \deg D_0\}.$$

If $\{W_n\}_{n\geq 0}$ is an OPS with respect to a semiclassical form w of class s, then $\{W_n\}_{n\geq 0}$ is called a semiclassical OPS of class s. In particular, when $s = 0$ (so that deg $\phi \leq 2$ and deg $\psi = 1$) one obtains, up to an affine change of the variable, the four well-known families of classical forms Hermite, $\mathcal{H}$; Laguerre, $\mathcal{L}(\alpha)$; Jacobi, $\mathcal{J}(\alpha,\beta)$ and Bessel, $\mathcal{B}(\alpha)$ (see [20]). Taking into account Jacobi linear forms $\mathcal{J}(\alpha,\beta)$ will be used in the sequel, we point out that $\phi(x) = x^2 - 1, \psi(x) = -(\alpha+\beta+2)x + (\alpha-\beta)$. Let us recall that a Jacobi form $\mathcal{J}(\alpha,\beta)$ has the following integral representation for $\Re(\alpha+1) > 0$ and $\Re(\beta+1) > 0$

$$\langle \mathcal{J}(\alpha,\beta), f\rangle = C_{\alpha,\beta}\int_{-1}^{+1}(1+x)^{\alpha}(1-x)^{\beta}f(x)\mathrm{d}x, \quad f \in \mathcal{P},$$

with

$$C_{\alpha,\beta} = \frac{1}{2^{\alpha+\beta+1}}\frac{\Gamma(\alpha+\beta+2)}{\Gamma(\alpha+1)\Gamma(\beta+1)}.$$

3. Third degree semiclassical linear forms

3.1. Third degree form. In this subsection, we briefly review the definitions and list some basic properties of the third degree regular forms. Afterwards, we will give some results concerning strict third degree classical forms which are needed later on in this paper.

Definition 3.1. A form w is called a third degree regular form (TDRF, in short) if it is regular and if there exist three polynomials A (monic), B and C such that

$$A(z)S^3(w)(z) + B(z)S^2(w)(z) + C(z)S(w)(z) + D(z) = 0, \tag{3.1}$$

where polynomial D depends on A, B, C and w.

Remark 3.1.
1. The polynomial D is explicitly given by $D(z) = \big(w^3\theta_0^3 A\big)(z) - \big(w^2\theta_0^2 B\big)(z) + \big(w\theta_0 C\big)(z)$.
2. A regular form w is called a second degree form if the corresponding Stieltjes function satisfies a quadratic equation with polynomial coefficients ([21])

$$B(z)S^2(w)(z) + C(z)S(w)(z) + D(z) = 0, \tag{3.2}$$

where B, C, D satisfy $B \neq 0, C^2 - 4BD \neq 0, D \neq 0$ due to regularity of w.

3. When the form w is a TDRF and not a second degree form, we shall call it a strict third degree regular form (STDRF, in short) ([4]).

REMARK 3.2. Among the most well-known forms which are STDRFs, we can find the Jacobi form $\mathcal{V} := \mathcal{J}(-\frac{2}{3}, -\frac{1}{3})$ ([4]). Thus, its formal Stieltjes function is

$$S(\mathcal{V})(z) = -(z+1)^{-2/3}(z-1)^{-1/3}, \tag{3.3}$$

and satisfies the cubic equation

$$(z+1)^2(z-1)S^3(\mathcal{V})(z) + 1 = 0. \tag{3.4}$$

Elementary transformations like association, perturbation, shift, multiplication and division by a polynomial, inversion, among others preserve the family of linear forms of third degree ([4, 7, 18]). Furthermore, the class of third degree forms is closed under rational spectral transformations of the form ([7]). In particual, we have the following result.

LEMMA 3.1. [4] *Let u and v be two regular linear forms satisfying $M(x)u = N(x)v$, where $M(x)$ and $N(x)$ are two polynomials. If one of the linear forms u and v is a third degree form then so is the other one. If u is a third degree linear form satisfying* (3.1) *then v is also a third degree linear form satisfying*

$$A_v S^3(v) + B_v S^2(v) + C_v S(v) + D_v = 0,$$

with

$$A_v = AN^3,$$

$$B_v = N^2\Big\{BM + 3A\big((v\theta_0 N) - (u\theta_0 M)\big)\Big\},$$

$$C_v = N\Big\{CM^2 + 2BM\big((v\theta_0 N) - (u\theta_0 M)\big) + 3A\big((v\theta_0 N) - (u\theta_0 M)\big)^2\Big\},$$

$$D_v = DM^3 + CM^2\big((v\theta_0 N) - (u\theta_0 M)\big) + BM\big((v\theta_0 N) - (u\theta_0 M)\big)^2 + A\big((v\theta_0 N) - (u\theta_0 M)\big)^3.$$

3.2. Strict third degree classical forms. As mentioned in the introduction, all the classical forms which are of strict third degree are determined, see [4]. More precisely, Hermite, Laguerre, and Bessel forms are not STDRFs, only Jacobi forms which satisfy a certain condition possess this property. Indeed,

THEOREM 3.1. [4] *Among the classical linear forms, only the Jacobi forms $\mathcal{J}(k + q/3, l - q/3)$ are STDRFs, provided $k + l \geq -1,\ k, l \in \mathbb{Z},\ q \in \{1, 2\}$.*

REMARK 3.3. Throughout this paper, we make use of the following notation: $\mathcal{V}_q^{k,l} := \mathcal{J}(k + q/3, l - q/3)$, with $k + l \geq -1,\ k, l \in \mathbb{Z},\ q \in \{1, 2\}$.

The next lemma gives us a fundamental relation to be used in the sequel. Precisely, the following lemma clarifies that the strict third degree classical linear forms $\mathcal{V}_q^{k,l}$ are rational perturbations of $h_{(-1)^{q-1}}\mathcal{V}$.

LEMMA 3.2. [14] *Let $q \in \{1, 2\}$ and $k, l \in \mathbb{Z}$ with $k + l \geq -1$. The forms $\mathcal{V}_q^{k,l}$ and $\mathcal{V}$ are related by*

$$f_q^{k,l}\mathcal{V}_q^{k,l} = g_q^{k,l} h_{(-1)^{q-1}}\mathcal{V}, \tag{3.5}$$

where $f_q^{k,l}$ and $g_q^{k,l}$ are polynomials defined by

$$f_q^{k,l}(x) := \Big\langle h_{(-1)^{q-1}}\mathcal{V}, (x+1)^{\frac{|k+1|+k+1}{2}}(x-1)^{\frac{|l|+l}{2}}\Big\rangle (x+1)^{\frac{|k+1|-(k+1)}{2}}(x-1)^{\frac{|l|-l}{2}}, \tag{3.6}$$

and

$$g_q^{k,l}(x) := \left\langle \mathcal{V}_q^{k,l}, (x+1)^{\frac{|k+1|-(k+1)}{2}}(x-1)^{\frac{|l|-l}{2}} \right\rangle (x+1)^{\frac{|k+1|+k+1}{2}}(x-1)^{\frac{|l|+l}{2}}. \tag{3.7}$$

4. A new family of strict third degree semiclassical linear forms of class one

In this section we establish several characterizations of the semiclassical forms of class one which are of strict third degree such that w is obtained through a canonical Christoffel transformation of the strict third degree classical linear forms.

THEOREM 4.1. *Let w be a regular form. The following statements are equivalent.*

(a) *w is a canonical Christoffel transformation of the strict third degree classical linear forms, i.e., there exist $c \in \mathbb{C}$ with $|c| > 1$ and $q \in \{1,2\}$, $(k,l) \in \mathbb{Z}^2$ with $k+l \geq -1$, such that*

$$w = (x-c)\mathcal{V}_q^{k,l}. \tag{4.1}$$

(b) *There exist $c \in \mathbb{C}$ with $|c| > 1$ and $q \in \{1,2\}$, $(k,l) \in \mathbb{Z}^2$ with $k+l \geq -1$, such that*

$$S(w)(z) = (z-c)S\big(\mathcal{V}_q^{k,l}\big)(z) + 1. \tag{4.2}$$

(c) *There exist $c \in \mathbb{C}$ with $|c| > 1$ and $q \in \{1,2\}$, $(k,l) \in \mathbb{Z}^2$ with $k+l \geq -1$, such that*

$$f_q^{k,l}(x)w = (x-c)g_q^{k,l}(x)(h_{(-1)^{q-1}}\mathcal{V}), \tag{4.3}$$

and

$$\begin{aligned}\big(w\theta_0 f_q^{k,l}\big)(z) = &-(z-c)\Big(\big((h_{(-1)^{q-1}}\mathcal{V})\theta_0 g_q^{k,l}\big) - \big(\mathcal{V}_q^{k,l}\theta_0 f_q^{k,l}\big)\Big)(z) \\ &+ \big((h_{(-1)^{q-1}}\mathcal{V})\theta_0((x-c)g_q^{k,l}(x))\big)(z) - f_q^{k,l}(z),\end{aligned} \tag{4.4}$$

where $f_q^{k,l}$ and $g_q^{k,l}$ are polynomials defined by (3.6) and (3.7), respectively.

(d) *There exist $c \in \mathbb{C}$ with $|c| > 1$ and $q \in \{1,2\}$, $(k,l) \in \mathbb{Z}^2$ with $k+l \geq -1$, such that*

$$\begin{aligned}(w)_n = &\sum_{\nu=0}^{n+1} \binom{n+1}{\nu} 2^{\nu-1} \frac{\Gamma(k+l+2)}{\Gamma(\nu+k+l+2)} F_{n+1,\nu}\left(k+\tfrac{q}{3}, l-\tfrac{q}{3}\right) \\ &- c\sum_{\nu=0}^{n} \binom{n}{\nu} 2^{\nu-1} \frac{\Gamma(k+l+2)}{\Gamma(\nu+k+l+2)} F_{n,\nu}\left(k+\tfrac{q}{3}, l-\tfrac{q}{3}\right),\ n \geq 0,\end{aligned}$$

where $F_{n,\nu}\left(k+\frac{q}{3}, l-\frac{q}{3}\right)$ is defined by

$$F_{n,\nu}\left(k+\tfrac{q}{3}, l-\tfrac{q}{3}\right) = (-1)^{n-\nu}\frac{\Gamma(\nu+k+\frac{q}{3}+1)}{\Gamma(k+\frac{q}{3}+1)} + (-1)^{\nu}\frac{\Gamma(\nu+l-\frac{q}{3}+1)}{\Gamma(l-\frac{q}{3}+1)}. \tag{4.5}$$

(e) *There exist $c \in \mathbb{C}$ with $|c| > 1$ and $q \in \{1,2\}$, $(k,l) \in \mathbb{Z}^2$ with $k+l \geq -1$, such that the form w is a strict third degree semiclassical form of class one satisfying*

$$\phi_w(z)S'(w)(z) = C_w(z)S(w)(z) + D_w(z),$$

with

$$\phi_w(z) = (z-c)(z^2-1),$$
$$C_w(z) = (k+l+1)z^2 + \big(l-k-\tfrac{2q}{3} - c(k+l)\big)z - c\big(l-k-\tfrac{2q}{3}\big) - 1,$$
$$D_w(z) = -\big(3l+k+2-\tfrac{2q}{3} - c(k+l)\big)z + c\big(l-k-\tfrac{2q}{3} + c(k+l+1)\big),$$

where $(w)_0 = \frac{3k-3l+2q}{3(k+l+2)} - c$ *and* $(w)_1 = 2\frac{(3k+3+q)(3k+6+q)+(3l+3-q)(3l+6-q)}{3(k+l+2)(k+l+3)} - 1 - c\frac{3k-3l+2q}{3(k+l+2)}$.

For the proof we need the following lemma:

Lemma 4.1. [12] *Let* ϖ_1 *and* ϖ_2 *be two semiclassical linear forms satisfying* (2.3) *with* $\deg\phi = \deg\psi + 1 = t$. *If* $(\varpi_1)_i = (\varpi_2)_i,\ 0 \le i \le t-2$, *then* $\varpi_1 = \varpi_2$.

Proof of Theorem 4.1. $(a) \Rightarrow (b)$ Applying the operator S to (4.1) and taking into account (2.1) we obtain the desired relation.
$(b) \Rightarrow (c)$ Multiplying both sides of (4.2) by $f_q^{k,l}(z)$, from (2.1) we deduce

$$\begin{aligned} f_q^{k,l}(z)S(w)(z) &= (z-c)S\big(f_q^{k,l}\mathcal{V}_q^{k,l}\big)(z) - (z-c)\big(\mathcal{V}_q^{k,l}\theta_0 f_q^{k,l}\big)(z) + f_q^{k,l}(z) \\ &\overset{by\ (3.5)}{=} (z-c)S\big(g_q^{k,l}h_{(-1)^{q-1}}\mathcal{V}\big)(z) - (z-c)\big(\mathcal{V}_q^{k,l}\theta_0 f_q^{k,l}\big)(z) + f_q^{k,l}(z) \\ &\overset{by\ (2.1)}{=} (z-c)g_q^{k,l}(z)S(h_{(-1)^{q-1}}\mathcal{V})(z) \\ &\quad + (z-c)\Big(\big((h_{(-1)^{q-1}}\mathcal{V})\theta_0 g_q^{k,l}\big) - \big(\mathcal{V}_q^{k,l}\theta_0 f_q^{k,l}\big)\Big)(z) + f_q^{k,l}(z), \end{aligned}$$

which readily gives

$$S\big(f_q^{k,l}(x)w\big)(z) = S\big((x-c)g_q^{k,l}(x)(h_{(-1)^{q-1}}\mathcal{V})\big)(z) + Q(z),$$

with

$$\begin{aligned} Q(z) =& (z-c)\Big(\big((h_{(-1)^{q-1}}\mathcal{V})\theta_0 g_q^{k,l}\big) - \big(\mathcal{V}_q^{k,l}\theta_0 f_q^{k,l}\big)\Big)(z) \\ &+ \big(w\theta_0 f_q^{k,l}(x)\big)(z) - \big((h_{(-1)^{q-1}}\mathcal{V})\theta_0((x-c)g_q^{k,l}(x))\big)(z) + f_q^{k,l}(z), \end{aligned}$$

or equivalently,

$$S\big(f_q^{k,l}(x)w - (x-c)g_q^{k,l}(x)(h_{(-1)^{q-1}}\mathcal{V})\big)(z) = Q(z) \in \mathcal{P}.$$

We may now invoke Remark 2.1 to deduce that

$$f_q^{k,l}(x)w - (x-c)g_q^{k,l}(x)(h_{(-1)^{q-1}}\mathcal{V}) = 0 \text{ in } \mathcal{P}',$$

and

$$Q(z) = 0.$$

Thus the result follows.
$(c) \Rightarrow (d)$ Applying the operator S to (4.3) and taking into account (2.1) we get

$$\begin{aligned} f_q^{k,l}(z)S(w)(z) =& (z-c)g_q^{k,l}(z)S\big(h_{(-1)^{q-1}}\mathcal{V}\big)(z) - \big(w\theta_0 f_q^{k,l}\big)(z) \\ &+ \big((h_{(-1)^{q-1}}\mathcal{V})\theta_0((x-c)g_q^{k,l}(x))\big)(z). \end{aligned}$$

Then, one has

$$\begin{aligned} f_q^{k,l}(z)S(w)(z) &\overset{by\ (2.1)-(3.5)}{=} (z-c)S\big(f_q^{k,l}\mathcal{V}_q^{k,l}\big)(z) - \big(w\theta_0 f_q^{k,l}\big)(z) \\ &\quad + \big((h_{(-1)^{q-1}}\mathcal{V})\theta_0((x-c)g_q^{k,l}(x))\big)(z) \\ &\quad - (z-c)\big((h_{(-1)^{q-1}}\mathcal{V})\theta_0 g_q^{k,l}\big)(z) \\ &\overset{by\ (2.1)}{=} (z-c)f_q^{k,l}(z)S\big(\mathcal{V}_q^{k,l}\big)(z) - \big(w\theta_0 f_q^{k,l}\big)(z) \\ &\quad + \big((h_{(-1)^{q-1}}\mathcal{V})\theta_0((x-c)g_q^{k,l}(z))\big)(z) \\ &\quad - (z-c)\Big(\big((h_{(-1)^{q-1}}\mathcal{V})\theta_0 g_q^{k,l}\big) - \big(\mathcal{V}_q^{k,l}\theta_0 f_q^{k,l}\big)\Big)(z). \end{aligned}$$

Therefore, using (4.4), the last equation becomes

$$f_q^{k,l}(z)S(w)(z) = (z-c)f_q^{k,l}(z)S\big(\mathcal{V}_q^{k,l}\big)(z) + f_q^{k,l}(z).$$

This clearly implies that

$$S(w)(z) = (z-c)S\big(\mathcal{V}_q^{k,l}\big)(z) + 1.$$

Then, using (2.1) and by Remark 2.1, we get

$$w = (x-c)\mathcal{V}_q^{k,l},$$

so,

$$(w)_n = \big(\mathcal{V}_q^{k,l}\big)_{n+1} - c\big(\mathcal{V}_q^{k,l}\big)_n,\ n \geq 0.$$

Hence, using (4.6) we have the desired relation.

$(d) \Rightarrow (e)$ Let us recall that the moments of the Jacobi form $\mathcal{V}_q^{k,l}$ with $k+l \geq -1,\ k,l \in \mathbb{Z}, q \in \{1,2\}$, are given by ([20])

$$\big(\mathcal{V}_q^{k,l}\big)_n = \sum_{\nu=0}^{n} \binom{n}{\nu} 2^{\nu-1} \frac{\Gamma(k+l+2)}{\Gamma(\nu+k+l+2)} F_{n,\nu}\left(k+\tfrac{q}{3}, l-\tfrac{q}{3}\right),\ n \geq 0, \tag{4.6}$$

where $F_{n,\nu}\left(k+\frac{q}{3}, l-\frac{q}{3}\right)$ is defined in (4.5), and Γ is the gamma function [20].
By hypothesis we have

$$\begin{aligned} (w)_n &= \big(\mathcal{V}_q^{k,l}\big)_{n+1} - c\big(\mathcal{V}_q^{k,l}\big)_n \\ &= \big((x-c)\mathcal{V}_q^{k,l}\big)_n,\ n \geq 0. \end{aligned}$$

Then,

$$w = (x-c)\mathcal{V}_q^{k,l}. \tag{4.7}$$

Using Lemma 3.1 we conclude that w is an STDRF.
Then, the relation (4.7) becomes

$$S(w)(z) = (z-c)S\big((h_{(-1)^{q-1}}\mathcal{V})\big)(z) + 1. \tag{4.8}$$

Taking formal derivatives in the last equation we get

$$S'(w)(z) = (z-c)S'\big(\mathcal{V}_q^{k,l}\big)(z) + S\big(\mathcal{V}_q^{k,l}\big)(z).$$

After combining the latter two expressions we obtain

$$S'(\mathcal{V}_q^{k,l})(z) = \frac{(z-c)S'(w)(z) - S(w)(z) + 1}{(z-c)^2}. \tag{4.9}$$

On the other hand, using the first order linear differential equation satisfied by the Stieltjes function of the Jacobi form ([19]), it is a straightforward exercise to prove that $S\big(\mathcal{V}_q^{k,l}\big)(z)$ satisfies

$$\Phi(z)S'\big(\mathcal{V}_q^{k,l}\big)(z) = C_{0,q}^{k,l}(z)S\big(\mathcal{V}_q^{k,l}\big)(z) + D_{0,q}^{k,l}(z), \tag{4.10}$$

with $\Phi, C_{0,q}^{k,l}$, and $D_{0,q}^{k,l}$ given by

$$\begin{aligned} \Phi(z) &= z^2 - 1, \\ C_{0,q}^{k,l}(z) &= (k+l)z + l - k - \frac{2q}{3}, \\ D_{0,q}^{k,l}(z) &= k + l + 1. \end{aligned} \tag{4.11}$$

Replacing (4.8) and (4.9) in (4.10), and multiplying both sides of the resulting equation by $(z-c)^2$, one obtains

$$\phi_w(z)S'(w)(z) = C_w(z)S(w)(z) + D_w(z), \tag{4.12}$$

where the polynomials ϕ_w, C_w and D_w are

$$\begin{aligned} \phi_w(z) &= (z-c)\Phi(z), \\ C_w(z) &= \Phi(z) + (z-c)C_{0,q}^{k,l}(z), \\ D_w(z) &= -\Phi(z) - (z-c)C_{0,q}^{k,l}(z) + (z-c)^2 D_{0,q}^{k,l}(z). \end{aligned}$$

Therefore, it follows from (4.11) that $S(w)(z)$ satisfies (4.12) with

$$\begin{aligned} \phi_w(z) &= (z-c)(z^2-1), \\ C_w(z) &= (k+l+1)z^2 + \big(l - k - \tfrac{2q}{3} - c(k+l)\big)z - c\big(l - k - \tfrac{2q}{3}\big) - 1, \\ D_w(z) &= -\big(3l + k + 2 - \tfrac{2q}{3} - c(k+l)\big)z + c\big(l - k - \tfrac{2q}{3} + c(k+l+1)\big). \end{aligned}$$

Further, we see that conditions

$$\begin{aligned} \Phi(c) &= c^2 - 1 \neq 0, \\ C_w(1) &= 2(1-c)(l - \tfrac{q}{3}) \neq 0, \\ C_w(-1) &= 2(1+c)(k + \tfrac{q}{3}) \neq 0, \end{aligned}$$

hold. Then ϕ_w, C_w, and D_w are coprime. Since $\deg D_w \leq 1$ and $\deg C_w = 2$ the class of the linear form w is one.

$(e) \Rightarrow (a)$ It is easy to verify that the form $(x-c)\mathcal{V}_q^{k,l}$ satisfies the same functional equation as w with $\deg \phi_w = \deg \psi_w + 1 = 3$, as well as

$$\big((x-c)\mathcal{V}_q^{k,l}\big)_0 = \frac{3k - 3l + 2q}{3(k+l+2)} - c = (w)_0$$

and

$$\big((x-c)\mathcal{V}_q^{k,l}\big)_1 = 2\frac{(3k+3+q)(3k+6+q) + (3l+3-q)(3l+6-q)}{3(k+l+2)(k+l+3)} - 1 - c\frac{3k-3l+2q}{3(k+l+2)} = (w)_1.$$

As a consequence, using Lemma 4.1 we conclude that $w = (x-c)\mathcal{V}_q^{k,l}$. Thus, the proof of Theorem 4.1 is completed. □

Corollary 4.1. *Let w be a linear form such that w is a canonical Christoffel transformation of the strict third degree classical linear forms. Then, there exist*

$c \in \mathbb{C}$ with $|c| > 1$ and $(k,l) \in \mathbb{Z}^2$ with $k+l \geq -1$, $q \in \{1,2\}$, such that its Stieltjes function $S(w)(z)$ satisfies the cubic equation

$$A_w(z)S^3(w)(z) + B_w(z)S^2(w)(z) + C_w(z)S(w)(z) + D_w(z) = 0,$$

with

$$\begin{aligned}
A_w(z) =&(z^2-1)\big(z+(-1)^{q-1}\big)(f_q^{k,l})^3(z),\\
B_w(z) =&3(x-c)(z^2-1)\big(z+(-1)^{q-1}\big)\left(\mathcal{V}_q^{k,l}\theta_0 f_q^{k,l}\right)(z) - 3(z^2-1)\big(z+(-1)^{q-1}\big)(f_q^{k,l})^3(z),\\
C_w(z) =&3(x-c)^2(z^2-1)\big(z+(-1)^{q-1}\big)\left(\left(\mathcal{V}_q^{k,l}\theta_0 f_q^{k,l}\right)(z) - \big((h_{(-1)^{q-1}}\mathcal{V})\theta_0 g_q^{k,l}\big)(z)\right)^2 f_q^{k,l}(z)\\
&-6(x-c)(z^2-1)\big(z+(-1)^{q-1}\big)\left(\mathcal{V}_q^{k,l}\theta_0 f_q^{k,l}\right)(z) + 3(z^2-1)\big(z+(-1)^{q-1}\big)(f_q^{k,l})^3(z),\\
D_w(z) =&(x-c)^3\Big[(g_q^{k,l})^3(z) + (z^2-1)\big(z+(-1)^{q-1}\big)\left(\left(\mathcal{V}_q^{k,l}\theta_0 f_q^{k,l}\right)(z) - \big((h_{(-1)^{q-1}}\mathcal{V})\theta_0 g_q^{k,l}\big)(z)\right)^3(z)\Big]\\
&-3(x-c)^2(z^2-1)\big(z+(-1)^{q-1}\big)\left(\left(\mathcal{V}_q^{k,l}\theta_0 f_q^{k,l}\right)(z) - \big((h_{(-1)^{q-1}}\mathcal{V})\theta_0 g_q^{k,l}\big)(z)\right)^2 f_q^{k,l}(z)\\
&+3(x-c)(z^2-1)\big(z+(-1)^{q-1}\big)(\left(\mathcal{V}_q^{k,l}\theta_0 f_q^{k,l}\right)(z) - (z^2-1)\big(z+(-1)^{q-1}\big)(f_q^{k,l})^3(z).
\end{aligned}$$

where $f_q^{k,l}$ and $g_q^{k,l}$ are polynomials given in (3.6) and (3.7), respectively.

Proof. Based on relations (3.4) and (3.5) and Lemma 3.1, the form $\mathcal{V}_q^{k,l}$ is an STDRF and its Stieltjes function $S\big(\mathcal{V}_q^{k,l}\big)(z)$ satisfies the cubic equation

$$A_q^{k,l}(z)S^3\big(\mathcal{V}_q^{k,l}\big)(z) + B_q^{k,l}(z)S^2\big(\mathcal{V}_q^{k,l}\big)(z) + C_q^{k,l}(z)S\big(\mathcal{V}_q^{k,l}\big)(z) + D_q^{k,l}(z) = 0, \quad (4.13)$$

with

$$\begin{aligned}
A_q^{k,l}(z) &= (z^2-1)\big(z+(-1)^{q-1}\big)(f_q^{k,l})^3(z),\\
B_q^{k,l}(z) &= 3(z^2-1)\big(z+(-1)^{q-1}\big)\left(\left(\mathcal{V}_q^{k,l}\theta_0 f_q^{k,l}\right)(z) - \big((h_{(-1)^{q-1}}\mathcal{V})\theta_0 g_q^{k,l}\big)(z)\right)(f_q^{k,l})^2(z),\\
C_q^{k,l}(z) &= 3(z^2-1)\big(z+(-1)^{q-1}\big)\left(\left(\mathcal{V}_q^{k,l}\theta_0 f_q^{k,l}\right)(z) - \big((h_{(-1)^{q-1}}\mathcal{V})\theta_0 g_q^{k,l}\big)(z)\right)^2 f_q^{k,l}(z),\\
D_q^{k,l}(z) &= (g_q^{k,l})^3(z) + (z^2-1)\big(z+(-1)^{q-1}\big)\left(\left(\mathcal{V}_q^{k,l}\theta_0 f_q^{k,l}\right)(z) - \big((h_{(-1)^{q-1}}\mathcal{V})\theta_0 g_q^{k,l}\big)(z)\right)^3(z).
\end{aligned} \quad (4.14)$$

On the other hand, according to Lemma 3.2 with $v = w, u = \mathcal{V}_q^{k,l}, N(z) = 1$ and $M(z) = z - c$, we obtain

$$\begin{aligned}
A_w(z) &= A_q^{k,l}(z),\\
B_w(z) &= (x-c)B_q^{k,l}(z) - 3A_q^{k,l}(z),\\
C_w(z) &= (x-c)^2C_q^{k,l}(z) - 2(x-c)B_q^{k,l}(z) + 3A_q^{k,l}(z),\\
D_w(z) &= (x-c)^3D_q^{k,l}(z) - (x-c)^2C_q^{k,l}(z) + (x-c)B_q^{k,l}(z) - A_q^{k,l}(z).
\end{aligned}$$

Thus, the desired relation holds. □

Concluding remarks

In this work we have analyzed an example of strict third degree linear forms which are also semiclassical of class one. An interesting question is to describe all strict third degree semiclassical linear forms of class one.

The preservation of such a family is ensured by rational spectral transformations of third degree linear forms, as highlighted in [7]. It is important to note, as outlined in [23], that these transformations encompass Christoffel, Geronimus, association, and anti-association transformations. Describing the transformations of linear forms that maintain the third degree character poses an interesting problem.

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Mohamed Khalfallah
University of Monastir, Faculty of Sciences of Monastir,
Department of Mathematics, 5019 Monastir, Tunisia
E-mail: khalfallah.mohamed91@gmail.com