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A NOTE ON FOUR DIMENSIONAL SUMMABILITY METHODS

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Abstract. Ishiguro studied some two dimensional summability methods in [7]. In this paper, we define the four dimensional Zweier matrix and extend the results given by Ishiguro [7] to four dimensional summability methods. We prove that an Abel summable double sequence is also summable the product of Abel and Zweier methods to the same limit. Besides this, we show the four dimensional Riesz and Zweier methods don't imply each other. In addition, we emphasize the four dimensional Zweier method implies the four dimensional Borel method.

1. Introduction

We denote the set of all complex valued double sequences by Ω which is a vector space with coordinatewise addition and scalar multiplication. Any vector subspace of Ω is called as *a double sequence space.* Consider the sequence $z = (z_{kl}) \in \Omega$. If for every $\varepsilon > 0$ there exists $n_0 = n_0(\varepsilon) \in \mathbb{N}$ and $\alpha \in \mathbb{C}$ such that $|z_{kl} - \alpha| < \varepsilon$ for all $k, l > n_0$, then we call that the double sequence z is *convergent* in the *Pringsheim's sense* (or $p-$convergent) to the limit α and write $p - \lim\limits_{k,l\to\infty} z_{kl} = \alpha$; where $\mathbb{C}$ denotes the complex field, [13]. The sequence z is called $p-$null whenever the limit α is zero. Also, we denote the space of $p-$null sequences by $\mathcal{C}_{p0}$.

Let (q_k), (t_l) be two sequences of non-negative numbers which are not all zero and $Q_m = \sum\limits_{k=0}^{m} q_k$, $q_0 > 0$, $T_n = \sum\limits_{l=0}^{n} t_l$, $t_0 > 0$. Then, the sequence $\{(R^{qt}a)_{mn}\}$ given by

$$(R^{qt}a)_{mn} = \frac{1}{Q_m T_n} \sum_{k=0}^{m} \sum_{l=0}^{n} q_k t_l a_{kl} \quad \text{for all} \quad m, n \in \mathbb{N} \tag{1}$$

is called the Riesz transform of a double sequence $a = (a_{kl})$. The Riesz mean R^{qt} with respect to the sequences $q = (q_k)$ and $t = (t_l)$ is RH-regular if and only if $\lim\limits_{m\to\infty} Q_m = \infty$ and $\lim\limits_{n\to\infty} T_n = \infty$, [19]. If $p - \lim(R^{qt}a)_{mn} = \alpha$, $\alpha \in \mathbb{C}$, then the sequence $a = (a_{kl})$ is said to be Riesz convergent to α, [1]. Note that in the case $q_k = 1$ for all $k \in \mathbb{N}$ and $t_l = 1$ for all $l \in \mathbb{N}$, the Riesz mean R^{qt} reduces to the four

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dimensional Cesàro mean $C(1,1)=(c_{mnkl})$ of orders 1 and 1 defined as follows;

$$c_{mnkl}=\begin{cases}\frac{1}{(m+1)(n+1)} & , \ 0\leq k\leq m, \ 0\leq l\leq n,\\ 0 & , \ \text{otherwise}\end{cases}$$

for all $m,n,k,l\in\mathbb{N}$, [11].

The four dimensional Borel method B is given by the matrix

$$b_{mnkl}=\frac{e^{-(m+n)m^k n^l}}{k!l!}$$

for all $m,n,k,l\in\mathbb{N}$, (see [12]).

Now, we define the four dimensional Zweier method Z given by the matrix $Z=(z_{mnkl})$ as follows;

$$z_{mnkl}=\begin{cases}1/4 & , \ m-1\leq k\leq m, \ n-1\leq l\leq n,\\ 0 & , \ \text{otherwise}\end{cases}$$

for all $m,n,k,l\in\mathbb{N}$.

If we take the sequences (q_k) and (t_l) as

$$q_k=\begin{cases}1 & , \ m-1\leq k\leq m,\\ 0 & , \ \text{otherwise}\end{cases} \quad \text{and} \quad t_l=\begin{cases}1 & , \ n-1\leq l\leq n,\\ 0 & , \ \text{otherwise}\end{cases}$$

for all $m,n,k,l\in\mathbb{N}$ in the relation (1), we obtain the method Z.

Let $D=(d_{mnkl})$ be a four dimensional matrix and $z=(z_{kl})\in\Omega$. Then, the double sequence $Dz=\{(Dz)_{mn}\}$ is called the $D-$transform of the sequence z, where

$$(Dz)_{mn}=\sum_{k,l} d_{mnkl}z_{kl}$$

such that the series on the right side converges for each $m,n\in\mathbb{N}$. We say that a double sequence z is D summable to the limit α if the $D-$transform Dz exists for each $m,n\in\mathbb{N}$ and is convergent in the Pringsheim's sense, that is,

$$p-\lim_{r,s\to\infty}\sum_{k=0}^{r}\sum_{l=0}^{s} d_{mnkl}z_{kl}=(Dz)_{mn} \ \text{ and } \ p-\lim_{m,n\to\infty}(Dz)_{mn}=\alpha. \tag{2}$$

Taking the matrices R^{qt}, $C(1,1)$, B and Z instead of the matrix D in the relation (2), respectively, we have R^{qt} summable, $C(1,1)$ summable, B summable and Z summable.

Karaev and Zeltser [8] gave the definition of Abel summability of a double sequence as: A double sequence (a_{kl}) is Abel summable (shortly A summable) to α if $\sum_{i,j=0}^{\infty} a_{ij}x^i y^j$ converges for all $x,y\in(0,1)$ and

$$\lim_{(x,y)\to(1^-,1^-)}(1-x)(1-y)\sum_{i,j} a_{ij}x^i y^j=\alpha.$$

Following Karaev and Zeltser [8], we easily say that a double sequence (a_{kl}) is $A.Z$ summable to α if the series $\sum_{i,j=0}^{\infty} a_{ij}x^i y^j$ converges for all $x,y\in(0,1)$ and

$$\lim_{(x,y)\to(1^-,1^-)}\frac{(1-x^2)(1-y^2)}{4}\sum_{i,j=0}^{\infty} a_{ij}x^i y^j=\alpha.$$

A four dimensional matrix D is said to be RH-regular if it maps every bounded p-convergent sequence into a p-convergent sequence with the same p-limit (see Robison [14]).

Robison and Hamilton presented a Silverman-Toeplitz type multidimensional characterization of regularity in [14] and [6], respectively.

THEOREM 1.1. [6,14] *A four dimensional matrix $A = (a_{mnkl})$ is RH-regular if and only if*

$$\begin{aligned}
RH_1 &: p-\lim_{m,n\to\infty} a_{mnkl} = 0 \text{ for each } k,l \in \mathbb{N},\\
RH_2 &: p-\lim_{m,n\to\infty} \sum_{k,l} a_{mnkl} = 1,\\
RH_3 &: p-\lim_{m,n\to\infty} \sum_{k} |a_{mnkl}| = 0 \text{ for each } l \in \mathbb{N},\\
RH_4 &: p-\lim_{m,n\to\infty} \sum_{l} |a_{mnkl}| = 0 \text{ for each } k \in \mathbb{N},\\
RH_5 &: \sum_{k,l} |a_{mnkl}| \text{ is } p\text{-convergent},\\
RH_6 &: \text{There exists finite positive integers } M \text{ and } N \text{ such that } \sum_{k,l>N} |a_{mnkl}| < M.
\end{aligned}$$

The two dimensional Zweier matrix as a summability method was studied by Szász [15] and Szász used the notation Y instead of the notation Z in [15]. Following Szász, Ishiguro [7] also studied this method. Our aim is to extend the results in Ishiguro [7] from two dimensional summability methods to four dimensional summability methods mentioned above.

Throughout the paper, we take $a_{kl} = 0$ for negative index. For relevant terminology and related topics on the normed/paranormed spaces of double sequences and the domain of triangle matrices in those spaces, and the matrix transformations, the reader can refer to Başar [2] and Başar and Yeşilkayagil Savaşcı [3].

2. Main results

THEOREM 2.1. *If a double sequence (a_{kl}) is Abel summable to α, then it is also summable $A.Z$ to the same limit.*

Proof. Assume that the double sequence (a_{kl}) is Abel summable to α, that is, the limit

$$\lim_{(x,y)\to(1^-,1^-)} (1-x)(1-y)\sum_{i,j} a_{ij}x^i y^j = \alpha$$

holds for all $x,y\in(0,1)$. Let $u_{mn}=(Za)_{mn}=\frac{1}{4}\sum\limits_{k=m-1}^{m}\sum\limits_{l=n-1}^{n}a_{kl}$ for all $m,n\in\mathbb{N}$. Then, we have

$$\begin{aligned}
\sum_{i,j=0}^{\infty}u_{ij}x^iy^j &= \sum_{i,j=0}^{\infty}\frac{a_{i-1,j-1}+a_{i-1,j}+a_{i,j-1}+a_{ij}}{4}x^iy^j\\
&= \frac{1}{4}\left[\sum_{i,j=1}^{\infty}a_{i-1,j-1}x^iy^j+\sum_{i=1}^{\infty}\sum_{j=0}^{\infty}a_{i-1,j}x^iy^j+\sum_{i=0}^{\infty}\sum_{j=1}^{\infty}a_{i-1,j}x^iy^j+\sum_{i,j=0}^{\infty}a_{ij}x^iy^j\right]\\
&= \frac{xy+x+y+1}{4}\sum_{i,j=0}^{\infty}a_{ij}x^iy^j\\
&= \frac{(1+x)(1+y)}{4}\sum_{i,j=0}^{\infty}a_{ij}x^iy^j \qquad (3)
\end{aligned}$$

and so

$$\begin{aligned}
\lim_{(x,y)\to(1^-,1^-)}(1-x)(1-y)\sum_{i,j=0}^{\infty}u_{ij}x^iy^j &= \lim_{(x,y)\to(1^-,1^-)}\frac{(1-x)(1-y)(1+x)(1+y)}{4}\sum_{i,j=0}^{\infty}a_{ij}x^iy^j\\
&= \alpha.
\end{aligned}$$

This completes the proof. □

THEOREM 2.2. *If a double sequence (a_{kl}) satisfies $p-\lim\limits_{k,l\to\infty}a_{kl}x^ky^l=0$ for all $x,y\in(0,1)$, and if (a_{kl}) is summable $A.Z$ to α, then it is summable A to the same value.*

Proof. The proof is obvious from the assumption and the relation (3). □

THEOREM 2.3. *R^{qt} does not imply Z.*

Proof. Define the double sequence (a_{kl}) by

$$a_{kl}:=\begin{cases}1 & , \quad k=3r \text{ or } l=3s,\\ 0 & , \quad \text{otherwise}\end{cases}$$

where $r,s\in\mathbb{N}$. Then, we have the equality

$$p-\lim_{m,n\to\infty}(R^{qt}a)_{mn}:=p-\lim_{m,n\to\infty}\frac{1}{Q_mT_n}\sum_{k=0}^{m}\sum_{l=0}^{n}q_kt_la_{kl}=0$$

for each $m,n\in\mathbb{N}$. But,

$$(Za)_{mn}=\begin{cases}1/4 & , \quad m=n=0,\\ 3/4 & , \quad m=1, \text{ and } n\in\{3s,3s+1\},\\ 1/2 & , \quad m=1, \text{ and } n=3s+2,\\ 1/2 & , \quad m\in\{2,3,4,...\} \text{ and } n\in\{3s,3s+2\},\\ 0 & , \quad m\in\{2,3,4,...\} \text{ and } n=3s+2,\end{cases}$$

for all $m,n,s\in\mathbb{N}$. Hence, the sequence $\{(Za)_{mn}\}$ has two limit points as $m,n\to\infty$ and so it is not $p-$convergent. Thus, the proof is completed. □

Now, we can give the following corollary as a direct consequence of Theorem 2.3:

COROLLARY 2.4. *$C(1,1)$ does not imply Z.*

THEOREM 2.5. *Z does not imply R^{qt}.*

Proof. Define the double sequence (a_{kl}) by

$$a_{kl} := \begin{cases} q_k t_l & , \quad 0 \le k \le m \text{ and } \ 0 \le l \le n, \\ 0 & , \quad \text{otherwise} \end{cases}$$

where $k, l, m, n \in \mathbb{N}$. For sufficiently large m and n, we have that

$$p - \lim_{m,n \to \infty} (Za)_{mn} = 0.$$

But, we easily obtain that

$$\begin{aligned} p - \lim_{m,n \to \infty} (R^{qt} a)_{mn} &= p - \lim_{m,n \to \infty} \frac{1}{Q_m T_n} \sum_{k=0}^{m} \sum_{l=0}^{n} a_{kl} q_k t_l \\ &= p - \lim_{m,n \to \infty} \frac{1}{Q_m T_n} \sum_{k=0}^{m} q_k^2 \sum_{l=0}^{n} t_l^2 = \infty. \end{aligned}$$

This completes the proof. □

Corollary 2.6. *Z does not imply $C(1,1)$.*

Proof. If we take the double sequence (a_{kl}) by

$$a_{kl} := \begin{cases} kl & , \quad 0 \le k \le m \text{ and } \ 0 \le l \le n, \\ 0 & , \quad \text{otherwise} \end{cases}$$

we obtain the desired result in a similar way to the proof of Theorem 2.5. □

Theorem 2.7. *Z implies B.*

Proof. Let $u_{mn} = (Za)_{mn} = \frac{1}{4} \sum\limits_{k=m-1}^{m} \sum\limits_{l=n-1}^{n} a_{kl}$ for all $m, n \in \mathbb{N}$. Then, we have $a_{kl} = 4 \sum_{i,j=0}^{k,l} (-1)^{k+l-(i+j)} u_{ij}$ for all $k, l \in \mathbb{N}$.

Now, assume that the double sequence (a_{kl}) is summable Z to α, that is

$$p - \lim_{m,n \to \infty} (Za)_{mn} = p - \lim_{m,n \to \infty} u_{mn} = \alpha. \tag{4}$$

With some calculation we obtain that

$$\begin{aligned} (Ba)_{mn} &= e^{-(m+n)} \sum_{k,l=0}^{\infty} \frac{m^k n^l}{k! l!} a_{kl} \\ &= 4 e^{-(m+n)} \sum_{k,l=0}^{\infty} \frac{m^k n^l}{k! l!} \sum_{i,j=0}^{k,l} (-1)^{k+l-(i+j)} u_{ij} \\ &= 4 e^{-(m+n)} \sum_{i,j=0}^{\infty} \left(\sum_{k=i}^{\infty} \sum_{l=j}^{\infty} (-1)^{k+l-(i+j)} \frac{m^k n^l}{k! l!} \right) u_{ij} \\ &= (Cu)_{mn}, \qquad (5) \end{aligned}$$

where $C = (c_{mnij})$ is defined by $c_{mnij} = 4 e^{-(m+n)} \sum_{k=i}^{\infty} \sum_{l=j}^{\infty} (-1)^{k+l-(i+j)} \frac{m^k n^l}{k! l!}$ for all $m, n, i, j \in \mathbb{N}$.

We want to show that the matrix C is $RH-$regular. The condition RH_1 is obvious.

For all $m, n, i, j \in \mathbb{N}$, we have the equality

$$
\begin{aligned}
\sum_{i,j=0}^{\infty} c_{mnij} &= 4e^{-(m+n)} \sum_{i,j=0}^{\infty} \left(\sum_{k=i}^{\infty} \sum_{l=j}^{\infty} (-1)^{k+l-(i+j)} \frac{m^k n^l}{k! l!} \right) \\
&= 4e^{-(m+n)} \sum_{i=0}^{\infty} \sum_{k=i}^{\infty} (-1)^i \frac{(-m)^k}{k!} \sum_{j=0}^{\infty} \sum_{l=j}^{\infty} (-1)^j \frac{(-n)^l}{l!} \\
&= 4e^{-(m+n)} \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} \frac{(m)^{2k}}{(2k)!} \frac{(n)^{2l}}{(2l)!} \\
&= 4e^{-(m+n)} \frac{e^m + e^{-m}}{2} \frac{e^n + e^{-n}}{2} \\
&= (1 + e^{-2m})(1 + e^{-2n}). \qquad (6)
\end{aligned}
$$

If we take $p-$limit as $m, n \to \infty$ in the relation (6), the condition RH_2 holds.

Since $\frac{m^{i+r}}{(i+r)!} > \frac{m^{i+r+1}}{(i+r+1)!}$ for all $r \in \mathbb{N}$, we obtain the inequality

$$
\begin{aligned}
\sum_{i=0}^{\infty} |c_{mnij}| &= 4e^{-(m+n)} \sum_{i=0}^{\infty} \left| \sum_{k=i}^{\infty} \sum_{l=j}^{\infty} (-1)^{k+l-(i+j)} \frac{m^k n^l}{k! l!} \right| \\
&= 4e^{-(m+n)} \sum_{i=0}^{\infty} \left| \sum_{k=i}^{\infty} (-1)^{k-i} \frac{m^k}{k!} \right| \left| \sum_{l=j}^{\infty} (-1)^j \frac{(-n)^l}{l!} \right| \\
&= 4e^{-(m+n)} \left| \sum_{l=j}^{\infty} (-1)^j \frac{(-n)^l}{l!} \right| \sum_{i=0}^{\infty} \left| \sum_{k=i}^{\infty} (-1)^{k-i} \frac{m^k}{k!} \right| \\
&< 4e^{-(m+2n)} \sum_{i=0}^{\infty} \left| \sum_{k=i}^{\infty} (-1)^{k-i} \frac{m^k}{k!} \right| \\
&= 4e^{-(m+2n)} \sum_{i=0}^{\infty} \left| \frac{m^i}{i!} - \frac{m^{i+1}}{(i+1)!} + \frac{m^{i+2}}{(i+2)!} - \frac{m^{i+3}}{(i+3)!} + \ldots \right| \\
&= 4e^{-(m+2n)} \sum_{i=0}^{\infty} \left| \frac{m^i}{i!} - \left(\frac{m^{i+1}}{(i+1)!} - \frac{m^{i+2}}{(i+2)!} + \frac{m^{i+3}}{(i+3)!} - \ldots \right) \right| \\
&< 4e^{-(m+2n)} \sum_{i=0}^{\infty} \left| \frac{m^i}{i!} \right| \\
&< 4e^{-2n}, \qquad (7)
\end{aligned}
$$

for all $m, n, i, j \in \mathbb{N}$. So, taking $p-$limit in the relation (7) as $m, n \to \infty$ we obtain the condition RH_3.

Using similar way in the relation (7), we easily have that the conditions RH_4-RH_6. So, the matrix C is $RH-$regular. Thus, by the relations (4) and (5), we obtain that

$$
p - \lim_{m,n\to\infty} (Ba)_{mn} = p - \lim_{m,n\to\infty} (Cu)_{mn} = \alpha,
$$

as desired. $\square$

THEOREM 2.8. *There is a double sequence summable B but not summable Z.*

Proof. Let us define the sequence $a = (a_{kl})$ as $a_{kl} = (-2)^{k+l}$ for all $k, l \in \mathbb{N}$. One can easily see that the sequence a is Borel summable but not Zweier summable. $\square$

3. Conclusion

Let $0 < r, s < 1$. The Euler mean of orders r and s for double sequences defined by the four dimensional matrix $E(r,s) = (e^{r,s}_{mnkl})$ defined in [16,18] as follows;

$$e^{r,s}_{mnkl} := \begin{cases} \binom{m}{k}\binom{n}{l} r^k s^l (1-r)^{m-k}(1-s)^{n-l} & , \ 0 \le k \le m, \ 0 \le l \le n, \\ \\ 0 & , \ \text{otherwise} \end{cases}$$

for all $m, n, k, l \in \mathbb{N}$.

In [9,10] Kiltho et al. defined the four dimensional Pascal matrix $P = (p_{mnkl})$ by

$$p_{mnkl} = \begin{cases} \binom{m}{m-k}\binom{n}{n-k} & , \ 0 \le k \le m, \ 0 \le l \le n, \\ 0 & , \ \text{otherwise} \end{cases}$$

for all $m, n, k, l \in \mathbb{N}$.

Let $L(\alpha, \beta)$ denote the method of doubly Laurent means of orders (α, β), defined by the four dimensional matrix $L(\alpha, \beta) = (l^{\alpha,\beta}_{mnkl})$ given by Talebi [17] as

$$l^{\alpha,\beta}_{mnkl} = \binom{m+k-1}{k}\binom{n+l-1}{l}(1-\alpha)^k(1-\beta)^l \alpha^m \beta^n$$

for all $m, n, k, l \in \mathbb{N}$.

Demiriz and Erdem [4,5] studied the four dimensional Euler-Totient matrix $\phi^* = (\phi^*_{mnkl})$ which is defined as follows;

$$\phi^*_{mnkl} = \begin{cases} \frac{\varphi(k)\varphi(l)}{mn} & , \ k|m, \ l|n, \\ 0 & , \ \text{otherwise} \end{cases}$$

for all $m, n, k, l \in \mathbb{N}$, where φ denotes the Euler function.

In this paper, we extended the results given by Ishiguro [7] from two dimensional summability methods to four dimensional Abel, Riesz, Cesàro, Borel and Zweier summability methods. I should note that one can investigate the relation between the methods mentioned above and four dimensional Zweier method.

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