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TWO TYPES OF ALGEBRAIC STRUCTURES BASED ON GENERALIZED RESIDUATED LATTICES

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Abstract. In this paper, we introduce two types of left and right algebraic structures. We investigate the relations between bi-interior(bi-closure) operators and bi-interior(bi-closure) systems. We explore how a bi-preordered space leads to the formation of right and left rough sets.

1. Introduction

Pawlak [11,12] introduced the rough set theory as a formal tool to deal information systems and decision rules in the data analysis. As an important mathematical tool for rough set theory, the notions of closure (interior) systems and closure (interior) operators facilitate to study topological structures, logic and concept lattices [1-6].

As a base lattice, Ward et al. [18] introduced a complete residuated lattice which is an algebraic structure for many valued logic [1,8,15]. Bělohlávek [1,2] investigated the properties of fuzzy relations and fuzzy closure systems on a residuated lattice which supports part of foundation of fuzzy concept lattices and theoretic computer science. Using fuzzy interior and fuzzy closure operators, many researchers investigated fuzzy rough sets based on a residuated lattice [5,6,13,16,17].

As a non-commutative algebraic structure, Turunen [15] introduced a generalized residuated lattice as a generalization of weak-pseudo-BL-algebras and left continuous pseudo-t-norms [7]. Ko et al. [9,10] introduced the notions of bi-closure operators, bi-closure systems and bi-completeness on a generalized residuated lattice.

In this paper, we introduce two types of left and right algebraic structures which are bi-preorders, bi-interior and bi-closure operators, bi-interior and bi-closure systems and rough sets based on a generalized residuated lattice.

In Section 2, we review the basic concepts and properties of generalized residuated lattices. In Theorem 3.5, we demonstrate that both bi-interior and bi-closure operators create corresponding bi-interior and bi-closure systems. Theorem 3.6 shows that these systems can in turn generate bi-interior and bi-closure operators. In Theorem 3.7, we

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explore how a bi-preordered space can lead to the formation of right and left rough sets.

2. Preliminaries

Definition 2.1. [9,10,15] A structure $(L,\vee,\wedge,\odot,\rightarrow,\Rightarrow,\bot,\top)$ is called a *generalized residuated lattice* if it satisfies the following conditions:
(GR1) $(L,\vee,\wedge,\top,\bot)$ is a bounded lattice where $\top$ is the upper bound and $\bot$ is the lower bound;
(GR2) $(L,\odot,\top)$ is a monoid;
(GR3) it satisfies a residuation , i.e.,

$$a\odot b\leq c \text{ iff } a\leq b\rightarrow c \text{ iff } b\leq a\Rightarrow c.$$

Remark 2.2. (1) A generalized residuated lattice is a residuated lattice ($\rightarrow=\Rightarrow$) iff $\odot$ is commutative.
(2) Let $(L,\leq,\odot)$ be a quantale [14]. For all $x,y\in L$, define

$$x\rightarrow y=\bigvee\{z\in L\mid z\odot x\leq y\},$$
$$x\Rightarrow y=\bigvee\{z\in L\mid x\odot z\leq y\}.$$

Then it satisfies Galois correspondence, that is,
$(x\odot y)\leq z$ iff $x\leq(y\rightarrow z)$ iff $y\leq(x\Rightarrow z)$.
Hence $(L,\vee,\wedge,\odot,\rightarrow,\Rightarrow,\bot,\top)$ is a generalized residuated lattice.
(3) [7,15] A pseudo MV-algebra is a generalized residuated lattice with the law of double negation.

In this paper, we always assume that $(L,\wedge,\vee,\odot,\rightarrow,\Rightarrow,\top,\bot)$ is a complete generalized residuated lattice with the law of double negation defined as $a=(a^*)^0=(a^0)^*$ where $a^0=a\rightarrow\bot$ and $a^*=a\Rightarrow\bot$.

Lemma 2.3. [9,10] *Let $x,y,z,x_i,y_i\in L$. Then the following hold.*
(1) *If $y\leq z$, then $(x\odot y)\leq(x\odot z)$, $x\rightarrow y\leq x\rightarrow z$ and $z\rightarrow x\leq y\rightarrow x$ where $\rightarrow\in\{\rightarrow,\Rightarrow\}$.*
(2) *$x\rightarrow(\bigwedge_{i\in\Gamma}y_i)=\bigwedge_{i\in\Gamma}(x\rightarrow y_i)$ and $(\bigvee_{i\in\Gamma}x_i)\rightarrow y=\bigwedge_{i\in\Gamma}(x_i\rightarrow y)$ for $\rightarrow\in\{\rightarrow,\Rightarrow\}$.*
(3) *$(x\odot y)\rightarrow z=x\rightarrow(y\rightarrow z)$ and $(x\odot y)\Rightarrow z=y\Rightarrow(x\Rightarrow z)$.*
(4) *$x\rightarrow(y\Rightarrow z)=y\Rightarrow(x\rightarrow z)$ and $x\Rightarrow(y\rightarrow z)=y\rightarrow(x\Rightarrow z)$.*
(5) *$x\odot(x\Rightarrow y)\leq y$ and $(x\rightarrow y)\odot x\leq y$.*
(6) *$(x\Rightarrow y)\odot z\leq x\Rightarrow y\odot z$ and $y\odot(x\rightarrow z)\leq x\rightarrow y\odot z$.*
(7) *$(x\Rightarrow y)\odot(y\Rightarrow z)\leq x\Rightarrow z$ and $(y\rightarrow z)\odot(x\rightarrow y)\leq x\rightarrow z$.*
(8) *$(x\Rightarrow z)\leq(y\odot x)\Rightarrow(y\odot z)$ and $(x\rightarrow z)\leq(x\odot y)\rightarrow(z\odot y)$.*
(9) *$x\rightarrow y\leq(y\rightarrow z)\Rightarrow(x\rightarrow z)$ and $(x\Rightarrow y)\leq(y\Rightarrow z)\rightarrow(x\Rightarrow z)$.*
(10) *$y\rightarrow z\leq(x\rightarrow y)\rightarrow(x\rightarrow z)$ and $(y\Rightarrow z)\leq(x\Rightarrow y)\Rightarrow(x\Rightarrow z)$.*
(11) *$x\rightarrow y=\top$ iff $x\leq y$, where $\rightarrow\in\{\rightarrow,\Rightarrow\}$.*
(12) *$x\rightarrow y=y^0\Rightarrow x^0$ and $x\Rightarrow y=y^*\rightarrow x^*$.*
(13) *$(x\rightarrow y)^*=x\odot y^*$ and $(x\Rightarrow y)^0=y^0\odot x$. Moreover, $(x\odot y)^*=y\Rightarrow x^*$ and $(x\odot y)^0=x\rightarrow y^0$.*
(14) *$\bigwedge_{i\in\Gamma}x_i^*=(\bigvee_{i\in\Gamma}x_i)^*$ and $\bigvee_{i\in\Gamma}x_i^*=(\bigwedge_{i\in\Gamma}x_i)^*$.*
(15) *$\bigwedge_{i\in\Gamma}x_i^0=(\bigvee_{i\in\Gamma}x_i)^0$ and $\bigvee_{i\in\Gamma}x_i^0=(\bigwedge_{i\in\Gamma}x_i)^0$.*
(16) *$x_i\rightarrow y_i\leq(\bigwedge_{i\in\Gamma}x_i)\rightarrow(\bigwedge_{i\in\Gamma}y_i)$, where $\rightarrow\in\{\rightarrow,\Rightarrow\}$.*

(17) $x_i \rightarrow y_i \leq (\bigvee_{i\in\Gamma} x_i) \rightarrow (\bigvee_{i\in\Gamma} y_i)$, *where* $\rightarrow \in \{\rightarrow, \Rightarrow\}$.

DEFINITION 2.4. [9,10] Let X be a set. A function $e^r_X : X \times X \rightarrow L$ is called a *right preorder* if it satisfies the following conditions :

(O) $e^r_X(x,x) = \top$ for all $x \in X$,

(R) $e^r_X(x,y) \odot e^r_X(y,z) \leq e^r_X(x,z)$ for all $x, y, z \in X$.

A function $e^l_X : X \times X \rightarrow L$ is called a *left preorder* if it satisfies (O) and

(L) $e^l_X(y,z) \odot e^l_X(x,y) \leq e^l_X(x,z)$ for all $x, y, z \in X$.

The triple (X, e^r_X, e^l_X) is called a *bi-preordered space.*

EXAMPLE 2.5. [9,10] (1) Define two functions $e^r_L, e^l_L : L \times L \rightarrow L$ as

$$e^r_L(x,y) = x \Rightarrow y,\ e^l_L(x,y) = x \rightarrow y.$$

By Lemma 2.3 (7), (L, e^r_L, e^l_L) is a bi-preordered space.

(2) Define two functions $e^r_{L^X}, e^l_{L^X} : L^X \times L^X \rightarrow L$ as

$$e^r_{L^X}(A,B) = \bigwedge_{x\in X}(A(x) \Rightarrow B(x)),$$
$$e^l_{L^X}(A,B) = \bigwedge_{x\in X}(A(x) \rightarrow B(x)).$$

By Lemma 2.3 (7), $(L^X, e^r_{L^X}, e^l_{L^X})$ is a bi-preordered space.

3. Two types of algebraic structures

DEFINITION 3.1. An operator $I^r : L^X \rightarrow L^X$ is called a *right interior operator* on X if it satisfies the following conditions:

(I1) $I^r(A) \leq A$,

(I2) if $A \leq B$, then $I^r(A) \leq I^r(B)$,

(I3) $I^r(I^r(A)) = I^r(A)$,

(IR) $I^r(A) \odot \alpha \leq I^r(A \odot \alpha)$ where $\alpha(x) = \alpha$.

The pair (X, I^r) is called a *right interior space.*

An operator $I^l : L^X \rightarrow L^X$ is called a *left interior operator* on X if it satisfies the conditions (I1),(I2),(I3) and

(IL) $\alpha \odot I^l(A) \leq I^l(\alpha \odot A)$ where $\alpha(x) = \alpha$.

The triple (X, I^r, I^l) is called a *bi-interior space.*

An operator $C^r : L^X \rightarrow L^X$ is called a *right closure operator* on X if it satisfies the following conditions:

(C1) $C^r(A) \geq A$,

(C2) if $A \leq B$, then $C^r(A) \leq C^r(B)$,

(C3) $C^r(C^r(A)) = C^r(A)$,

(CR) $C^r(\alpha \rightarrow A) \leq \alpha \rightarrow C^r(A)$ where $\alpha(x) = \alpha$.

The pair (X, C^r) is called a *right closure space.*

An operator $C^l : L^X \rightarrow L^X$ is called a *left closure operator* on X if it satisfies the conditions (C1),(C2) (C3) and

(CL) $C^l(\alpha \Rightarrow A) \leq \alpha \Rightarrow C^l(A)$ where $\alpha(x) = \alpha$.

The triple (X, C^r, C^l) is called a *bi-closure space.*

DEFINITION 3.2. Let (X, I^r, I^l) be a bi-interior space and (X, C^r, C^l) be a bi-closure space.

(1) The pair $(I^r(A), C^r(A))$ is a right rough set for $A \in L^X$.

(2) The pair $(I^l(A), C^l(A))$ is a left rough set for $A \in L^X$.

(3) Define $\alpha^r : L^X \to L$ as

$$\alpha^r(A) = \bigwedge_{x \in X} (C(A)(x) \Rightarrow I(A)(x))$$

for all for $A \in L^X$, $C \in \{C^r, C^l\}$ and $I \in \{I^r, I^l\}$. The map α^r is called a *right accuracy measure*

Define $\alpha^l : L^X \to L$ as

$$\alpha^l(A) = \bigwedge_{x \in X} (C(A)(x) \to I(A)(x))$$

for all $A \in L^X$, $C \in \{C^r, C^l\}$ and $I \in \{I^r, I^l\}$. The map α^l is called a *left accuracy measure.*

DEFINITION 3.3. (1) A family F^r is called a *right interior system* on X if $(A_i \odot k), \bigvee_{i \in \Gamma} A_i \in F^r$ for all $A_i \in F^r$ and $k \in L$.

(2) A family F^l is called a *left interior system* on X if $(k \odot A_i), \bigvee_{i \in \Gamma} A_i \in F^l$ for all $A_i \in F^l$ and $k \in L$.

The triple (X, F^r, F^l) is called a *bi-interior systems.*

(3) A family G^r is called a *right closure system* on X if $(k \to A_i) \in G^r, \bigwedge_{i \in \Gamma} A_i$ for all $A_i \in G^r$ and $k \in L$.

(4) A family G^l is called a *left closure system* on X if $(k \Rightarrow A_i) \in G^l, \bigwedge_{i \in \Gamma} A_i$ for all $A_i \in G^l$ and $k \in L$.

The triple (X, G^r, G^l) is called a *bi-closure system.*

THEOREM 3.4. (1) *An operator $I^l : L^X \to L^X$ is a left interior operator on X iff I^l satisfies* (I1), (I2), (I3) *and* $I^l(\alpha \Rightarrow A) \leq \alpha \Rightarrow I^l(A)$.

(2) *An operator $I^r : L^X \to L^X$ is a right interior operator on X iff I^r satisfies* (I1), (I2), (I3) *and* $I^r(\alpha \to A) \leq \alpha \to I^r(A)$.

(3) *An operator $I^l : L^X \to L^X$ is a left interior operator on X iff I^l satisfies* (I1), (I3) *and* $e^l_{L^X}(A, B) \leq e^l_{L^X}(I^l(A), I^l(B))$ *for all* $A, B \in L^X$.

(4) *An operator $I^r : L^X \to L^X$ is a right interior operator on X iff I^r satisfies* (I1), (I3) *and* $e^r_{L^X}(A, B) \leq e^r_{L^X}(I^r(A), I^r(B))$ *for all* $A, B \in L^X$.

(5) *An operator $C^r : L^X \to L^X$ is a right closure operator on X iff C^r satisfies* (C1), (C2), (C3) *and*

$$C^r(A) \odot \alpha \leq C^r(A \odot \alpha).$$

(6) *An operator $C^l : L^X \to L^X$ is a left closure operator on X iff C^l satisfies* (C1), (C2), (C3) *and*

$$\alpha \odot C^l(A) \leq C^l(\alpha \odot A).$$

(7) *An operator $C^r : L^X \to L^X$ is a right closure operator on X iff C^r satisfies* (C1), (C3) *and* $e^r_{L^X}(A, B) \leq e^r_{L^X}(C^r(A), C^r(B))$ *for all* $A, B \in L^X$.

(8) *An operator $C^l : L^X \to L^X$ is a left closure operator on X iff C^l satisfies* (C1), (C3) *and* $e^l_{L^X}(A, B) \leq e^l_{L^X}(C^l(A), C^l(B))$ *for all* $A, B \in L^X$.

Proof. (1) Since $\alpha \odot I^l(\alpha \Rightarrow A) \leq I^l(\alpha \odot (\alpha \Rightarrow A)) \leq I^l(A)$, then $I^l(\alpha \Rightarrow A) \leq \alpha \Rightarrow I^l(A)$.

Conversely, since $I^l(\alpha \Rightarrow \alpha \odot A) \leq \alpha \Rightarrow I^l(\alpha \odot A)$ iff $\alpha \odot I^l(\alpha \Rightarrow \alpha \odot A) \leq I^l(\alpha \odot A)$, we have

$$\alpha \odot I^l(A) \leq \alpha \odot I^l(\alpha \Rightarrow \alpha \odot A) \leq I^l(\alpha \odot A).$$

(3) Put $\alpha = e^l_{L^X}(A,B)$. By (IL) and (I2), $\alpha \odot I^l(A) \leq I^l(\alpha \odot A) = I^l(e^l_{L^X}(A,B) \odot A) \leq I^l(B)$. Hence $e^l_{L^X}(A,B) \leq e^l_{L^X}(I^l(A), I^l(B))$.

Conversely, if $A \leq B$, then $\top = e^l_{L^X}(A,B) \leq e^l_{L^X}(I^l(A), I^l(B))$. Thus, $I^l(A) \leq I^l(B)$.

Since $\alpha \leq e^l_{L^X}(A, \alpha \odot A)$, we have $\alpha \leq e^l_{L^X}(A, \alpha \odot A) \leq e^l_{L^X}(I^l(A), I^l(\alpha \odot A))$. Thus $\alpha \odot I^l(A) \leq I^l(\alpha \odot A)$.

(5) Let C^r be a right closure operator on X. Since $C^r(\alpha \to A \odot \alpha) \leq \alpha \to C^r(A \odot \alpha)$ iff $C^r(\alpha \to A \odot \alpha) \odot \alpha \leq C^r(A \odot \alpha)$, we have

$$C^r(A) \odot \alpha \leq C^r(\alpha \to A \odot \alpha) \odot \alpha \leq C^r(A \odot \alpha).$$

Conversely, $C^r(\alpha \to A) \odot \alpha \leq C^r((\alpha \to A) \odot \alpha) \leq C^r(A)$, then $C^r(\alpha \to A) \leq \alpha \to C^r(A)$.

(2), (4) and (6) are similarly proved as (1),(3) and (5), respectively.

(7) Put $\alpha = e^r_{L^X}(A,B)$. Since, by (CR), $C^r(e^r_{L^X}(A,B) \to B) \leq e^r_{L^X}(A,B) \to C^r(B)$, $e^r_{L^X}(A,B) \leq C^r(e^r_{L^X}(A,B) \to B) \Rightarrow C^r(B)$. Since $A \odot e^r_{L^X}(A,B) \leq B$, then $A \leq e^r_{L^X}(A,B) \to B$. Hence

$$\begin{aligned} e^r_{L^X}(A,B) &\leq C^r(e^r_{L^X}(A,B) \to B) \Rightarrow C^r(B) \\ &\leq C^r(A) \Rightarrow C^r(B). \end{aligned}$$

Thus $e^r_{L^X}(A,B) \leq e^r_{L^X}(C^r(A), C^r(B))$.

Conversely, if $A \leq B$, then $\top = e^r_{L^X}(A,B) \leq e^r_{L^X}(C^r(A), C^r(B))$. Thus, $C^r(A) \leq C^r(B)$.

Since $\alpha \leq e^r_{L^X}(\alpha \to A, A)$, we have $\alpha \leq e^r_{L^X}(\alpha \to A, A) \leq e^r_{L^X}(C^r(\alpha \to A), C^r(A))$. Thus $C^r(\alpha \to A) \leq \alpha \to C^r(A)$.

(8) Put $\alpha = e^l_{L^X}(A,B)$. Since, by (CL), $C^l(e^l_{L^X}(A,B) \Rightarrow B) \leq e^l_{L^X}(A,B) \Rightarrow C^l(B)$, $e^l_{L^X}(A,B) \leq C^l(e^l_{L^X}(A,B) \Rightarrow B) \to C^l(B)$. Since $e^l_{L^X}(A,B) \odot A \leq B$, then $A \leq e^l_{L^X}(A,B) \Rightarrow B$. Hence

$$\begin{aligned} e^l_{L^X}(A,B) &\leq C^l(e^l_{L^X}(A,B) \Rightarrow B) \to C^l(B) \\ &\leq C^l(A) \to C^l(B). \end{aligned}$$

Thus $e^l_{L^X}(A,B) \leq e^l_{L^X}(C^l(A), C^l(B))$.

Conversely, (C2) if $A \leq B$, then $\top = e^l_{L^X}(A,B) \leq e^l_{L^X}(C^l(A), C^l(B))$. Thus, $C^l(A) \leq C^l(B)$.

(CL) Since $\alpha \leq e^l_{L^X}(\alpha \Rightarrow A, A)$, we have $\alpha \leq e^l_{L^X}(\alpha \Rightarrow A, A) \leq e^l_{L^X}(C^l(\alpha \Rightarrow A), C^l(A))$. It implies $\alpha \odot C^l(\alpha \Rightarrow A) \leq C^l(A)$. Thus $C^l(\alpha \Rightarrow A) \leq \alpha \Rightarrow C^l(A)$.

□

THEOREM 3.5. (1) *Let (X, I^r, I^l) be a bi-interior space. Let $F^r_{I^r} = \{A \in L^X \mid I^r(A) = A\}$ and $F^l_{I^l} = \{A \in L^X \mid I^l(A) = A\}$. Then $(X, F^r_{I^r}, F^l_{I^l})$ is a bi-interior system such that $F^r_{I^r} = \{I^r(B) \mid B \in L^X\}$ and $F^l_{I^l} = \{I^l(B) \mid B \in L^X\}$.*

(2) *Let (X, C^r, C^l) be a bi-closure space. Let $G^r_{C^r} = \{A \in L^X \mid C^r(A) = A\}$ and $G^l_{C^l} = \{A \in L^X \mid C^l(A) = A\}$. Then $(X, G^r_{C^r}, G^l_{C^l})$ is a bi-closure system such that $G^r_{C^r} = \{C^r(B) \mid B \in L^X\}$ and $G^l_{C^l} = \{C^l(B) \mid B \in L^X\}$.*

(3) *Let (X, I^r, I^l) be a bi-interior space. Let $C^r(A) = I^l(A^*)^0$ and $C^l(A) = I^r(A^0)^*$ for all $A \in X$. Then (X, C^r, C^l) is a bi-closure space.*

Proof. (1) For all $A_i \in F^r_{I^r}$, since $A_i \odot \alpha = I^r(A_i) \odot \alpha \leq I^r(A_i \odot \alpha) \leq A_i \odot \alpha$, then $A_i \odot \alpha \in F^r_{I^r}$. Since $\bigvee_{i\in\Gamma} A_i = \bigvee_{i\in\Gamma} I^r(A_i) \leq I^r(\bigvee_{i\in\Gamma} A_i) \leq \bigvee_{i\in\Gamma} A_i$, then $\bigvee_{i\in\Gamma} A_i \in$

$F^r_{I^r}$. Put $F^r = \{I^r(B) \mid B \in L^X\}$. Let $I^r(B) \in F^r$. Then $I^r(I^r(B)) = I^r(B)$;i.e. $I^r(B) \in F^r_{I^r}$. Let $A \in F^r_{I^r}$. Then $A = I^r(A) \in F^r$. Hence $F^r_{I^r}$ is a right interior system such that $F^r_{I^r} = \{I^r(B) \mid B \in L^X\}$. Other case is similarly proved.

(2) For all $A_i \in G^r_{C^r}$, since $C^r(\alpha \to A_i) \leq \alpha \to C^r(A_i) = \alpha \to A_i$, then $\alpha \to A_i \in G^r_{C^r}$. Since $\bigwedge_{i\in\Gamma} A_i = \bigwedge_{i\in\Gamma} C^r(A_i) \geq C^r(\bigwedge_{i\in\Gamma} A_i) \geq \bigwedge_{i\in\Gamma} A_i$, then $\bigwedge_{i\in\Gamma} A_i \in G^r_{C^r}$.

Other cases are similarly proved as (1) and the above method.

(3) Since $I^l(A^*) \leq A^*$, we have $A \leq C^r(A)$. $C^r(C^r(A)) = C^r(I^l(A^*)^0) = I^l(I^l(A^*))^0 \leq I^l(A^*)^0 = C^r(A)$. By Lemma 2.3(13),

$$\begin{aligned} &C^r(\alpha \to A) = I^l((\alpha \to A)^*)^0 = I^l(\alpha \odot A^*)^0 \\ &\leq (\alpha \odot I^l(A^*))^0 = \alpha \to I^l(A^*)^0 = \alpha \to C^r(A). \end{aligned}$$

Since $I^r(A^0) \leq A^0$, we have $A \leq C^l(A)$. $C^l(C^l(A)) = C^l(I^r(A^0)^*) = I^r(I^r(A^0))^* \leq I^r(A^0)^* = C^l(A)$. By Lemma 2.3(13),

$$\begin{aligned} &C^l(\alpha \Rightarrow A) = I^r((\alpha \Rightarrow A)^0)^* = I^r(A^0 \odot \alpha)^* \\ &\leq (I^r(A^0) \odot \alpha)^* = \alpha \Rightarrow I^r(A^0)^* = \alpha \Rightarrow C^l(A). \end{aligned}$$

□

THEOREM 3.6. (1) *Let* (X, F^r, F^l) *be a bi-interior system on* X*, Define* $I^r_{F^r}, I^l_{F^l} : L^X \to L^X$ *as*

$$\begin{aligned} I^r_{F^r}(A) &= \bigvee\{A_i \mid A_i \leq A,\ A_i \in F^r\} \\ I^l_{F^l}(A) &= \bigvee\{A_i \mid A_i \leq A,\ A_i \in F^l\}. \end{aligned}$$

Then $(X, I^r_{F^r}, I^l_{F^l})$ *is a bi-interior space such that*

$$\begin{aligned} I^r_{F^r}(A) &= \bigvee_{A_i\in F^r}(A_i \odot e^r_{L^X}(A_i, A)), \\ I^l_{F^l}(A) &= \bigvee_{A_i\in F^l}(e^l_{L^X}(A_i, A) \odot A_i). \end{aligned}$$

(2) *Let* (X, G^r, G^l) *be a bi-closure system. Define* $C^r_{G^r}, C^l_{G^l} : L^X \to L^X$ *as*

$$\begin{aligned} C^r_{G^r}(A) &= \bigwedge\{A_i \mid A \leq A_i,\ A_i \in G^r\}, \\ C^l_{G^l}(A) &= \bigwedge\{A_i \mid A \leq A_i,\ A_i \in G^l\}. \end{aligned}$$

Then $(X, C^r_{G^r}, C^l_{G^l})$ *is a bi-closure space such that*

$$\begin{aligned} C^r_{G^r}(A) &= \bigwedge_{A_i\in G^r}(e^r_{L^X}(A, A_i) \to A_i), \\ C^l_{G^l}(A) &= \bigwedge_{A_i\in G^l}(e^l_{L^X}(A, A_i) \Rightarrow A_i). \end{aligned}$$

(3) $F^r = \{B_i \mid i \in \Gamma\}$ *is a right interior system on* X *iff* $G^l = \{B^*_i \mid B_i \in F^r\}$ *is a left closure system on* X*.*

(4) $F^l = \{B_i \mid i \in \Gamma\}$ *is a left interior system on* X *iff* $G^r = \{B^0_i \mid B_i \in F^l\}$ *is a right closure system on* X*.*

Proof. (1) Since $I^l_{F^l}(A) = \bigvee\{A_i \mid A_i \leq A,\ A_i \in F^l\} \leq A$, (I1) and (I2) are easy.

Since $I^l_{F^l}(A) \leq I^l_{F^l}(A)$ and $I^l_{F^l}(A) \in F^l$, $I^l_{F^l}(I^l_{F^l}(A)) \geq I^l_{F^l}(A)$. By (I1), $I^l_{F^l}(I^l_{F^l}(A)) = I^l_{F^l}(A)$. (IL), for all $\alpha, A \in L^X$,

$$\begin{aligned} &\alpha \odot I^l_{F^l}(A) = \alpha \odot \bigvee\{A_i \mid A_i \leq A,\ A_i \in F^l\} \\ &\leq \bigvee\{\alpha \odot A_i \mid \alpha \odot A_i \leq \alpha \odot A,\ \alpha \odot A_i \in F^l\} \\ &= I^l_{F^l}(\alpha \odot A). \end{aligned}$$

Put $I^l(A) = \bigvee_{A_i\in F^l}(e^l_{L^X}(A_i, A) \odot A_i)$. Since $\bigvee_{A_i\in F^l}(e^l_{L^X}(A_i, A) \odot A_i) \leq A$ and $\bigvee_{A_i\in F^l}(e^l_{L^X}(A_i, A) \odot A_i) \in F^l$, $I^l(A) \leq I^l_{F^l}(A)$.

Since $I^l_{F^l}(A) \in F^l$, $I^l(A) \geq e^l_{L^X}(I^l_{F^l}(A), A) \odot I^l_{F^l}(A) = I_{F^l}(A)$.

Other case is similarly proved.

(2) Since $C^r_{G^r}(A) = \bigwedge\{A_i \mid A \leq A_i,\ A_i \in G^r\}$, then (C1) and (C2) hold.

Since $C^r_{G^r}(A) \in G^r$ and $C^r_{G^r}(A) \leq C^r_{G^r}(A)$, $C^r_{G^r}(C^r_{G^r}(A)) \leq C^r_{G^r}(A)$. By (C1), $C^r_{G^r}(C^r_{G^r}(A)) = C^r_{G^r}(A)$. (CL), for all $\alpha, A \in L^X$,

$$\begin{aligned}
&\alpha \to C^r_{G^r}(A) = \alpha \to \bigwedge\{A_i \mid A \leq A_i,\ A_i \in G^r\}\\
&= \bigwedge\{\alpha \to A_i \mid A \leq A_i,\ A_i \in G^r\}\\
&\geq \bigwedge\{\alpha \to A_i \mid \alpha \to A \leq \alpha \to A_i,\ \alpha \to A_i \in G^r\}\\
&\geq C^r_{G^r}(\alpha \to A).
\end{aligned}$$

Then $C^r_{G^r}$ is a right closure operator on X. Similarly, $C^l_{G^l}$ is a left closure operator on X.

Put $C^l(A) = \bigwedge_{A_i \in G^l}(e^l_{L^X}(A, A_i) \Rightarrow A_i)$. Since $A \leq e^l_{L^X}(A, A_i) \Rightarrow A_i \in G^l$ for $A_i \in G^l$, $C^l_{G^l} \leq C^l$.

Since $C^l(A) \leq e^l_{L^X}(A, C^l_{G^l}(A)) \Rightarrow C^l_{G^l}(A)$, $C^l(A) \leq C^l_{G^l}(A)$.

(3) and (6) are easily proved that for $B^*_i \in G^l$ with $B_i \in F^r$, by Lemma 2.3 (13), $\alpha \Rightarrow B^*_i = (B_i \odot \alpha)^* \in G^l$.

(4) and (5) are easily proved that, by Lemma 2.3 (13), $\alpha \to B^0_i = (\alpha \odot B_i)^0$.

□

Theorem 3.7. *Let (X, e^r_X, e^l_X) be a bi-preordered space.*

(1) *Define*

$$\begin{aligned}
I^l_{e^r_X}(A) &= \bigwedge_{x \in X}(e^r_X(-, x) \to A(x))\\
C^l_{e^r_X}(A) &= \bigvee_{x \in X}((A(x) \odot e^r_X(x, -)).
\end{aligned}$$

Then $(I^l_{e^r_X}(A), C^l_{e^r_X}(A))$ is a left rough set for $A \in L^X$ and e^r_X.

(2) *Define*

$$\begin{aligned}
I^r_{e^r_X}(A) &= \bigwedge_{x \in X}(e^r_X(x, -) \Rightarrow A(x))\\
C^r_{e^r_X}(A) &= \bigvee_{x \in X}(e^r_X(-, x) \odot A(x)).
\end{aligned}$$

Then $(I^r_{e^r_X}(A), C^r_{e^r_X}(A))$ is a right rough set for $A \in L^X$ and e^r_X.

(3) *Define*

$$\begin{aligned}
I^l_{e^l_X}(A) &= \bigwedge_{x \in X}(e^l_X(x, -) \to A(x))\\
C^l_{e^l_X}(A) &= \bigvee_{x \in X}((A(x) \odot e^l_X(-, x)).
\end{aligned}$$

Then $(I^l_{e^l_X}(A), C^l_{e^l_X}(A))$ is a left rough set for $A \in L^X$ and e^l_X.

(4) *Define*

$$\begin{aligned}
I^r_{e^l_X}(A) &= \bigwedge_{x \in X}(e^l_X(-, x) \Rightarrow A(x))\\
C^r_{e^l_X}(A) &= \bigvee_{x \in X}(e^l_X(x, -) \odot A(x)).
\end{aligned}$$

Then $(I^r_{e^l_X}(A), C^r_{e^l_X}(A))$ is a right rough set for $A \in L^X$ and e^l_X.

(5) *Let $A, B \in L^X$. Then $A \leq I^l_{e^r_X}(C^l_{e^r_X}(A))$ and $C^l_{e^r_X}(I^l_{e^r_X}(B)) \leq B$. Moreover, $C^l_{e^r_X}(A) \leq B$ iff $A \leq I^l_{e^r_X}(B)$.*

(6) *Let $A, B \in L^X$. Then $A \leq I^r_{e^r_X}(C^r_{e^r_X}(A))$ and $C^r_{e^r_X}(I^r_{e^r_X}(B)) \leq B$. Moreover, $C^r_{e^r_X}(A) \leq B$ iff $A \leq I^r_{e^r_X}(B)$.*

(7) *Let $A, B \in L^X$. Then $A \leq I^l_{e^l_X}(C^l_{e^l_X}(A))$ and $C^l_{e^l_X}(I^l_{e^l_X}(B)) \leq B$. Moreover, $C^l_{e^l_X}(A) \leq B$ iff $A \leq I^l_{e^l_X}(B)$.*

(8) *Let* $A, B \in L^X$. *Then* $A \leq I^r_{e^l_X}(C^r_{e^l_X}(A))$ *and* $C^r_{e^l_X}(I^r_{e^l_X}(B)) \leq B$. *Moreover,* $C^r_{e^l_X}(A) \leq B$ *iff* $A \leq I^r_{e^l_X}(B)$.

Proof. (1) Since $a \to b \odot c \geq b \odot (a \to c)$,

$$\begin{array}{l} I^l_{e^r_X}(\alpha \odot A)(y) = \bigwedge_{x\in X}(e^r_X(y,x) \to \alpha \odot A(x)) \\ \geq \alpha \odot (\bigwedge_{x\in X}(e^r(y,x) \to A(x)), \end{array}$$

$$\begin{array}{l} I^l_{e^r_X}(I^l_{e^r_X}(A))(y) = \bigwedge_{x\in X}(e^r(y,x) \to I^l_{e^r_X}(A)(x)) \\ = \bigwedge_{x\in X}(e^r(y,x) \to (\bigwedge_{z\in X}(e^r(x,z) \to A(z)) \\ = \bigwedge_{z\in X}(\bigvee_{x\in X}(e^r(y,x) \odot e^r(x,z)) \to A(z)) \\ = \bigwedge_{z\in X}(e^r(y,z) \to A(z)) = I^l_{e^r_X}(A)(y). \end{array}$$

Other cases (I1) and (I2) are easily proved. Thus $I^l_{e^r_X}$ is a left interior operator.

(C1) and (C2) are easily proved. (C3) and (CL) follow

$$\begin{array}{l} C^l_{e^r_X}(C^l_{e^r_X}(A))(y) = \bigvee_{x\in X}(C^l_{e^r_X}(A)(x) \odot_{e^r_X} (x,y)) \\ = \bigvee_{x\in X}(\bigvee_{z\in X}(A(z) \odot e^r_X(z,x) \odot e^r_X(x,y)) \\ = \bigvee_{z\in X}(A(z) \odot \bigvee_{x\in X}(e^r_X(z,x) \odot e^r_X(x,y))) \\ = \bigvee_{z\in X}(A(z) \odot e^r_X(z,y)) = C^l_{e^r_X}(A)(y). \end{array}$$

Since $(a \Rightarrow b) \odot c \leq a \Rightarrow b \odot c$, $C^l_{e^r_X}(\alpha \Rightarrow A) \leq \alpha \Rightarrow C^l(A)$. Hence $C^l_{e^r_X}$ is a left closure operator.

(2),(3) and (4) are similarly proved.

(5) For all $y \in X, A \in L^X$,

$$\begin{array}{l} C^l_{e^r_X}(I^l_{e^r_X}(A))(y) = \bigvee_{x\in X}(I^l_{e^r_X}(A)(x) \odot_{e^r_X} (x,y)) \\ = \bigvee_{x\in X}(\bigwedge_{z\in X}(e^r_X(x,z) \to A(z)) \odot e^r_X(x,y)) \\ \leq \bigvee_{x\in X}((e^r_X(x,y) \to A(y)) \odot e^r_X(x,y)) \leq A(y). \end{array}$$

$$\begin{array}{l} I^l_{e^r_X}(C^r_{e^r_X}(A))(y) = \bigwedge_{x\in X}(e^r(y,x) \to C^l_{e^r_X}(A)(x)) \\ = \bigwedge_{x\in X}(e^r(y,x) \to (\bigvee_{z\in X}(A(z) \odot e^r_X(z,x)) \\ \geq \bigwedge_{x\in X}(e^r(y,x) \to (A(y) \odot e^r_X(y,x)) \geq A(y). \end{array}$$

□

Example 3.8. Let $M = \{(x,y) \in R^2 \mid y > 0\}$ be a set and we define an operation $\otimes : M \times M \to M$ as follows:

$$(x_1, y_1) \otimes (x_2, y_2) = (x_1 + y_1 x_2, y_1 y_2).$$

Then $(M, \otimes)$ is a group with $e = (0,1)$, $(x,y)^{-1} = (-\frac{x}{y}, \frac{1}{y})$.

For $(x_1,y_1), (x_2,y_2) \in M$, we define

$$\begin{array}{l} (x_1,y_1) \leq (x_2,y_2) \\ \Leftrightarrow y_1 < y_2 \ \text{ or } \ y_1 = y_2, x_1 \leq x_2. \end{array}$$

Then $(M, \leq \otimes)$ is a lattice-group.

Let $(L, \odot, \Rightarrow, \rightarrow, (1, \frac{1}{2}), (0,1))$ be a pseudo MV-algebra where $(1, \frac{1}{2})$ is the least element and $(0,1)$ is the greatest element from the following statements:

$$\begin{aligned}(x_1,y_1)\odot(x_2,y_2) &= (x_1,y_1)\otimes(x_2,y_2)\vee(1,\tfrac{1}{2})\\ &= (x_1+y_1x_2, y_1y_2)\vee(1,\tfrac{1}{2}),\\ (x_1,y_1)\Rightarrow(x_2,y_2) &= ((x_1,y_1)^{-1}\otimes(x_2,y_2))\wedge(0,1)\\ &= (\tfrac{-x_1+x_2}{y_1}, \tfrac{y_2}{y_1})\wedge(0,1),\\ (x_1,y_1)\rightarrow(x_2,y_2) &= ((x_2,y_2)\otimes(x_1,y_1)^{-1})\wedge(0,1)\\ &= (x_2-\tfrac{x_1y_2}{y_1}, \tfrac{y_2}{y_1})\wedge(0,1).\end{aligned}$$

It is not commutative because

$$\begin{aligned}&(\tfrac{2}{3},\tfrac{3}{4})\odot(4,\tfrac{1}{2}) = (3+\tfrac{2}{3},\tfrac{3}{8})\\ &\neq (4,\tfrac{1}{2})\odot(\tfrac{2}{3},\tfrac{3}{4}) = (4+\tfrac{1}{3},\tfrac{3}{8}).\end{aligned}$$

Furthermore, we have $(x,y) = (x,y)^{*\circ} = (x,y)^{\circ *}$ from:

$$\begin{aligned}(x,y)^* &= (x,y)\Rightarrow(1,\tfrac{1}{2}) = (\tfrac{-x+1}{y}, \tfrac{1}{2y}),\\ (x,y)^\circ &= (x,y)\rightarrow(1,\tfrac{1}{2}) = (1-\tfrac{x}{2y}, \tfrac{1}{2y}).\end{aligned}$$

Let $X = \{x,y,z\}$ and $A \in L^X$ as follows:

$$A(x) = (1,0.6), A(y) = (0.2,0.8), A(z) = (0,0.6).$$

Define $e^r_A, e^l_A : X\times X \to L$ as $e^r_A(x,y) = A(x)\Rightarrow A(y)$ and $e^l_A(x,y) = A(x)\rightarrow A(y)$ such that

$$e^r_A = \begin{pmatrix} (0,1) & (0,1) & (-\frac{5}{3},1)\\ (1,\frac{3}{4}) & (0,1) & (-\frac{1}{4},\frac{3}{4})\\ (0,1) & (0,1) & (0,1)\end{pmatrix}$$

$$e^l_A = \begin{pmatrix} (0,1) & (0,1) & (-1,1)\\ (\frac{17}{20},\frac{3}{4}) & (0,1) & (-\frac{3}{20},\frac{3}{4})\\ (0,1) & (0,1) & (0,1)\end{pmatrix}.$$

By Lemma 2.3(7), (X, e^r_A, e^l_A) is a bi-preordered space. Moreover, $e^r_A(x,y) = A(x) \Rightarrow A(y) = A^*(y) \rightarrow A^*(x) = e^l_{A^*}(y,x)$ and $e^l_A(x,y) = A(x)\rightarrow A(y) = A^0(y) \Rightarrow A^0(x) = e^r_{A^0}(y,x)$.

(1) Put $F^r = \{A\odot k \mid k\in L\}$ and $G^r = \{k\rightarrow A \mid k \in L\}$. Since $k_i \le A \Rightarrow \bigvee_{i\in I}(A\odot k_i)$, $\bigvee_{i\in I} k_i \le A \Rightarrow \bigvee_{i\in I}(A\odot k_i)$. Then $A\odot(\bigvee_{i\in I}k_i) \le \bigvee_{i\in I}(A\odot k_i)$. Moreover, $\bigvee_{i\in I}(A\odot k_i) \le A\odot(\bigvee_{i\in I}k_i)$. Hence F^r is a right interior system.

Since $k_1 \rightarrow (k_2\rightarrow A) = (k_1\odot k_2)\rightarrow A \in G^r$ and $\bigwedge_{i\in I}(k_i\rightarrow A) = (\bigvee_{i\in I}k_i)\rightarrow A \in G^r$, G^r is a left closure system.

For $B = ((1,\frac{4}{5}),(0,\frac{2}{3}),(2,\frac{3}{4})) \in L^X$, since $A\odot k_1 \le B$ and $B \le k_2 \rightarrow A$, $k_1 = \bigwedge_{x\in X}(A(x)\Rightarrow B(x)) = (-\frac{1}{4},\frac{5}{6})$, $k_2 = \bigwedge_{x\in X}(B(x)\Rightarrow A(x)) = (0,\frac{3}{4})$ and

$$\begin{aligned}&I^r_{F^r}(B) = \textstyle\bigvee_{D\in F^r}(D(-)\odot e^r_{L^X}(D,B))\\ &= \textstyle\bigvee\{D \mid D\le B, D\in F^r\} = A\odot k_1\\ &= A\odot(-\tfrac{1}{4},\tfrac{5}{6}) = ((1,\tfrac{1}{2}),(0,\tfrac{2}{3}),(1,\tfrac{1}{2})),\\ &C^r_{G^r}(B) = \textstyle\bigwedge_{D\in G^r}(e^r_{L^X}(B,D)\rightarrow D(-))\\ &= \textstyle\bigwedge\{D \mid B\le D, D\in G^r\} = k_2\rightarrow A\\ &= (0,\tfrac{3}{4})\rightarrow A = ((1,\tfrac{4}{5}),(0,\tfrac{2}{3}),(0,\tfrac{4}{5})).\end{aligned}$$

The pair $(I^r_{F^r}(B), C^r_{G^r}(B))$ is a right rough set for B.

Two maps $\alpha^r, \alpha^l : L^X \to L$ are right and left accuracy measures for $B \in L^X$,

$$\begin{array}{l} \alpha^r(B) = \bigwedge_{x\in X}(C^r_{G^r}(B)(x) \Rightarrow I^r_{F^r}(B)(x)) = (0, \frac{5}{8}), \\ \alpha^l(A) = \bigwedge_{x\in X}(C^r_{G^r}(B)(x) \to I^r_{F^r}(B)(x)) = (\frac{3}{8}, \frac{5}{8}). \end{array}$$

(2) Put $F^l = \{k \odot A \mid k \in L\}$ and $G^l = \{k \Rightarrow A \mid k \in L\}$. Since $k_i \leq A \to \bigvee_{i\in I}(k_i \odot A)$, $\bigvee_{i\in I} k_i \leq A \to \bigvee_{i\in I}(k_i \odot A)$. Then $(\bigvee_{i\in I} k_i) \odot A \leq \bigvee_{i\in I}(k_i \odot A)$. Moreover, $\bigvee_{i\in I}(k_i \odot A) \leq (\bigvee_{i\in I} k_i) \odot A$. Hence F^l is a left interior system.

Since $k_1 \Rightarrow (k_2 \Rightarrow A) = (k_2 \odot k_1) \Rightarrow A \in G^l$ and $\bigwedge_{i\in I}(k_i \Rightarrow A) = (\bigvee_{i\in I} k_i) \Rightarrow A \in G^l$, G^l is a left closure system.

For $B = ((1, \frac{4}{5}), (0, \frac{2}{3}), (2, \frac{3}{4})) \in L^X$, $k_3 = \bigwedge_{x\in X}(A(x) \to B(x)) = (-\frac{1}{6}, \frac{5}{6})$, $k_4 = \bigwedge_{x\in X}(B(x) \to A(x)) = (\frac{1}{4}, \frac{3}{4})$ and

$$\begin{array}{l} I^l_{F^l}(B) = \bigvee_{B\in F^l}(e^l_{L^X}(B, A) \odot B(-)) \\ = \bigvee\{B \mid B \leq A, B \in F^l\} = k_3 \odot A \\ = (-\frac{1}{6}, \frac{5}{6}) \odot A = ((\frac{2}{3} \vee 1, \frac{1}{2}), (0, \frac{2}{3}), (-\frac{1}{6} \vee 1, \frac{1}{2})) \\ = ((1, \frac{1}{2}), (0, \frac{2}{3}), (1, \frac{1}{2})), \\ C^l_{G^l}(B) = \bigwedge_{B\in G^l}(e^l_{L^X}(A, B) \Rightarrow B(-)) \\ = \bigwedge\{B \mid A \leq B, B \in G^l\} = k_4 \Rightarrow A \\ = (\frac{1}{4}, \frac{3}{4}) \Rightarrow A = ((1, \frac{4}{5}), (0, 1), (-\frac{1}{3}, \frac{4}{5})). \end{array}$$

The pair $(I^l_{F^l}(B), C^l_{G^l}(B))$ is a left rough set for B and e^l_X

Two maps $\alpha^r, \alpha^l : L^X \to L$ are right and left accuracy measures for $B \in L^X$,

$$\begin{array}{l} \alpha^r(B) = \bigwedge_{x\in X}(C^l_{G^l}(B)(x) \Rightarrow I^l_{F^l}(B)(x)) = (0, \frac{5}{8}), \\ \alpha^l(A) = \bigwedge_{x\in X}(C^l_{G^l}(B)(x) \to I^l_{F^l}(B)(x)) = (\frac{3}{8}, \frac{5}{8}). \end{array}$$

References

[1] R. Bělohlávek, *Fuzzy Relational Systems*, Kluwer Academic Publishers, New York, 2002.
https://www.kybernetika.cz/content/2003/1/119/paper.pdf

[2] R. Bělohlávek, *Fuzzy closure operator*, J. Math. Anal. Appl., **262** (2001), 473–486.

[3] L. Biacino and G. Gerla, *Closure systems and L-subalgebras*, Inf. Sci., **33** (1984), 181–195.

[4] L. Biacino and G. Gerla, *An extension principle for closure operators*, J. Math. Anal. Appl., **198** (1996), 1–24.

[5] L. Guo, G. Q. Zhang, and Q. Li, *Fuzzy closure systems on L-ordered sets*, Math. Log. Quart., **57** (3) (2011), 281–291.
https://doi.org/10.1002/malq.201010007

[6] J. Fang and Y. Yue, *L-fuzzy closure systems*, Fuzzy Sets Syst., **161** (2010), 1242–1252.
https://doi.org/10.1016/j.fss.2009.10.002

[7] G. Georgescu and A. Popescue, *Non-commutative Galois connections*, Soft Comput., **7** (2003), 458–467.

[8] P. Hájek, *Metamathematics of Fuzzy Logic*, Kluwer Academic Publishers, Dordrecht, 1998.

[9] J. M. Ko and Y. C. Kim, *Bi-closure systems and bi-closure operators on generalized residuated lattices*, J. Intell. Fuzzy Syst., **36** (2019), 2631–2643.
https://doi.org/10.3233/JIFS-18493

[10] J. M. Ko and Y. C. Kim, *Various operations and right completeness in generalized residuated lattices*, J. Intell. Fuzzy Syst., **40** (2021), 149–164.
https://doi.org/10.3233/JIFS-190424

[11] Z. Pawlak, *Rough sets*, Internat. J. Comput. Inform. Sci., **11** (1982), 341–356.

[12] Z. Pawlak, *Rough sets: Theoretical Aspects of Reasoning about Data, System Theory, Knowledge Engineering and Problem Solving*, Kluwer Academic Publishers, Dordrecht, 1991.

[13] J. Qiao and B. Q. Hu, *Granular variable precision L-fuzzy rough sets based on residuated lattices*, Fuzzy Sets Syst., **336** (2018), 148–166.

[14] C. J. Mulvey, *Quantales*, Suppl. Rend. Circ. Mat. Palermo Ser. II, **12** (1986), 99–104.
[15] E. Turunen, *Mathematics Behind Fuzzy Logic*, Springer-Verlag, 1999.
[16] C. Y. Wang, *Topological characterizations of generalized fuzzy rough sets*, Fuzzy Sets Syst., **312** (2017), 109–125.
https://doi.org/10.1016/j.fss.2016.02.005
[17] C. Y. Wang, *Topological structures of L-fuzzy rough sets and similarity sets of L-fuzzy relations*, Int. J. Approx. Reason., **83** (2017), 160–175.
https://doi.org/10.1016/j.ijar.2017.01.002
[18] M. Ward and R. P. Dilworth, *Residuated lattices*, Trans. Amer. Math. Soc., **45** (1939), 335–354.

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