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TROTTER-KATO THEOREM IN THE WEAK TOPOLOGY

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Abstract. In this paper, we prove Trotter-Kato theorem in the weak topology if X^* is a uniformly convex Banach space.

1. Introduction

Let X be a Banach space. A family $\{T(t) : t \geq 0\}$ of bounded linear operators from X into itself is called a contraction C_0 semigroup on X if $T(0) = I$, $T(t+s) = T(t)T(s)$ for $t, s \geq 0$, for each $x \in X$ $T(t)x$ is continuous in $t \geq 0$ and $\|T(t)x\| \leq \|x\|$ for $t \geq 0$ and $x \in X$.

The linear operator A, defined by

$$Ax = \lim_{h \to 0} \frac{1}{h}(T(h)x - x)$$

for $x \in D(A) = \{x \in X : \lim_{h \to 0}(T(h)x - x)/h \text{ exists}\}$, is called the generator of a contraction C_0 semigroup $\{T(t) : t \geq 0\}$ and $D(A)$ is the domain of A.

The resolvent set of A is denoted by $\rho(A)$ and for $\lambda \in \rho(A)$ $R(\lambda, A) = (\lambda I - A)^{-1}$ is the resolvent operator of A, and we define the Yosida approximation of A by

$$A_\lambda = \lambda A R(\lambda, A) = \lambda^2 R(\lambda, A) - \lambda I.$$

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For more information about C_0 semigroups and their generators, we refer [2, 5].

The object of this paper is to discuss when the Trotter-Kato approximation theorem holds for the weak operator topology. This type of result plays a tool for the numerical study of partial and stochastic (partial) differential equations and is one of methods to use study a complicated operator. For the strong operator topology, the convergence of a sequence of C_0 semigroups $\{T_n(t) : t \geq 0\}$ is related to the convergence of their generators A_n and their resolvents $R(\lambda, A_n)$. Replacing the strong convergence of the resolvents by the weak convergence does not imply the weak convergence of corresponding C_0 semigroups, while the inverse is true by the Laplace transform representation of the resolvent of the generator and Lebesgue's theorem. In [1], the weak convergence of generators or resolvents of their generators does not imply the weak convergence of C_0 semigroups even if the generators are bounded.

In [3], G. Marinoschi has proved that the weak version of Trotter-Kato approximation theorem with some restrictions on generators is valid for a Hilbert space. In this paper we extend this result to a Banach space X whose dual space X^* is uniformly convex for contraction C_0 semigroups.

With the uniform convexity of X^* we have the uniform continuity of the dual mapping. The inner product of a Hilbert space is replaced by the dual mapping and the uniform continuity of the dual mapping is essential for the proof of our main result.

2. Weak Convergence

Let X be a Banach space and let X^* be its dual space. We denote the value $x^*(x)$ of $x^* \in X^*$ at $x \in X$ by the duality pairing $< x, x^* >$ or $< x^*, x >$.

We recall that for each $x \in X$ the dual mapping $J : X \to X^*$ is defined by

$$J(x) = \{x^* \in X^* :< x, x^* >= \|x\|^2 = \|x^*\|^2\}.$$

Note that $J(x)$ is a subset of X^* and $J(x) \neq \emptyset$for all $x \in X$, by Hahn-Banach Theorem. Hence J can be viewed as a multi-valued function. Under some restrictions on X^*, J can be a single-valued. If X is a Hilbert space, then J is the identity mapping on X. With the uniform convexity of X^*, we have the following properties of the dual mapping (see [4]). For example, Hilbert spaces are uniformly convex and L^p spaces $(1 < p < \infty)$ are also uniformly convex.

THEOREM 1. *If X^* is a uniformly convex Banach space, then the dual mapping J is single-valued and uniformly continuous on every bounded subsets of X.*

THEOREM 2. *If $u : (a,\ b) \to X$ has a weak derivative $u'(t)$ and $\|u(t)\|$ is differentiable, then*

$$\frac{d}{dt}\|u(t)\|^2 = 2 < u'(t),\ f > \ \ \text{for } f \in J(u(t)).$$

We set $w - \lim$ by the weak limit and the Yosida approximation of A_n by $A_{n,\lambda}$ for any $\lambda > 0$.

THEOREM 3. *Let X be a Banach space whose dual space X^* is uniformly convex. Let $\{T_n(t) : t \geq 0\}$ be a sequence of contraction C_0 semigroups with generators A_n and let $\{T(t) : t \geq 0\}$ be a contraction C_0 semigroup with generator A. Suppose that*

$$w - \lim_{n\to\infty} R(\lambda, A_n)^k x = R(\lambda, A)^k x, \quad k = 1, 2, \cdots$$

for $x \in X$ and

$$D = \{x \in \cap_{n=1}^{\infty} D(A_n) : \sup_{n\geq 1} \|A_n x\| < \infty\}$$

is dense in X. Then

$$w - \lim_{n\to\infty} T_n(t)x = T(t)x$$

for all $x \in X$ and the convergence is uniform on bounded $t-$ intervals.

Proof. Let $0 \leq t \leq T$ and $x \in D$. Then

$$\begin{aligned} & | < T_n(t)x - T(t)x, \phi > | \\ \leq & \ | < T_n(t)x - e^{tA_{n,\lambda}}x, \phi > | + | < e^{tA_{n,\lambda}}x - e^{tA_\lambda}x, \phi > | \\ & + | < e^{tA_\lambda}x - T(t)x, \phi > | \end{aligned}$$

for each $\phi \in X^*$.

By Hille-Yosida's Theorem, we have

$$\lim_{\lambda\to\infty} \|e^{tA_\lambda}x - T(t)x\| = 0 \text{ and } \lim_{\lambda\to\infty} \|e^{tA_{n,\lambda}}x - T_n(t)x\| = 0,$$

uniformly on bounded $t-$intervals. Hence we have

$$\lim_{\lambda\to\infty} < T_n(t)x - e^{tA_{n,\lambda}}x, \phi >= 0 \text{ and } \lim_{\lambda\to\infty} < T(t)x - e^{tA_\lambda}x, \phi >= 0,$$

uniformly on $[0,T]$ for $x \in D$ and $\phi \in X^*$.

We will show that the convergence $\lim_{\lambda\to\infty} < T_n(t)x - e^{tA_{n,\lambda}}x, \phi >= 0$ is uniform with respect to n.

Let $u_{n,\lambda}(t) = e^{tA_{n,\lambda}}x$ and $u_n(t) = T_n(t)x$. Then we have the following properties which are given in the proof of Hille-Yosida's theorem.

$$\begin{aligned}
&\tfrac{d}{dt}u_{n,\lambda}(t) = A_{n,\lambda}u_{n,\lambda}(t), \quad u_{n,\lambda}(0) = x \\
&\tfrac{d}{dt}u_n(t) = A_n u_n(t), \quad u_n(0) = x \\
&\|u_{n,\lambda}(t)\| \le \|x\| \\
&\|\tfrac{d}{dt}u_{n,\lambda}(t)\| = \|A_{n,\lambda}u_{n,\lambda}(t)\| \le \|A_{n,\lambda}x\| \\
&\|R(\lambda, A_n)x\| \le \tfrac{1}{\lambda}\|x\| \\
&< A_{n,\lambda}x, J(x) > \le 0 \text{ for } x \in X \\
&\|A_{n,\lambda}x\| = \|\lambda A_n R(\lambda, A_n)x\| \le \|A_n x\| \text{ for } x \in D(A_n)
\end{aligned}$$

By Theorem 2, we have

$$\frac{1}{2}\frac{d}{dt}\|u_{n,\lambda}(t) - u_{n,\mu}(t)\|^2 =< A_{n,\lambda}u_{n,\lambda}(t) - A_{n,\mu}u_{n,\mu}(t), J(u_{n,\lambda}(t) - u_{n,\mu}(t)) >$$

for $\lambda, \mu > 0$.

Consider the following estimates.

$$\begin{aligned}
\|(u_{n,\lambda}(t) - \lambda R(\lambda, A_n)u_{n,\lambda}(t) &- (u_{n,\mu}(t) - \mu R(\mu, A_n)u_{n,\mu}(t))\| \\
&= \| - \frac{1}{\lambda}(\lambda^2 R(\lambda, A_n)u_{n,\lambda}(t) - \lambda u_{n,\lambda}(t)) \\
&\quad + \frac{1}{\mu}(\mu^2 R(\mu, A_n)u_{n,\mu}(t) - \mu u_{n,\mu}(t))\| \\
&= \| - \frac{1}{\lambda}A_{n,\lambda}u_{n,\lambda}(t) + \frac{1}{\mu}A_{n,\mu}u_{n,\mu}(t)\| \\
&\le \frac{1}{\lambda}\|A_{n,\lambda}u_{n,\lambda}(t)\| + \frac{1}{\mu}\|A_{n,\mu}u_{n,\mu}(t)\| \\
&\le (\frac{1}{\lambda} + \frac{1}{\mu})\|A_n x\|.
\end{aligned}$$

Let $\varepsilon > 0$ be given. Since $x \in D$ and J is uniform continuous,

$$\|J(u_{n,\lambda}(t) - u_{n,\mu}(t)) - J(\lambda R(\lambda, A_n)u_{n,\lambda}(t) - \mu R(\mu, A_n)u_{n,\mu}(t))\| < \varepsilon$$

for sufficiently large λ and μ.

By the uniform continuity of J, we have

$$\begin{aligned}
&< A_{n,\lambda}u_{n,\lambda}(t) - A_{n,\mu}u_{n,\mu}(t), J(u_{n,\lambda}(t) - u_{n,\mu}(t)) > \\
= \ & < A_{n,\lambda}u_{n,\lambda}(t) - A_{n,\mu}u_{n,\mu}(t), J(u_{n,\lambda}(t) - u_{n,\mu}(t)) \\
&\quad -J(\lambda R(\lambda, A_n)u_{n,\lambda}(t) - \mu R(\mu, A_n)u_{n,\mu}(t)) > \\
&\ + < A_{n,\lambda}u_{n,\lambda}(t) - A_{n,\mu}u_{n,\mu}(t), J(\lambda R(\lambda, A_n)u_{n,\lambda}(t) \\
&\quad -\mu R(\mu, A_n)u_{n,\mu}(t)) > \\
\leq \ & (\|A_{n,\lambda}u_{n,\lambda}(t)\| + \|A_{n,\mu}u_{n,\mu}(t)\|)\|J(u_{n,\lambda}(t) - u_{n,\mu}(t)) \\
&\quad -J(\lambda R(\lambda, A_n)u_{n,\lambda}(t) - \mu R(\mu, A_n)u_{n,\mu}(t))\| \\
&\ + < A_n(\lambda R(\lambda, A_n)u_{n,\lambda}(t) - \mu R(\mu, A_n)u_{n,\mu}(t)), J(\lambda R(\lambda, A_n)u_{n,\lambda}(t) \\
&\quad -\mu R(\mu, A_n)u_{n,\mu}(t)) > \\
\leq \ & 2\|A_n x\|\varepsilon
\end{aligned}$$

for sufficiently large λ and μ. Let $M = \sup_{n\geq 1} \|A_n x\|$. Then we have

$$\frac{d}{dt}\|u_{n,\lambda}(t) - u_{n,\mu}(t)\|^2 \leq 4M\varepsilon.$$

Integrate this inequality from 0 to t, then we have $\|u_{n,\lambda}(t) - u_{n,\mu}(t)\|^2 \leq 4MT\varepsilon$. Letting $\mu \to \infty$, then $\|u_{n,\lambda}(t) - u_n(t)\|^2 \leq 4MT\varepsilon$. Since ε is arbitrary, we have

$$\lim_{\lambda\to\infty} \|u_{n,\lambda}(t) - u_n(t)\| = 0,$$

uniformly with respect to n.

It remains to show that

$$\lim_{n\to\infty} | < e^{tA_{n,\lambda_0}}x - e^{tA_{\lambda_0}}, \phi > | = 0$$

for sufficiently large λ_0.

Since $A_{n,\lambda_0}x = \lambda_0^2 R(\lambda_0, A_n)x - \lambda_0 x$ and $w - \lim_{n\to\infty} R(\lambda_0, A_n)^k x = R(\lambda_0, A)^k x$, $k = 1, 2, \cdots$, $< A_{n,\lambda_0}^k x, \phi > = < (\lambda_0^2 R(\lambda_0, A_n) - \lambda_0)^k x, \phi >$ converges to $< (\lambda_0^2 R(\lambda_0, A) - \lambda_0)^k x, \phi > = < A_{\lambda_0}^k x, \phi >$ as $n \to \infty$. Also since $e^{tA_{n,\lambda_0}} = \sum_{k=0}^{\infty} \frac{t^k}{k!} A_{n,\lambda_0}^k$, we have

$$\lim_{n\to\infty} < e^{tA_{n,\lambda_0}}x - e^{tA_{\lambda_0}}, \phi > = 0.$$

Therefore, we have $w - \lim_{n\to\infty} T_n(t)x = T(t)x$ for $x \in D$.

Let $x \in X$. Since D is dense in X, there exist x_l in D such that $\lim_{l\to\infty} x_l = x$, Since $\{T_n(t) : t \geq 0\}$ and $\{T(t) : t \geq 0\}$ are contraction

C_0 semigroups,

$$\begin{aligned} & | < T_n(t) - T(t)x, \phi > | \\ & \quad \leq \; | < T_n(t)x - T_n(t)x_l, \phi > | + | < T_n(t)x_l - T(t)x_l, \phi > | \\ & \qquad + | < T(t)x_l - T(t)x, \phi > | \\ & \quad \leq \; 2\|x_l - x\|\|\phi\| + | < T_n(t)x_l - T(t)x_l, \phi > \end{aligned}$$

Choose x_{l_0} such that $2\|x_{l_0} - x\|\|\phi\| < \varepsilon/2$. Then $| < T_n(t)x_{l_0} - T(t)x_{l_0}, \phi > | < \varepsilon/2$ for sufficiently large n. Therefore

$$\lim_{n\to\infty} < T_n(t)x - T(t)x, \phi >= 0.$$

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