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FIXED POINT IN BANACH *-ALGEBRAS WITH AN APPLICATION TO FUNCTIONAL INTEGRAL EQUATION OF FRACTIONAL ORDER

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Abstract. In this paper, we investigate the solvability of an operator equation involving four operators in the setting of Banach *-algebras using Schauder's fixed point theorem. Moreover, we have given an application of our result to the following functional integral equation of fractional order:

$$\xi(t) = g(t, \xi(\psi_1(t)))I^\alpha f_1(t, I^\beta u(t, \xi(\psi_2(t)))) + h(t, \xi(\psi_3(t)))I^\gamma f_2(t, I^\delta v(t, \xi^*(\psi_4(t))))$$

for proving the existence as well as the uniqueness of the solution in Banach *-algebras under some generalized conditions.

1. Introduction

Topological fixed point theorems, including the Schauder fixed point principle, the Leray-Schauder nonlinear alternative, and the topological transversality principle, serve as powerful tools in analyzing nonlinear differential and integral equations. These theorems are instrumental in establishing the existence of solutions under specific compactness conditions, thereby providing crucial insights into the behaviour of nonlinear systems. Fixed point theorems in Banach algebras was introduced by Dhage [7] in 1988. After that several researchers (refer to [1], [19], [33]) have developed different important findings in this field. The term $\mathcal{D}$-Lipschitzian was defined by Dhage [8] in 2003 by generalizing the concept of Lipschitzian mappings. In 2012, Pathak et al. [28] defined the concept of $\mathcal{P}$-Lipschitzian mappings and established some fixed point results with examples. Similar type of fixed point results are done by Dhage and many other researchers including two operators as well as three operators (refer to [5], [12], [14], [18]).

In recent years, there has been a rise in interest among researcher to explore quadratic functional integral equations, marking this field as one of the most dynamic areas within integral equations and functional integral equations. Numerous interesting existence results have emerged, showing the significance of this research domain.

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For an overview of some of the latest findings in integral equations as well as fixed point theory, we refer to [2,3,15–17,21,22,24–27,29,31,32] and the references therein.

Motivated by these findings, in this paper, we have derived a fixed point result in Banach *-algebra involving four operators with some generalized conditions. As an application of this result, we have given an existence and uniqueness result of the solution to the following nonlinear quadratic functional integral equation of fractional order:

$$\xi(t) = g(t, \xi(\psi_1(t)))I^{\alpha} f_1(t, I^{\beta} u(t, \xi(\psi_2(t)))) + h(t, \xi(\psi_3(t)))I^{\gamma} f_2(t, I^{\delta} v(t, \xi^*(\psi_4(t)))), \tag{1}$$

where $\alpha, \beta, \gamma, \delta \in (0, 1)$ with $g, h : [0, T] \times \mathbb{R} \to \mathbb{R} \setminus \{0\},\ f_1, f_2, u, v : [0, T] \times \mathbb{R} \to \mathbb{R}$ and $\psi_1, \psi_2, \psi_3, \psi_4 : [0, T] \to [0, T]$.

2. Preliminaries

In this section, we present the basic definitions and required results for our paper.

Definition 2.1. [20] The Riemann-Liouville fractional integral of the function $f \in L^1(J)$ of order $\alpha \in \mathbb{R}^+$ is defined by

$$I_x^{\alpha} f(t) = \int_x^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s) ds,$$

where $x, t \in J$, $\Gamma(.)$ is Euler's gamma function, $L^1(J)$ is the class of Lebesgue integrable functions on the interval $J = [0, T]$.

Definition 2.2. [4] In an algebra $\mathbb{A}$, for $x, x^* \in \mathbb{A}$, an involution is a self mapping on $\mathbb{A}$ with $x \to x^*$ such that

(i) $(x + y)^* = x^* + y^*$,
(ii) $(x^*)^* = x$,
(iii) $(xy)^* = y^* x^*$,
(iv) $(\alpha x^*) = \bar{\alpha} x^*$

for all $x, y \in \mathbb{A}$ and for all scalars α, where x^* is called the adjoint of x.

An algebra $\mathbb{A}$ with an involution is called a *-algebra. A Banach *-algebra is a Banach algebra $\mathbb{A}$ with an involution '*' defined on it.

Example 2.3. [4] Let $\mathbb{A}$ be the algebra of all $n \times n$ complex matrices and let $a = (a_{ij}) \in \mathbb{A}$. Then $\mathbb{A}$ is a Banach *-algebra, where $a^* = (\overline{a_{ji}})$.

Definition 2.4. [8] A mapping T on a Banach space X is called $\mathcal{D}$-Lipschitzian if there exists a continuous and non-decreasing function $\phi : \mathbb{R}^+ \cup \{0\} \to \mathbb{R}^+ \cup \{0\}$ such that

$$||T\xi - T\eta|| \leq \phi(||\xi - \eta||),$$

for all $\xi, \eta \in X$, where $\phi(0) = 0$.

The function ϕ is called a $\mathcal{D}$-function of T on X. It is clear that every Lipschitzian mapping is $\mathcal{D}$-Lipschitzian, but the converse is not always true.

Definition 2.5. [28] A mapping T on a Banach space X is called a $\mathcal{P}$-Lipschitzian if there exists a non-decreasing function $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ such that

$$||T\xi - T\eta|| \leq \phi(||\xi - \eta||),$$

for all $\xi, \eta \in X$.
The function ϕ is also called a $\mathcal{P}$-function of T on X. Every $\mathcal{D}$-Lipschitzian mapping is a $\mathcal{P}$-Lipschitzian mapping, but the converse is not true.

Example 2.6. [28] Consider $X = \mathbb{R}$. Let the mapping $T : X \to X$ be defined by

$$T(\xi) = \begin{cases} \sin\xi, & \xi \geq 0, \\ \frac{1}{1+|\xi|}, & \xi < 0 \end{cases}$$

and $\phi : \mathbb{R}^+ \cup \{0\} \to \mathbb{R}^+ \cup \{0\}$ be defined by

$$\phi(t) = \begin{cases} e^t, & t > 0, \\ 2, & t = 0. \end{cases}$$

Here, T is a $\mathcal{P}$-Lipschitzian mapping, but not $\mathcal{D}$-Lipschitzian.

For a Banach space X, an operator $T : X \to X$ is called a compact operator if $\overline{T(X)}$ is a compact subset of X. Again, T is called totally bounded if for any bounded subset Y of X, $T(Y)$ is a totally bounded set of X. T is called completely continuous if it is continuous as well as totally bounded. A compact operator is totally bounded. However, the converse holds for bounded subsets of X.

Theorem 2.7. *(Schauder's fixed point theorem, [30]) Let X be a Banach space over $\mathbb{K}$ ($\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$) and Δ is a non-empty closed, convex and bounded subset of X. Then any compact operator $T : \Delta \to \Delta$ has atleast one fixed point.*

3. Main Results

Extending the results of Dhage [9] and Pathak et al. [28], we obtain the following fixed point results involving four operators in the setting of Banach *-algebras.

Theorem 3.1. *Let Δ be a closed, convex and bounded subset of a Banach *-algebra X such that if $\xi \in \Delta$, then $\xi^* \in \Delta$. Let $P, R : X \to X$, $Q, S : \Delta \to X$ be four operators such that*
(i) P and R are $\mathcal{P}$-Lipschitzians with $\mathcal{P}$-functions ϕ_P and ϕ_R respectively,
(ii) Q, S are completely continuous,
(iii) $M\phi_P(r) + N\phi_R(r) < r$, $r > 0$ where $M = ||Q(\Delta)||$ and $N = ||S(\Delta)||$,
(iv) $||S\xi^* - S\xi_k^*|| \leq ||Q\xi - Q\xi_k||$ for every $\xi, \xi_k \in \Delta$,
(v) $\xi = P\xi Q\eta + R\xi S\eta^* \implies \xi \in \Delta$ for all $\eta \in \Delta$.
Then the operator equation $\xi = P\xi Q\xi + R\xi S\xi^*$ has a solution.

Proof. Let $\eta \in \Delta$ and define a mapping $P_\eta : X \to X$ by

$$P_\eta(\xi) = P\xi Q\eta + R\xi S\eta^*, \ \xi \in X.$$

Now for $\xi_1, \xi_2 \in X$,

$$\begin{aligned} ||P_\eta(\xi_1) - P_\eta(\xi_2)|| &\leq ||P\xi_1 - P\xi_2|| \, ||Q\eta|| + ||R\xi_1 - R\xi_2|| \, ||S\eta^*|| \\ &\leq M\phi_P(||\xi_1 - \xi_2||) + N\phi_R(||\xi_1 - \xi_2||) < ||\xi_1 - \xi_2||. \end{aligned}$$

By hypothesis (iii), P_η is a contraction on X and so, there exists a unique fixed point $z \in X$ such that

$$P_\eta(z) = z,$$
$$i.e., \;\; PzQ\eta + RzS\eta^* = z.$$

By (v) we have $z \in \Delta$.
We define a mapping $\Omega : \Delta \to X$ such that

$$\Omega\eta = w,$$

where $w \in X$ is the unique solution of the equation:

$$w = PwQ\eta + RwS\eta^*, \;\; \eta \in \Delta.$$

We consider a sequence $\{\eta_n\}$ in Δ converging to a point η. Since Δ is closed, $\eta \in \Delta$. Now,

$$\begin{aligned} ||\Omega\eta_n - \Omega\eta|| &\le ||P\Omega\eta_n Q\eta_n - P\Omega\eta Q\eta|| + ||R\Omega\eta_n S\eta_n^* - R\Omega\eta S\eta^*|| \\ &\le ||P\Omega\eta_n Q\eta_n - P\Omega\eta Q\eta_n|| + ||P\Omega\eta Q\eta_n - P\Omega\eta Q\eta|| \\ &+ ||R\Omega\eta_n S\eta_n^* - R\Omega\eta S\eta_n^*|| + ||R\Omega\eta S\eta_n^* - R\Omega\eta S\eta^*|| \\ &\le ||P\Omega\eta_n - P\Omega\eta||\;||Q\eta_n|| + ||P\Omega\eta||\;||Q\eta_n - Q\eta|| \\ &+ ||R\Omega\eta_n - R\Omega\eta||\;||S\eta_n^*|| + ||R\Omega\eta||\;||S\eta_n^* - S\eta^*||. \end{aligned}$$

Since, $M\phi_P(r) + N\phi_R(r) < r$, $r > 0$, there exists $\lambda \in (0, 1)$ such that

$$M\phi_P(r) + N\phi_R(r) = \lambda r.$$

Then the above inequality becomes

$$||\Omega\eta_n - \Omega\eta|| \le \lambda||\Omega\eta_n - \Omega\eta|| + ||P\Omega\eta||\;||Q\eta_n - Q\eta|| + ||R\Omega\eta||\;||S\eta_n^* - S\eta^*||.$$

Taking limit superior as $n \to \infty$ on both sides of the above inequality we get,

$$\lim_{n\to\infty} \sup ||\Omega\eta_n - \Omega\eta|| = 0.$$

This shows that Ω is continuous on Δ. Now we show that P, R are compact operators on Δ. For any $w \in \Delta$ we have

$$\begin{aligned} ||Pw|| &\le ||Pa|| + ||Pw - Pa|| \\ &\le ||Pa|| + \alpha||w - a|| \\ &\le c_1, \end{aligned}$$

where $c_1 = ||Pa|| + \alpha\; diam(\Delta)$ for some fixed $a \in \Delta$ and $diam(\Delta) = \sup\{||\xi - \eta|| : \xi, \eta \in \Delta\}$.
Similarly, $|||Rw|| \le c_2$ where $c_2 = ||Rb|| + diam(\Delta)$ for some fixed $b \in \Delta$.
Since, Q is completely continuous, $Q(\Delta)$ is totally bounded. Then there exists a set $Y = \{\eta_1, \eta_2, ..., \eta_n\}$ in Δ such that

$$Q(\Delta) \subset \bigcup_{i=1}^{n} B_\delta(x_i),$$

where $x_i = Q(\eta_i)$, $\delta = (\frac{1-(\alpha M+\beta N)}{c_1+c_2})\varepsilon$ and $B_\delta(x_i)$ is an open ball in X centered at x_i of radius δ. Hence, for any $\eta \in \Delta$ we have an $\eta_k \in Y$ such that

$$||Q\eta - Q\eta_k|| < \left(\frac{1-(\alpha M+\beta N)}{c_1+c_2}\right)\varepsilon.$$

Now,

$$\begin{aligned}||\Omega\eta - \Omega\eta_k|| &\leq ||PwQ\eta - Pw_kQ\eta_k|| + ||RwS\eta^* - Rw_kS\eta_k^*||\\ &\leq ||PwQ\eta - Pw_kQ\eta|| + ||Pw_kQ\eta - Pw_kQ\eta_k||\\ &+ ||RwS\eta^* - Rw_kS\eta^*|| + ||Rw_kS\eta^* - Rw_kS\eta_k^*||\\ &\leq ||Pw - Pw_k||\ ||Q\eta|| + ||Pw_k||\ ||Q\eta - Q\eta_k||\\ &+ ||Rw - Rw_k||\ ||S\eta^*|| + ||Rw_k||\ ||S\eta^* - S\eta_k^*||\\ &\leq (\alpha M + \beta N)||w - w_k|| + (c_1 + c_2)||Q\eta - Q\eta_k||\\ &\leq \frac{c_1 + c_2}{1 - (\alpha M + \beta N)}||Q\eta - Q\eta_k||\\ &< \varepsilon.\end{aligned}$$

This is true for every $\eta \in \Delta$ and so

$$\Omega(\Delta) \subset \bigcup_{i=1}^{n} B_\varepsilon(w_i),$$

where $w_i = \Omega(\eta_i)$. Hence, $\Omega(\Delta)$ is totally bounded. Since Ω is continuous, it is a compact operator on Δ. Now applying the Schauder's fixed point theorem, Ω has a fixed point in Δ.Then

$$\xi = \Omega\xi = P(\Omega\xi)Q\xi + R(\Omega\xi)S\xi^* = P\xi Q\xi + R\xi S\xi^*,$$

and so, the operator equation $\xi = P\xi Q\xi + R\xi S\xi^*$ has a solution in Δ. □

REMARK 3.2. Taking P as a $\mathcal{D}$-Lipschitzian mapping and $R = S = O$ (zero operator), our result reduces to the Theorem 2.1 of [10], in the setting of Banach algebra. Again, considering X as a unital Banach algebra with unit element e, and $S(\xi) = e$ for all $\xi \in \Delta$, we get Theorem 4.1 of [28].

THEOREM 3.3. *Let Δ be a closed, convex and bounded subset of a Banach *-algebra X such that if $\xi \in \Delta$, then $\xi^* \in \Delta$. Let $P, R : X \to X$ and $Q, S : \Delta \to X$ be four operators satisfying*
(i) P and R are $\mathcal{P}$-Lipschitzians with $\mathcal{P}$-functions ϕ_P and ϕ_R respectively,
(ii) $\left(\frac{I}{P+R}\right)^{-1}$ exists on $Q(\Delta)$, where I is the identity operator on X,
(iii) Q, S are completely continuous, and
(iv) $M\phi_P(r) + N\phi_R(r) < r$, $r > 0$ where $M = ||Q(\Delta)||$ and $N = ||S(\Delta)||$.
(v) $Q\xi = S\xi^*$ for any $\xi \in \Delta$.
Then the operator equation $P\xi Q\xi + R\xi S\xi^* = \xi$ has a solution in Δ.

Proof. Define an operator $T : \Delta \to X$ by

$$T = \left(\frac{I}{P+R}\right)^{-1} Q,$$

Since by (ii), $\left(\frac{I}{P+R}\right)^{-1}$ exists on $Q(\Delta)$, the composition $\left(\frac{I}{P+R}\right)^{-1} Q$ exists on Δ. Now, we show that

$$Q(\Delta) \subseteq \left(\frac{I}{P+R}\right)(X).$$

Let $\eta \in \Delta$ be fixed and define an operator P_η on X by

$$P_\eta(\xi) = P\xi Q\eta + R\xi S\eta^*,\ \xi \in X.$$

As in Theorem 3.1, P_η is a contraction on X and so, there exists a unique fixed point z in X such that

$$z = PzQ\eta + RzS\eta^*.$$

Using (v), we get,

$$z = (Pz + Rz)Q\eta$$
$$i.e., Q\eta = \Big(\frac{I}{P+R}\Big)z.$$

Thus, the operator T is well defined.
Now, we show that $\left(\frac{I}{P+R}\right)^{-1}$ is continuous on $Q(\Delta)$. Let $\{\xi_n\}$ be a sequence in $Q(\Delta)$ with $\xi_n \to \xi$ as $n \to \infty$.
For each n, we take

$$\Big(\frac{I}{P+R}\Big)^{-1}(\xi_n) = \eta_n \implies \xi_n P\eta_n + \xi_n R\eta_n = \eta_n.$$

Let

$$\Big(\frac{I}{P+R}\Big)^{-1}(\xi) = \eta \implies \xi P\eta + \xi R\eta = \eta.$$

Now,

$$\begin{aligned}
||\eta_n - \eta|| &= ||\xi_n P\eta_n + \xi_n R\eta_n - \xi P\eta - \xi R\eta|| \\
&\le ||\xi_n P\eta_n - \xi P\eta|| + ||\xi_n R\eta_n - \xi R\eta|| \\
&\le ||\xi_n P\eta_n - \xi_n P\eta|| + ||\xi_n P\eta - \xi P\eta|| + ||\xi_n R\eta_n - \xi_n R\eta|| + ||\xi_n R\eta - \xi R\eta|| \\
&\le ||\xi_n||\ ||P\eta_n - P\eta|| + ||P\eta||\ ||\xi_n - \xi|| + ||\xi_n||\ ||R\eta_n - R\eta|| + ||R\eta||\ ||\xi_n - \xi|| \\
&\le M\phi_P(||\eta_n - \eta||) + ||P\eta||\ ||\xi_n - \xi|| + N\phi_R(||\eta_n - \eta||) + ||R\eta||\ ||\xi_n - \xi||.
\end{aligned}$$

Hence

$$\limsup_n ||\eta_n - \eta|| \le M\phi_P(\limsup_n ||\eta_n - \eta||) + N\phi_R(\limsup_n ||\eta_n - \eta||).$$

If $\limsup_n ||\eta_n - \eta|| > 0$, we get a contradiction to (iv). Therefore, $\limsup_n ||\eta_n - \eta|| = 0$ and so,

$$\lim_n ||\Big(\frac{I}{P+R}\Big)^{-1}(\xi_n) - \Big(\frac{I}{P+R}\Big)^{-1}(\xi)|| = \lim_n ||\eta_n - \eta|| = 0.$$

Hence the operator $\left(\frac{I}{P+R}\right)^{-1}$ is continuous on $Q(\Delta)$. Since T is a composition of continuous and a completely continuous operator, so it is completely continuous on Δ. Hence by Schauder's fixed point theorem we get the solution. □

4. Application

In this section, we show the existence of solution of the functional integral equation of fractional order given by (1). For this, we consider the following conditions:

(C_1) The functions $g, h : [0, T] \times \mathbb{R} \to \mathbb{R} \setminus \{0\}$ are continuous and there exist two positive functions $L(t)$ and $K(t)$ with norms $||L||$ and $||K||$ respectively, such that

$$|g(t, x) - g(t, y)| \le L(t)|x - y| \text{ and } |h(t, x) - h(t, y)| \le K(t)|x - y|$$

for all $t \in [0,T]$ and $x, y \in \mathbb{R}$.
(C_2) The functions $f_1, f_2, u, v : [0,T] \times \mathbb{R} \to \mathbb{R}$ are measurable in t for any $\xi \in \mathbb{R}$ and continuous in ξ for almost all $t \in [0,T]$. There exist functions $a(.), b(.), c(.), d(.), m(.)$ and $n(.)$ such that

$$|f_1(t,\xi)| \leq a(t) + b(t)|\xi|, \ |u(t,\xi)| \leq m(t)$$

and

$$|f_2(t,\xi)| \leq c(t) + d(t)|\xi|, \ |v(t,\xi^*)| \leq n(t) \text{ for all } (t,\xi) \in [0,T] \times \mathbb{R},$$

where $a(.), c(.), m(.), n(.) \in L^1$ and $b(.), d(.)$ are measurable and bounded. Also,

$$I_q^{\gamma_1} a(.) \leq M_1, \ I_q^{\gamma_1} m(.) \leq M_2 \text{ and } I_q^{\gamma_2} c(.) \leq N_1, \ I_q^{\gamma_2} n(.) \leq N_2,$$

for all $\gamma_1 \leq \alpha$, $\gamma_2 \leq \gamma$ and $q \geq 0$.
(C_3) There exists a number $r > 0$ such that

$$\frac{G\Big(M_1 \frac{T^{\alpha-\gamma_1}}{\Gamma(\alpha-\gamma_1+1)} + ||b|| M_2 \frac{T^{\alpha+\beta-\gamma_1}}{\Gamma(\alpha+\beta-\gamma_1+1)}\Big) + H\Big(N_1 \frac{T^{\gamma-\gamma_2}}{\Gamma(\gamma-\gamma_2+1)} + ||d|| N_2 \frac{T^{\gamma+\delta-\gamma_2}}{\Gamma(\gamma+\delta-\gamma_2+1)}\Big)}{1 - ||L||\Big(M_1 \frac{T^{\alpha-\gamma_1}}{\Gamma(\alpha-\gamma_1+1)} + ||b|| M_2 \frac{T^{\alpha+\beta-\gamma_1}}{\Gamma(\alpha+\beta-\gamma_1+1)}\Big) - ||K||\Big(N_1 \frac{T^{\gamma-\gamma_2}}{\Gamma(\gamma-\gamma_2+1)} + ||d|| N_2 \frac{T^{\gamma+\delta-\gamma_2}}{\Gamma(\gamma+\delta-\gamma_2+1)}\Big)} \leq r,$$

where $G = \sup_{t\in[0,T]} |g(t,0)|$, $H = \sup_{t\in[0,T]} |h(t,0)|$ and

$$\begin{aligned} &||L||\Big(M_1 \frac{T^{\alpha-\gamma_1}}{\Gamma(\alpha-\gamma_1+1)} + ||b|| M_2 \frac{T^{\alpha+\beta-\gamma_1}}{\Gamma(\alpha+\beta-\gamma_1+1)}\Big) \\ &+ ||K||\Big(N_1 \frac{T^{\gamma-\gamma_2}}{\Gamma(\gamma-\gamma_2+1)} + ||d|| N_2 \frac{T^{\gamma+\delta-\gamma_2}}{\Gamma(\gamma+\delta-\gamma_2+1)}\Big) < 1. \end{aligned}$$

(C_4) The functions f_1, f_2, u, v defined above satisfy

$$\begin{aligned} &I^\gamma\Big(|f_2(t, I^\delta v(t, \xi^*(\psi_4(t)))) - f_2(t, I^\delta v(t, \xi_k^*(\psi_4(t))))|\Big) \\ &\leq I^\alpha\Big(|f_1(t, I^\beta u(t, \xi(\psi_2(t)))) - f_1(t, I^\beta u(t, \xi_k(\psi_2(t))))|\Big) \end{aligned}$$

(C_5) $\psi_i : [0,T] \to [0,T]$ are continuous functions with $\psi_i(0) = 0$, $i = 1,2,3,4$.

THEOREM 4.1. *Assume that the conditions $(C_1) - (C_5)$ hold. Then the nonlinear functional integral equation of fractional order (1) has atleast one solution defined on $[0,T]$.*

Proof. We consider $X = C(J,\mathbb{R})$ with $J = [0,T]$ and define a subset Δ of X such that

$$\Delta = \{\xi \in X, ||\xi|| \leq r\},$$

where r satisfies the first inequality in (C_3). Clearly Δ is closed, convex and bounded in X.
Now we define four operators; $P, R : X \to X$ and $Q, S : \Delta \to X$ by:
$P\xi(t) = g(t, \xi(\psi_1(t)))$, $Q\xi(t) = I^\alpha f_1(t, I^\beta u(t, \xi(\psi_2(t))))$, $R\xi(t) = h(t, \xi(\psi_3(t)))$ and $S\xi(t) = I^\gamma f_2(t, I^\delta v(t, \xi(\psi_4(t))))$, where $t \in J$, $\xi \in X$.
Then the integral equation (1) can be written as:

$$\xi(t) = P\xi(t) Q\xi(t) + R\xi(t) \dot{S}\xi^*(t), \ t \in J.$$

We show that P, Q, R and S satisfy all the conditions of Theorem 3.1.
Step 1: We first show that P and R are $\mathcal{D}$-Lipschitzian on X. Let $\xi, \eta \in X$ and using (C_1),

$$|P\xi(t)-P\eta(t)| = |g(t,\xi(\psi_1(t)))-g(t,\eta(\psi_1(t)))| \leq L(t)\ |\xi(\psi_1(t))-\eta(\psi_1(t))| \leq ||L||\ ||\xi-\eta||$$

which implies that, $||P\xi - P\eta|| \leq ||L||\ ||\xi - \eta||$ for all $\xi, \eta \in X$. Hence P is $\mathcal{D}$-Lipschitzian on X with $\mathcal{D}$-function $\phi_P(t) = ||L||t,\ t \in \mathbb{R}^+$. Similarly, we can show that R is also a $\mathcal{D}$-Lipschitzian on X with $\mathcal{D}$-function $\phi_R(t) = ||K||t,\ t \in \mathbb{R}^+$.
Step 2: We show that Q is continuous on Δ. Let $\{\xi_n\}$ be a sequence in Δ converging to a point $\xi \in \Delta$. Let us assume that $t \in J$ and since $u(t, \xi(t))$ is continuous in X, then $u(t, \xi_n(t))$ converges to $u(t, \xi(t))$. Then by Lebesgue dominated theorem and using (C_2), we get,

$$\lim_{n\to\infty} I^\beta u(s, \xi_n(\psi_2(s))) = I^\beta u(s, \xi(\psi_2(s))).$$

Since $f_1(t, \xi(t))$ is continuous in X,

$$\lim_{n\to\infty} Q\xi_n(t) = \lim_{n\to\infty} I^\alpha f_1(t, I^\beta u(t, \xi_n(\psi_2(t)))) = I^\alpha f_1(t, I^\beta u(t, \xi(\psi_2(t)))) = Q\xi(t).$$

Hence, $Q\xi_n \to Q\xi$ as $n \to \infty$ uniformly on $\mathbb{R}^+$ and so Q is continuous operator on Δ. Next we show that Q is a compact operator on Δ. Let $\xi \in \Delta$ be arbitrry. Proceeding as in Theorem 3.1 of [2] and using (C_2) we have,

$$||Q\xi(t)|| \leq M_1 \frac{T^{\alpha-\gamma_1}}{\Gamma(\alpha-\gamma_1+1)} + ||b||M_2 \frac{T^{\alpha-\beta-\gamma_1}}{\Gamma(\alpha-\beta-\gamma_1+1)} = k.$$

Thus, $||Q\xi(t)|| \leq k$ for all $\xi \in \Delta$. Hence, Q is uniformly bounded on Δ.
Now, we show that $Q(\Delta)$ is equicontinuous on X. Let $t_1, t_2 \in J$ and $\xi \in \Delta$. Without loss of generality, let $t_1 < t_2$. Then as in Theorem 3.1 of [2] we get,

$$|Q\xi(t_2)-Q\xi(t_1)| \leq ||a|| \Big\{ \frac{|t_2^\alpha - t_1^\alpha - 2(t_2-t_1)^\alpha|}{\Gamma(\alpha+1)} \Big\} + ||b||M_2 \Big\{ \frac{|t_2^\alpha - t_1^\alpha - 2(t_2-t_1)^\alpha| T^{\beta-\gamma}}{\Gamma(\alpha+1)\Gamma(\beta-\gamma+1)} \Big\}.$$

Therefore, for $\varepsilon > 0$, there exists $\delta > 0$ such that

$$|t_2 - t_1| < \delta \implies |Q\xi(t_2) - Q\xi(t_1)| < \varepsilon,$$

for all $t_1, t_2 \in J$ and $\xi \in \delta$. Hence $Q(\Delta)$ is equicontinuous in X and so, it is compact. Thus Q is completely continuous on Δ.
Similarly, S is also completely continuous on Δ.
Step 3: Let $\xi \in X$ and $\eta \in \Delta$ be arbitrary elements such that $\xi = P\xi Q\eta + R\xi S\eta^*$. Then

$$\begin{aligned}
|\xi(t)| &\leq |P\xi(t)|\ |Q\eta(t)| + |R\xi(t)|\ |S\eta^*(t)| \\
&\leq |g(t, \xi(\psi_1(t)))| \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} |f_1(s, I^\beta u(s, \eta(\psi_2(s))))| ds \\
&+ |h(t, \xi(\psi_3(t)))| \int_0^t \frac{(t-s)^{\gamma-1}}{\Gamma(\gamma)} |f_2(s, I^\delta v(s, \eta^*(\psi_4(s))))| ds \\
&\leq \Big\{ |g(t, \xi(\psi_1(t))) - g(t,0)| + |g(t,0)| \Big\} \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \Big\{ a(s) + b(s) I^\beta |u(s, \eta(\psi_2(s)))| \Big\} ds \\
&+ \Big\{ |h(t, \xi(\psi_3(t))) - h(t,0)| + |h(t,0)| \Big\} \int_0^t \frac{(t-s)^{\gamma-1}}{\Gamma(\gamma)} \Big\{ c(s) + d(s) I^\delta |v(s, \eta^*(\psi_4(s)))| \Big\} ds
\end{aligned}$$

$$\leq \Big\{||L||\ |\xi(\psi_1(t))| + G\Big\} \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \Big\{a(s) + b(s) I^{\beta} m(s)\Big\} ds$$
$$+ \Big\{||K||\ |\xi(\psi_3(t))| + H\Big\} \int_0^t \frac{(t-s)^{\gamma-1}}{\Gamma(\gamma)} \Big\{c(s) + d(s) I^{\delta} n(s)\Big\} ds$$
$$\leq \{||L||r_1 + G\}\{I^{\alpha}a(t) + ||b||I^{\alpha+\beta}m(t)\} + \{||K||r_1 + H\}\{I^{\gamma}c(t) + ||d||I^{\gamma+\delta}n(t)\}$$
$$\leq \{||L||r_1 + G\}\{I^{\alpha-\gamma_1}I^{\gamma_1}a(t) + ||b||I^{\alpha+\beta-\gamma_1}I^{\gamma_1}m(t)\}$$
$$+ \{||K||r_1 + H\}\{I^{\gamma-\gamma_2}I^{\gamma_2}c(t) + ||d||I^{\gamma+\delta-\gamma_2}I^{\gamma_2}n(t)\}$$
$$\leq \{||L||r_1 + G\}\{M_1 \int_0^t \frac{(t-s)^{\alpha-\gamma_1-1}}{\Gamma(\alpha-\gamma_1)} ds + ||b||M_2 \int_0^t \frac{(t-s)^{\alpha+\beta-\gamma_1-1}}{\Gamma(\alpha+\beta-\gamma_1)} ds\}$$
$$+ \{||K||r_1 + H\}\{N_1 \int_0^t \frac{(t-s)^{\gamma-\gamma_2-1}}{\Gamma(\gamma-\gamma_2)} ds + ||d||N_2 \int_0^t \frac{(t-s)^{\gamma+\delta-\gamma_2-1}}{\Gamma(\gamma+\delta-\gamma_2)} ds\}$$
$$\leq \{||L||r_1 + G\}\{M_1 \frac{(s)^{\alpha-\gamma_1}}{\Gamma(\alpha-\gamma_1+1)} + ||b||M_2 \frac{(s)^{\alpha+\beta-\gamma_1}}{\Gamma(\alpha+\beta-\gamma_1+1)}\}$$
$$+ \{||K||r_1 + H\}\{N_1 \frac{(s)^{\gamma-\gamma_2}}{\Gamma(\gamma-\gamma_2+1)} + ||d||N_2 \frac{(s)^{\gamma+\delta-\gamma_2}}{\Gamma(\gamma+\delta-\gamma_2+1)}\}$$
$$\leq \{||L||r_1 + G\}\{M_1 \frac{T^{\alpha-\gamma_1}}{\Gamma(\alpha-\gamma_1+1)} + ||b||M_2 \frac{T^{\alpha+\beta-\gamma_1}}{\Gamma(\alpha+\beta-\gamma_1+1)}\}$$
$$+ \{||K||r_1 + H\}\{N_1 \frac{T^{\gamma-\gamma_2}}{\Gamma(\gamma-\gamma_2+1)} + ||d||N_2 \frac{T^{\gamma+\delta-\gamma_2}}{\Gamma(\gamma+\delta-\gamma_2+1)}\}$$

So,

$$r_1 \leq \{||L||r_1 + G\}\{M_1 \frac{T^{\alpha-\gamma_1}}{\Gamma(\alpha-\gamma_1+1)} + ||b||M_2 \frac{T^{\alpha+\beta-\gamma_1}}{\Gamma(\alpha+\beta-\gamma_1+1)}\}$$
$$+ \{||K||r_1 + H\}\{N_1 \frac{T^{\gamma-\gamma_2}}{\Gamma(\gamma-\gamma_2+1)} + ||d||N_2 \frac{T^{\gamma+\delta-\gamma_2}}{\Gamma(\gamma+\delta-\gamma_2+1)}\}$$

$$\implies r_1 \leq \frac{G\Big(M_1 \frac{T^{\alpha-\gamma_1}}{\Gamma(\alpha-\gamma_1+1)} + ||b||M_2 \frac{T^{\alpha+\beta-\gamma_1}}{\Gamma(\alpha+\beta-\gamma_1+1)}\Big) + H\Big(N_1 \frac{T^{\gamma-\gamma_2}}{\Gamma(\gamma-\gamma_2+1)} + ||d||N_2 \frac{T^{\gamma+\delta-\gamma_2}}{\Gamma(\gamma+\delta-\gamma_2+1)}\Big)}{1 - ||L||\Big(M_1 \frac{T^{\alpha-\gamma_1}}{\Gamma(\alpha-\gamma_1+1)} + ||b||M_2 \frac{T^{\alpha+\beta-\gamma_1}}{\Gamma(\alpha+\beta-\gamma_1+1)}\Big) - ||K||\Big(N_1 \frac{T^{\gamma-\gamma_2}}{\Gamma(\gamma-\gamma_2+1)} + ||d||N_2 \frac{T^{\gamma+\delta-\gamma_2}}{\Gamma(\gamma+\delta-\gamma_2+1)}\Big)}$$

Taking supremum over t and using (C_3) we get,

$$||\xi(t)|| \leq r.$$

Hence, $\xi \in \Delta$.

Step 4: Next we show that $M\phi_P(r) + N\phi_R(r) < r$, $r > 0$. From step 2 we have,

$$M = ||Q(\Delta)|| \leq M_1 \frac{T^{\alpha-\gamma_1}}{\Gamma(\alpha-\gamma_1+1)} + ||b||M_2 \frac{T^{\alpha+\beta-\gamma_1}}{\Gamma(\alpha+\beta-\gamma_1+1)}.$$

In a similar way, we can show that

$$N = ||S(\Delta)|| \leq N_1 \frac{T^{\gamma-\gamma_2}}{\Gamma(\gamma-\gamma_2+1)} + ||d||N_2 \frac{T^{\gamma+\delta-\gamma_2}}{\Gamma(\gamma+\delta-\gamma_2+1)}.$$

Using (C_3) we get,

$$M\phi_P + N\phi_R < 1,$$

with $\phi_P = ||L||$ and $\phi_R = ||K||$.

Step 5: Finally we show that $||S\xi^* - S\xi_k^*|| \leq ||Q\xi - Q\xi_k||$ for every $\xi, \xi_k \in \Delta$.
Now, by (C_4),

$$\begin{aligned}|S\xi^*(t) - S\xi_k^*(t)| &\leq I^\gamma\Big(|f_2(t, I^\delta v(t, \xi^*(\psi_4(t)))) - f_2(t, I^\delta v(t, \xi_k^*(\psi_4(t))))|\Big) \\ &\leq |I^\alpha\Big(f_1(t, I^\beta u(t, \xi(\psi_2(t)))) - f_1(t, I^\beta u(t, \xi_k(\psi_2(t))))\Big)| \\ &= |Q\xi(t) - Q\xi_k(t)|.\end{aligned}$$

Taking supremum over t, we get

$$||S\xi^* - S\xi_k^*|| \leq ||Q\xi - Q\xi_k||.$$

Hence all the conditions of Theorem 3.1 is satisfied. So the operator equation $\xi = P\xi Q\xi + R\xi S\xi^*$ has a solution in Δ and hence the functional integral equation of fractional order (1) has a solution in J. □

Uniqueness of the solution:

Let us consider the following condition:

(C_6) Let $f_1, f_2 : [0,T]\times\mathbb{R} \to \mathbb{R}$ and $u, v : [0,T]\times\mathbb{R} \to \mathbb{R}$ be continuous functions satisfying the Lipschitz condition and there exists the positive functions $\Omega_1(t), \Omega_2(t), \Theta_1(t)$, $\Theta_2(t)$ with norms $||\Omega_1||$, $||\Omega_2||$, $||\Theta_1||$ and $||\Theta_2||$ such that

$$|f_1(t,\xi) - f_1(t,\eta)| \leq \Omega_1(t)|\xi - \eta|, \quad |u(t,\xi) - u(t,\eta)| \leq \Theta_1(t)|\xi - \eta|$$

and

$$|f_2(t,\xi) - f_2(t,\eta)| \leq \Omega_2(t)|\xi - \eta|, \quad |v(t,\xi) - v(t,\eta)| \leq \Theta_2(t)|\xi - \eta|,$$

for all $t \in [0,T]$ and $\xi, \eta \in \mathbb{R}$, where $F_1 = \sup\limits_{t\in[0,T]} |f_1(t,0)|$, $F_2 = \sup\limits_{t\in[0,T]} |f_2(t,0)|$, $U = \sup\limits_{t\in[0,T]} |u(t,0)|$ and $V = \sup\limits_{t\in[0,T]} |v(t,0)|$.

THEOREM 4.2. *Let the conditions of Theorem 4.1 be satisfied with replacing* (C_2) *by* (C_6). *Then the solution of the equation (1) is unique, if*

$$\begin{aligned}\Bigg(&||L||\Big(||\Omega_1||\frac{T^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}(||\Theta_1||\ ||\xi|| + U) + F_1\frac{T^\alpha}{\Gamma(\alpha+1)}\Big) \\ &+ (||L||\ ||\xi|| + G)||\Omega_1||\ ||\Theta_1||\frac{T^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)} \\ &+ ||K||\Big(||\Omega_2||\frac{T^{\gamma+\delta}}{\Gamma(\gamma+\delta+1)}(||\Theta_2||\ ||\xi|| + V) + F_2\frac{T^\gamma}{\Gamma(\gamma+1)}\Big) \\ &+ (||K||\ ||\xi|| + H)||\Omega_2||\ ||\Theta_2||\frac{T^{\gamma+\delta}}{\Gamma(\gamma+\delta+1)}\Bigg) < 1\end{aligned}$$

Proof. If possible, let the equation (1) has two solutions ξ and η. Then

$$\begin{aligned}&|\xi(t) - \eta(t)| \\ &\leq |g(t,\xi(\psi_1(t)))I^\alpha f_1(t, I^\beta u(t,\xi(\psi_2(t)))) - g(t,\eta(\psi_1(t)))I^\alpha f_1(t, I^\beta u(t,\eta(\psi_2(t))))| \\ &+ |h(t,\xi(\psi_3(t)))I^\gamma f_2(t, I^\delta v(t,\xi^*(\psi_4(t)))) - h(t,\eta(\psi_3(t)))I^\gamma f_2(t, I^\delta v(t,\eta^*(\psi_4(t))))| \\ &\leq |g(t,\xi(\psi_1(t))) - g(t,\eta(\psi_1(t)))|\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}|f_1(s, I^\beta u(s,\xi(\psi_2(s))))|ds\end{aligned}$$

$$+ |g(t,\xi(\psi_1(t)))| \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} |f_1(s, I^\beta u(s,\xi(\psi_2(s)))) - f_1(s, I^\beta u(s,\eta(\psi_2(s))))|$$
$$+ |h(t,\xi(\psi_3(t))) - h(t,\eta(\psi_3(t)))| \int_0^t \frac{(t-s)^{\gamma-1}}{\Gamma(\gamma)} |f_2(s, I^\delta v(s,\xi(\psi_4(s))))| ds$$
$$+ |h(t,\xi(\psi_3(t)))| \int_0^t \frac{(t-s)^{\gamma-1}}{\Gamma(\gamma)} |f_2(s, I^\delta v(s,\xi(\psi_4(s)))) - f_2(s, I^\delta v(s,\eta(\psi_4(s))))|$$
$$\leq |L(t)| \, |\xi(\psi_1(t)) - \eta(\psi_1(t))| \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} |f_1(s, I^\beta u(s,\xi(\psi_2(s)))) - f_1(s,0) + f_1(s,0)| ds$$
$$+ |g(t,\xi(\psi_1(t))) - g(t,0) + g(t,0)| \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} |\Omega_1(s)| \, |I^\beta u(s,\xi(\psi_2(s))) - I^\beta u(s,\eta(\psi_2(s)))| ds$$
$$+ |K(t)| \, |\xi(\psi_3(t)) - \eta(\psi_3(t))| \int_0^t \frac{(t-s)^{\gamma-1}}{\Gamma(\gamma)} |f_2(s, I^\delta v(s,\xi(\psi_4(s)))) - f_2(s,0) + f_2(s,0)| ds$$
$$+ |h(t,\xi(\psi_3(t))) - h(t,0) + h(t,0)| \int_0^t \frac{(t-s)^{\gamma-1}}{\Gamma(\gamma)} |\Omega_2(s)| \, |I^\delta v(s,\xi(\psi_4(s))) - I^\delta v(s,\eta(\psi_4(s)))| ds$$
$$\leq |L(t)| \, |\xi(\psi_1(t)) - \eta(\psi_1(t))| \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \Big(|\Omega_1(s)| I^\beta |u(s,\xi(\psi_2(s)))| + F_1 \Big) ds$$
$$+ \Big(|L(t)| \, |\xi(\psi_1(t))| + G \Big) \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} |\Omega_1(s)| \int_0^s \frac{(s-p)^{\beta-1}}{\Gamma(\beta)} |u(p,\xi(\psi_2(p))) - u(p,\eta(\psi_2(p)))| dsdp$$
$$+ |K(t)| \, |\xi(\psi_3(t)) - \eta(\psi_3(t))| \int_0^t \frac{(t-s)^{\gamma-1}}{\Gamma(\gamma)} \Big(|\Omega_2(s)| I^\delta |v(s,\xi(\psi_4(s)))| + F_2 \Big) ds$$
$$+ \Big(|K(t)| \, |\xi(\psi_3(t))| + H \Big) \int_0^t \frac{(t-s)^{\gamma-1}}{\Gamma(\gamma)} |\Omega_2(s)| \int_0^s \frac{(s-p)^{\delta-1}}{\Gamma(\delta)} |v(p,\xi(\psi_4(p))) - v(p,\eta(\psi_4(p)))| dsdp$$
$$\leq ||L|| \, ||\xi - \eta|| \Big(||\Omega_1|| \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \int_0^s \frac{(s-p)^{\beta-1}}{\Gamma(\beta)} (|\Theta_1(p)| \, |\xi(\psi_2(p))| + U)$$
$$+ \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} F_1 ds \Big) dsdp$$
$$+ \Big(||L|| \, ||\xi|| + G \Big) ||\Omega_1|| \, ||\Theta_1|| \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \int_0^s \frac{(s-p)^{\beta-1}}{\Gamma(\beta)} |\xi(\psi_2(p)) - \eta(\psi_2(p))| dsdp$$
$$+ ||K|| \, ||\xi - \eta|| \Big(||\Omega_2|| \int_0^t \frac{(t-s)^{\gamma-1}}{\Gamma(\gamma)} \int_0^s \frac{(s-p)^{\delta-1}}{\Gamma(\delta)} (|\Theta_2(p)| \, |\xi(\psi_4(p))| + V)$$
$$+ \int_0^t \frac{(t-s)^{\gamma-1}}{\Gamma(\gamma)} F_2 ds \Big) dsdp$$
$$+ \Big(||K|| \, ||\xi|| + H \Big) ||\Omega_2|| \, ||\Theta_2|| \int_0^t \frac{(t-s)^{\gamma-1}}{\Gamma(\gamma)} \int_0^s \frac{(s-p)^{\delta-1}}{\Gamma(\delta)} |\xi(\psi_4(p)) - \eta(\psi_4(p))| dsdp$$
$$\leq ||L|| \, ||\xi - \eta|| \Big(||\Omega_1|| \frac{T^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)} (||\Theta_1|| \, ||\xi|| + U) + F_1 \frac{T^\alpha}{\Gamma(\alpha+1)} \Big)$$
$$+ (||L|| \, ||\xi|| + G) ||\Omega_1|| \, ||\Theta_1|| \frac{T^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)} ||\xi - \eta||$$
$$+ ||K|| \, ||\xi - \eta|| \Big(||\Omega_2|| \frac{T^{\gamma+\delta}}{\Gamma(\gamma+\delta+1)} (||\Theta_2|| \, ||\xi|| + V) + F_2 \frac{T^\gamma}{\Gamma(\gamma+1)} \Big)$$
$$+ (||K|| \, ||\xi|| + H) ||\Omega_2|| \, ||\Theta_2|| \frac{T^{\gamma+\delta}}{\Gamma(\gamma+\delta+1)} ||\xi - \eta||,$$

Taking supremum over t we get,

$$||\xi - \eta|| \leq \Bigg(||L|| \Big(||\Omega_1|| \frac{T^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)} (||\Theta_1|| \, ||\xi|| + U) + F_1 \frac{T^\alpha}{\Gamma(\alpha+1)} \Big)$$

$$+ (||L||\ ||\xi|| + G)||\Omega_1||\ ||\Theta_1||\frac{T^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}$$
$$+ ||K||\Big(||\Omega_2||\frac{T^{\gamma+\delta}}{\Gamma(\gamma+\delta+1)}(||\Theta_2||\ ||\xi|| + V) + F_2\frac{T^{\gamma}}{\Gamma(\gamma+1)}\Big)$$
$$+ (||K||\ ||\xi|| + H)||\Omega_2||\ ||\Theta_2||\frac{T^{\gamma+\delta}}{\Gamma(\gamma+\delta+1)}\Bigg)||\xi - \eta||$$

Hence the solution is unique. □

5. Conclusion

For nonlinear differential and integral equations, fixed point theory offers powerful techniques for establishing the existence and uniqueness of solutions. In this paper, we have derived a fixed point result using four operators in Banach *-algebra which generalizes different existing fixed point results in the setting of Banach algebra. Also, we have applied our result to solve a functional integral equation of fractional order which shows the existence and uniqueness of the solution. In [2], Alissa et al. derived an application regarding fractional hybrid differential equations using fixed point theorem of Dhage [9]. In [21], Metwali et al. gave an application to prove the existence of solution for an initial value problem of fractional order using Darbo fixed point theorem associated with the fractional calculus and measure of noncompactness. Similar applications of our results can be investigated to some other types of functional integral equations and hybrid differential equations of fractional order as well as integral inequalities involving k-fractional order integral operators (refer to [6], [23]).

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