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FUZZY CONNECTIONS AND COMPLETENESS IN COMPLETE RESIDUATED LATTICES

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Abstract. In this paper, we investigate the properties of fuzzy Galois (dual Galois, residuated, dual residuated) connections in a complete residuated lattice L.

1. Introduction

Galois connection is an important mathematical tool for algebraic structure, data analysis and knowledge processing [1-5,7-11]. Orlowska and Rewitzky [9] investigated the algebraic structures of operators of Galois-style (Galois, dual Galois, residuated, dual residuated) connections on set. Hájek [6] introduced a complete residuated lattice L which is an algebraic structure for many valued logic. Recently, Yao and Lu [11] introduced Galois connections and fuzzy completeness in a complete residuated lattice L. Bělohlávek [1-3] developed the notion of fuzzy contexts using Galois connections with $R \in L^{X\times Y}$ on a complete residuated lattice L.

In this paper, we investigate the properties of fuzzy Galois (dual Galois, residuated, dual residuated) connections as an extension of Yao and Lu [11] in a complete residuated lattice L.

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2. Preliminaries

Definition 2.1. *([6,11]) An algebra* $(L, \wedge, \vee, \odot, \to, 0, 1)$ *is called a complete residuated lattice if it satisfies the following conditions:*

(C1) $L = (L, \leq, \vee, \wedge, 1, 0)$ *is a complete lattice with the greatest element* 1 *and the least element* 0*;*
(C2) $(L, \odot, 1)$ *is a commutative monoid;*
(C3) $x \odot y \leq z$ *iff* $x \leq y \to z$ *for* $x, y, z \in L$.

Remark 2.2. *([6]) (1) A completely distributive lattice* $L = (L, \leq, \vee, \wedge = \odot, \to, 1, 0)$ *is a complete residuated lattice defined by*

$$x \to y = \bigvee\{z \mid x \wedge z \leq y\}.$$

In particular, the unit interval $([0,1], \vee, \wedge = \odot, \to, 0, 1)$ *is a complete residuated lattice defined by*

$$x \to y = \bigvee\{z \mid x \wedge z \leq y\}.$$

(2) The unit interval $([0,1], \vee, \wedge, \odot, \to, 0, 1)$ *with a left-continuous t-norm* $\odot$ *is a complete residuated lattice defined by*

$$x \to y = \bigvee\{z \mid x \odot z \leq y\}.$$

In this paper, we assume $(L, \wedge, \vee, \odot, \to, 0, 1)$ is a complete residuated latttice.

Definition 2.3. *([11]) Let* X *be a set. A function* $e_X : X \times X \to L$ *is called:*

(E1) reflexive if $e_X(x, x) = 1$ *for all* $x \in X$,
(E2) transitive if $e_X(x, y) \odot e_X(y, z) \leq e_X(x, z)$, *for all* $x, y, z \in X$.
(E3) if $e_X(x, y) = e_X(y, x) = 1$, *then* $x = y$.

If e_X *satisfies (E1) and (E2),* e_X *is a fuzzy preorder and* (X, e_X) *is a fuzzy preorder set. If* e *satisfies (E1), (E2) and (E3),* e_X *is a fuzzy partially order and* (X, e_X) *is a fuzzy partially order set (simply, fuzzy poset).*

Example 2.4. *(1) We define a function* $e_L : L \times L \to L$ *as* $e_L(x, y) = x \to y$. *Then* e_L *is a partial order.*

(2) We define a function $e_{L^X} : L^X \times L^X \to L$ *as* $e_{L^X}(A, B) = \bigwedge_{x \in X}(A(x) \to B(x))$. *Then* (L^X, e_{L^X}) *is a fuzzy poset.*

(3) If (X, e_X) *is a fuzzy poset and we define a function* $e_X^{-1}(x, y) = e_X(y, x)$, *then* (X, e_X^{-1}) *is a fuzzy poset.*

DEFINITION 2.5. *Let* (X, e_X) *and* (Y, e_Y) *be a fuzzy poset and* $f : X \to Y$ *and* $g : Y \to X$ *maps.*

(1) (e_X, f, g, e_Y) *is called a Galois connection if for all* $x \in X, y \in Y$,

$$e_Y(y, f(x)) = e_X(x, g(y)).$$

(2) (e_X, f, g, e_Y) *is called a dual Galois connection if for all* $x \in X, y \in Y$,

$$e_Y(f(x), y) = e_X(g(y), x).$$

(3) (e_X, f, g, e_Y) *is called a residuated connection if for all* $x \in X, y \in Y$,

$$e_Y(f(x), y) = e_X(x, g(y)).$$

(4) (e_X, f, g, e_Y) *is called a dual residuated connection if for all* $x \in X, y \in Y$,

$$e_Y(y, f(x)) = e_X(g(y), x).$$

(5) f *is an order preserving map if* $e_Y(f(x_1), f(x_2)) \geq e_X(x_1, x_2)$ *for all* $x_1, x_2 \in X$.

(6) f *is an order reversing map if* $e_Y(f(x_1), f(x_2)) \geq e_X(x_2, x_1)$ *for all* $x_1, x_2 \in X$.

3. Fuzzy connections and completeness in complete residuated lattices

DEFINITION 3.1. *([10]) Let* (X, e_X) *be a fuzzy poset and* $A \in L^X$.

(1) *A point* x_0 *is called a join (or supremum) of* A, *denoted by* $x_0 = \sqcup A$, *if it satisfies*

(J1) $A(x) \leq e_X(x, x_0)$,

(J2) $\bigwedge_{x \in X}(A(x) \to e_X(x, y)) \leq e_X(x_0, y)$.

A point x_1 *is called a meet (or infimum) of* A, *denoted by* $x_1 = \sqcap A$, *if it satisfies*

(M1) $A(x) \leq e_X(x_1, x)$,

(M2) $\bigwedge_{x \in X}(A(x) \to e_X(y, x)) \leq e_X(y, x_1)$.

The pair (X, e_X) *is called a fuzzy complete lattice if for all* $A \in L^X$, $\sqcup A$ *and* $\sqcap A$ *exist.*

(2) $x_0 = \max A$ *is called a maximal element if* $A(x_0) = 1$ *and* $A(y) \leq e_X(y, x_0)$, *for all* $y \in X$.

(3) $x_1 = \min A$ *is called a minimal element if* $A(x_1) = 1$ *and* $A(y) \leq e_X(x_1, y)$, *for all* $y \in X$.

REMARK 3.2. *Let (X, e_X) be a fuzzy poset and $A \in L^X$. If x_0 is a join of A, then it is unique because $e_X(x_0, y) = e_X(y_0, y)$ for all $y \in X$, put $y = x_0$ or $y = y_0$, then $e_X(x_0, y_0) = e_X(y_0, x_0) = 1$ implies $x_0 = y_0$. Similarly, if a meet of A exist, then it is unique.*

THEOREM 3.3. *Let (X, e_X) be a fuzzy poset and $A \in L^X$.*
(1) x_0 is a join of A iff $\bigwedge_{x\in X}(A(x) \to e_X(x, y)) = e_X(x_0, y)$.
(2) x_1 is a meet of A iff $\bigwedge_{x\in X}(A(x) \to e_X(y, x)) = e_X(y, x_1)$.
(3) $x_0 = \max A$ iff $A(x_0) = 1$ and $x_0 = \sqcup A$.
(4) $x_1 = \min A$ iff $A(x_1) = 1$ and $x_1 = \sqcap A$.

Proof. (1) ($\Rightarrow$) Let x_0 be a join of A. Then $A(x) \le e_X(x, x_0)$. Thus, $A(x) \odot e_X(x_0, y) \le e_X(x, x_0) \odot e_X(x_0, y) \le e_X(x, y)$. Hence $e_X(x_0, y) \le \bigwedge_{x\in X}(A(x) \to e_X(x, y))$. By (J2), the equality holds.

($\Leftarrow$) Since $\bigwedge_{x\in X}(A(x) \to e_X(x, x_0)) = e_X(x_0, x_0) = 1$, then $A(x) \le e_X(x, x_0)$. Hence the result holds.

(2) ($\Rightarrow$) Let x_1 be a meet of A. Then $A(x) \le e_X(x_1, x)$. Thus, $e_X(y, x_1) \odot A(x) \le e_X(y, x_1) \odot e_X(x_1, x) \le e_X(y, x)$. Hence $e_X(y, x_1) \le \bigwedge_{x\in X}(A(x) \to e_X(x, y))$. By (M2), the equality holds.

($\Leftarrow$) Since $\bigwedge_{x\in X}(A(x) \to e_X(x_1, x)) = e_X(x_1, x_1) = 1$, then $A(x) \le e_X(x_1, x)$. Hence the result holds.

(3) Let $x_0 = \max A$. Then

$$\bigwedge_{z\in X}(A(z) \to e_X(z, x)) \le A(x_0) \to e_X(x_0, x) = e_X(x_0, x),$$

$$\bigwedge_{z\in X}(A(z) \to e_X(z, x)) \ge \bigwedge_{z\in X}(e_X(z, x_0) \to e_X(z, x)) = e_X(x_0, x).$$

Thus $e_X(x_0, x) = \bigwedge_{z\in X}(A(z) \to e_X(z, x))$. So, $x_0 = \sqcup A$.

Let $A(x_0) = 1$ and $x_0 = \sqcup A$. Then $e_X(x_0, x_0) = \bigwedge_{z\in X}(A(z) \to e_X(z, x_0)) = 1$ implies $A(z) \le e_X(z, x_0)$. Then $x_0 = \max A$.

(4) It is similarly proved as (3). □

REMARK 3.4. *Let (X, e_X) be a fuzzy poset and $A \in L^X$.*

(1) *Since $\bigwedge_{x\in X}(e_X(x, y) \to e_X(x, z)) = e_X(y, z)$, then, by Theorem 3.3 (1), $y = \sqcup(e_X)^y$ where $(e_X)^y(x) = e_X(x, y)$.*
(2) *Since $\bigwedge_{z\in X}(e_X(y, z) \to e_X(x, z)) = e_X(x, y)$, then, by Theorem 3.3 (3), $y = \sqcap(e_X)_y$ where $(e_X)_y(x) = e_X(y, x)$.*

REMARK 3.5. *Let (L, e_L) be a fuzzy poset and $A \in L$.*

(1) *If* x_0 *is a join of* A*, then* $\bigwedge_{x\in L}(A(x) \to e_L(x,y)) = \bigwedge_{x\in L}(A(x) \to (x \to y)) = \bigvee_{x\in L}(x \odot A(x)) \to y = e_L(x_0,y) = x_0 \to y$*. So,* $x_0 = \sqcup A = \bigvee_{x\in L}(x \odot A(x))$*.*

(2) *If* x_1 *is a meet of* A *iff* $\bigwedge_{x\in L}(A(x) \to e_L(y,x)) = \bigwedge_{x\in L}(A(x) \to (y \to x)) = y \to \bigwedge_{x\in L}(A(x) \to x) = e_X(y,x_1) = y \to x_1$*, then* $x_1 = \sqcap A = \bigwedge_{x\in L}(A(x) \to x)$*.*

EXAMPLE 3.6. *Let* $X = \{a,b,c\}$ *be a set. Define a binary operation* $\odot$ *(called Łukasiewicz conjection) on* $L = [0,1]$ *by*

$$x \odot y = \max\{0, x+y-1\},\ x \to y = \min\{1-x+y, 1\}.$$

Let $(X = \{a,b,c\}, e_X)$ *be a fuzzy poset as follows:*

$$e_X(a,a) = 1,\ e_X(a,b) = 0.6,\ \ e_X(a,b) = 0.5$$
$$e_X(b,a) = 0.8,\ e_X(b,b) = 1,\ \ e_X(b,c) = 0.7$$
$$e_X(c,a) = 0.9,\ e_X(c,b) = 0.6,\ \ e_X(c,c) = 1.$$

(1) *For* $A = (e_X)^a = (1, 0.8, 0.9)^t$*, we have* $a = \sqcup A = \max A$ *from*

$$\bigwedge(A(x) \to e_X(x,z)) = e_X(a,z).$$

(2) *For* $A = (e_X)^b = (0.6, 1, 0.6)^t$*, we have* $b = \sqcup A = \max A$ *from*

$$\bigwedge(A(x) \to e_X(x,z)) = e_X(b,z).$$

(3) *For* $A = (e_X)^c = (0.5, 0.7, 1)^t$*, we have* $c = \sqcup A$ *from*

$$\bigwedge_{x\in X}(A(x) \to e_X(x,y)) = e_X(\sqcup A, y) = e_X(c,y).$$

(4) *For* $A = (e_X)_a = (1, 0.6, 0.5)^t$*, we have* $a = \sqcap A = \min A$ *from*

$$\bigwedge(A(x) \to e_X(z,x)) = e_X(z,a).$$

(5) *For* $A = (e_X)_b = (0.8, 1, 0.7)^t$*, we have* $b = \sqcap A = \min A$ *from*

$$\bigwedge(A(x) \to e_X(z,x)) = e_X(z,b).$$

(6) *For* $A = (e_X)_c = (0.9, 0.6, 1)^t$*, we have* $c = \sqcap A = \min A$ *from*

$$\bigwedge_{x\in X}(A(x) \to e_X(z,x)) = e_X(z,c).$$

(7) *For* $A=(0.5,0.8,0.6)^t$, $\sqcap A$ *and* $\sqcup A$ *do not exist from:*

$$0.9=\bigwedge(A(x)\to e_X(x,c))\neq e_X(y,c),\forall y\in\{a,b,c\}$$

$$0.8=\bigwedge(A(x)\to e_X(a,x))\neq e_X(a,y),\forall y\in\{a,b,c\}.$$

Hence (X,e_X) *is not fuzzy complete.*

THEOREM 3.7. *Let* (X,e_X) *and* (Y,e_Y) *be complete.*

(1) (e_X,f,g,e_Y) *is a Galois connection iff* f *is an order reversing map and* $g(y)=\max f^{\leftarrow}((e_Y)_y)$ *iff* g *is an order reversing map and* $f(x)=\max g^{\leftarrow}((e_X)_x)$.
(2) (e_X,f,g,e_Y) *is a dual Galois connection iff* f *is an order reversing map and* $g(y)=\min f^{\leftarrow}((e_Y)^y)$ *iff* g *is an order reversing map and* $f(x)=\min g^{\leftarrow}((e_X)^x)$.
(3) (e_X,f,g,e_Y) *is a residuated connection iff* f *is an order preserving map and* $g(y)=\max f^{\leftarrow}((e_Y)^y)$ *iff* g *is an order preserving map and* $f(x)=\min g^{\leftarrow}((e_X)_x)$.
(4) (e_X,f,g,e_Y) *is a dual residuated connection iff* f *is an order preserving map and* $g(y)=\min f^{\leftarrow}((e_Y)_y)$ *iff* g *is an order preserving map and* $f(x)=\max g^{\leftarrow}((e_X)^x)$.

Proof. (1) We only show that (e_X,f,g,e_Y) is a Galois connection iff f is an order reversing map and $g(y)=\max f^{\leftarrow}((e_Y)_y)$ because other case is similarly proved.

($\Rightarrow$) Since $e_X(x,g(f(x)))=e_Y(f(x),f(x))=1$, we have

$$\begin{aligned} e_Y(f(x),f(y)) &= e_X(y,g(f(x))) \\ &\geq e_X(y,x)\odot e_X(x,g(f(x)))=e_X(y,x). \end{aligned}$$

Hence f is an order reversing map. Moreover, $g(y)=\max f^{\leftarrow}((e_Y)_y)$ because

$$\begin{aligned} f^{\leftarrow}((e_Y)_y)(g(y)) &= (e_Y)_y(f(g(y)) \\ &= e_Y(y,f(g(y))) \\ &= e_X(g(y),g(y))=1, \end{aligned}$$

$$f^{\leftarrow}((e_Y)_y)(x)=(e_Y)_y(f(x))=e_Y(y,f(x))=e_X(x,g(y)).$$

($\Leftarrow$) Since $g(y)=\max f^{\leftarrow}((e_Y)_y)$, we have

$$e_Y(y,f(x))=(e_Y)_y(f(x))=f^{\leftarrow}((e_Y)_y)(x)\leq e_X(x,g(y)).$$

Since $g(y) = \max f^{\leftarrow}((e_Y)_y)$,

$$f^{\leftarrow}((e_Y)_y)(g(y)) = (e_Y)_y(f(g(y)) = e_Y(y, f(g(y))) = 1.$$

$$e_X(x, g(y)) \leq e_Y(f(g(y)), f(x)) \odot e_Y(y, f(g(y))) \leq e_Y(y, f(x)).$$

Thus $e_X(x, g(y)) = e_Y(y, f(x))$.

(2) We only show that (e_X, f, g, e_Y) is a dual Galois connection iff f is an order reversing map and $g(y) = \min f^{\leftarrow}((e_Y)^y)$ because other case is similarly proved.

($\Rightarrow$) Since $e_X(g(f(x)), x) = e_Y(f(x), f(x)) = 1$, we have

$$\begin{aligned} e_Y(f(x), f(y)) &= e_X(g(f(y)), x) \\ &\geq e_X(g(f(y)), y) \odot e_X(y, x) = e_X(y, x). \end{aligned}$$

Hence f is an order reversing map. Moreover, $g(y) = \min f^{\leftarrow}((e_Y)^y)$ because

$$\begin{aligned} f^{\leftarrow}((e_Y)^y)(g(y)) &= (e_Y)^y(f(g(y)) \\ &= e_Y(f(g(y)), y) = e_X(g(y), g(y)) = 1 \\ f^{\leftarrow}((e_Y)^y)(x) &= (e_Y)^y(f(x)) = e_Y(f(x), y) = e_X(g(y), x). \end{aligned}$$

($\Leftarrow$) Since $g(y) = \min f^{\leftarrow}((e_Y)^y)$, we have

$$e_Y(f(x), y) = (e_Y)^y(f(x)) = f^{\leftarrow}((e_Y)^y)(x) \leq e_X(g(y), x).$$

Since $g(y) = \min f^{\leftarrow}((e_Y)^y)$,

$$f^{\leftarrow}((e_Y)^y)(g(y)) = (e_Y)^y(f(g(y)) = e_Y(f(g(y)), y) = 1.$$

$$e_X(g(y), x) \leq e_Y(f(x), f(g(y))) \odot e_Y(f(g(y)), y) \leq e_Y(f(x), y).$$

Thus, $e_Y(f(x), y) = e_X(g(y), x)$.

(3) It follows from Theorem 3.4 in [11].

(4) First, we show that (e_X, f, g, e_Y) is a dual residuated connection iff f is an order preserving map and $g(y) = \min f^{\leftarrow}((e_Y)_y)$.

($\Rightarrow$) Since $e_X(g(f(x)), x) = e_Y(f(x), f(x)) = 1$, we have

$$\begin{aligned} e_Y(f(x), f(y)) &= e_X(g(f(x)), y) \\ &\geq e_X(x, y) \odot e_X(g(f(x)), x) = e_X(x, y). \end{aligned}$$

We obtain $g(y) = \min f^{\leftarrow}((e_Y)_y)$ because

$$\begin{aligned} f^{\leftarrow}((e_Y)_y)(g(y)) &= (e_Y)_y(f(g(y))) = e_Y(y, f(g(y))) \\ &= e_X(g(y), g(y)) = 1, \\ f^{\leftarrow}((e_Y)_y)(x) &= (e_Y)_y(f(x)) = e_Y(y, f(x)) \\ &= e_X(g(y), x). \end{aligned}$$

($\Leftarrow$) Since $g(y) = \min f^{\leftarrow}((e_Y)_y)$, we have

$$e_Y(y, f(x)) = (e_Y)_y(f(x)) = f^{\leftarrow}((e_Y)_y)(x) \le e_Y(g(y), x).$$

Since $g(y) = \min f^{\leftarrow}((e_Y)_y)$,

$$f^{\leftarrow}((e_Y)_y)(g(y)) = (e_Y)_y(f(g(y)) = e_Y(y, f(g(y))) = 1.$$

$$e_X(g(y), x) \le e_Y(f(g(y)), f(x)) \odot e_Y(y, f(g(y))) \le e_Y(y, f(x)).$$

Hence $e_X(g(y), x) = e_Y(y, f(x))$.

Second, we show that (e_X, f, g, e_Y) is a dual residuated connection iff g is an order preserving map and $f(x) = \max g^{\leftarrow}((e_X)^x)$.

($\Rightarrow$) Since $e_Y(y, f(g(y))) = e_X(g(y), g(y)) = 1$, we have

$$e_X(g(x), g(y)) = e_Y(x, f(g(y))) \ge e_Y(x, y) \odot e_Y(y, g(f(y))) = e_Y(x, y).$$

We obtain $f(x) = \max g^{\leftarrow}((e_X)^x)$ because

$$\begin{aligned} g^{\leftarrow}((e_X)^x)(f(x)) &= (e_X)^x(g(f(x))) = e_Y(g(f(x)), x) \\ &= e_Y(f(x), f(x)) = 1, \\ g^{\leftarrow}((e_X)^x)(y) &= (e_X)^x(g(y)) = e_X(g(y), x) = e_Y(y, f(x)). \end{aligned}$$

($\Leftarrow$) Since $f(x) = \max g^{\leftarrow}((e_X)^x)$, we have

$$e_X(g(y), x) = (e_X)^x(g(y)) = g^{\leftarrow}((e_X)^x)(y) \le e_Y(y, f(x)).$$

Since $f(x) = \max g^{\leftarrow}((e_X)^x)$,

$$g^{\leftarrow}((e_X)^x)(f(x)) = (e_X)^x(g(f(x)) = e_X(g(f(x)), x) = 1.$$

$$e_Y(y, f(x)) \le e_X(g(y), g(f(x))) \odot e_X(g(f(x)), x) \le e_X(g(y), x).$$

Hence $e_X(g(y), x) = e_Y(y, f(x))$. □

THEOREM 3.8. *Let (X, e_X) and (Y, e_Y) be complete.*

(1) *(e_X, f, g, e_Y) is a Galois connection iff $f(\sqcup A) = \sqcap f^{\rightarrow}(A)$ for all $A \in L^X$ iff $g(\sqcup B) = \sqcap g^{\rightarrow}(B)$ for all $B \in L^Y$.*
(2) *(e_X, f, g, e_Y) is a dual Galois connection iff $f(\sqcap A) = \sqcup f^{\rightarrow}(A)$ for all $A \in L^X$ iff $g(\sqcap B) = \sqcup g^{\rightarrow}(B)$ for all $B \in L^Y$.*
(3) *(e_X, f, g, e_Y) is a residuated connection iff $f(\sqcup A) = \sqcup f^{\rightarrow}(A)$ for all $A \in L^X$ iff $g(\sqcap B) = \sqcap g^{\rightarrow}(B)$ for all $B \in L^Y$.*
(4) *(e_X, f, g, e_Y) is a dual residuated connection iff, for all $A \in L^X$, $f(\sqcap A) = \sqcap f^{\rightarrow}(A)$ iff $g(\sqcup B) = \sqcup g^{\rightarrow}(B)$ for all $B \in L^Y$.*

Proof. (1) First, we will show that (e_X, f, g, e_Y) is a Galois connection iff $f(\sqcup A) = \sqcap f^{\rightarrow}(A)$ for all $A \in L^X$.

($\Rightarrow$) Put $y_0 = \sqcap f^{\rightarrow}(A)$. Then

$$\begin{aligned} e_Y(y, y_0) &= \bigwedge_{z\in Y} (f^{\rightarrow}(A)(z) \rightarrow e_Y(y,z)) \\ &= \bigwedge_{z\in Y} ((\bigvee_{f(x)=z} A(x) \rightarrow e_Y(y, f(x))) \\ &= \bigwedge_{z\in Y} \bigwedge_{f(x)=z} (A(x) \rightarrow e_Y(y, f(x))) \\ &= \bigwedge_{x\in X} (A(x) \rightarrow e_X(x, g(y))) \\ &= e_X(\sqcup A, g(y)) = e_Y(y, f(\sqcup A)). \end{aligned}$$

Hence $y_0 = f(\sqcup A) = \sqcap f^{\rightarrow}(A)$.

($\Leftarrow$) Put $A = (e_X)^y$. Since $\sqcup(e_X)^y = y$ from Remark 3.4(1), we have $f(y) = f(\sqcup(e_X)^y) = \sqcap f^{\rightarrow}((e_X)^y)$. By the definition of $\sqcap f^{\rightarrow}((e_X)^y)$,

$$e_Y(f(y), f(x)) \geq f^{\rightarrow}((e_X)^y)(f(x)) = \bigvee_{f(z)=f(x)} (e_X)^y(z) \geq e_X(x, y).$$

Thus, f is order-reversing.

Define $g : Y \rightarrow X$ as $g(y) = \sqcup f^{\leftarrow}((e_Y)_y)$. By the definition of $g(y_1) = \sqcup f^{\leftarrow}((e_Y)_{y_1})$, we have

$$\begin{aligned} e_X(g(y_1), g(y_2)) &= \bigwedge_{z\in Y} (f^{\leftarrow}((e_Y)_{y_1})(z) \rightarrow e_X(z, g(y_2))) \\ &\geq \bigwedge_{z\in Y} (f^{\leftarrow}((e_Y)_{y_1})(z) \rightarrow f^{\leftarrow}((e_Y)_{y_2})(z)) \\ &= \bigwedge_{z\in Y} (e_Y(y_1, f(z)) \rightarrow e_Y(y_2, f(z))) \\ &\geq e_Y(y_2, y_1). \end{aligned}$$

Thus, g is order-reversing. Since

$$f(g(y)) = f(\sqcup f^{\leftarrow}((e_Y)_y)) = \sqcap f^{\rightarrow}(f^{\leftarrow}((e_Y)_y))$$

$$\begin{aligned} e_Y(y, f(g(y))) &= \bigwedge_{z\in Y}(f^{\rightarrow}(f^{\leftarrow}((e_Y)_y))(z) \rightarrow e_Y(y,z)) \\ &= \bigwedge_{z\in Y}(\bigvee_{f(x)=z}(f^{\leftarrow}((e_Y)_y))(z) \rightarrow e_Y(y,z)) \\ &= \bigwedge_{z\in Y}(\bigvee_{f(x)=z} e_Y(y,f(x)) \rightarrow e_Y(y,z)) \\ &= \bigwedge_{x\in X}(e_Y(y,f(x)) \rightarrow e_Y(y,f(x))) = 1. \end{aligned}$$

Since $g(f(x)) = \sqcup f^{\leftarrow}((e_Y)_{f(x)})$,

$$\begin{aligned} e_X(x, g(f(x)) &\geq f^{\leftarrow}((e_Y)_{f(x)})(x) = (e_Y)_{f(x)}(f(x)) = 1, \\ e_X(x, g(y)) &\leq e_Y(f(g(y)), f(x)) = e_Y(f(g(y)), f(x)) \\ &\odot\ e_Y(y, f(g(y))) \leq e_Y(y, f(x)), \\ e_Y(y, f(x)) &\leq e_X(g(f(x)), g(y)) = e_X(g(f(x)), g(y)) \\ &\odot\ e_Y(x, g(f(x))) \leq e_X(x, g(y)). \end{aligned}$$

Thus $e_X(x, g(y)) = e_Y(y, f(x))$.

Second, (e_X, f, g, e_Y) is a Galois connection iff $g(\sqcup B) = \sqcap g^{\rightarrow}(B)$ for all $B \in L^Y$.

($\Rightarrow$) Put $x_0 = \sqcap g^{\rightarrow}(B)$. Then

$$\begin{aligned} e_X(x, x_0) &= \bigwedge_{z\in X}(g^{\rightarrow}(B)(z) \rightarrow e_X(x,z)) \\ &= \bigwedge_{z\in X}(\bigvee_{g(y)=z} B(y) \rightarrow e_X(x,g(y))) \\ &= \bigwedge_{z\in X}\bigwedge_{g(y)=z}(B(y) \rightarrow e_Y(y,f(x))) \\ &= \bigwedge_{y\in Y}(B(y) \rightarrow e_Y(y,f(x))) \\ &= e_Y(\sqcup_l B, f(x)) = e_X(x, g(\sqcup_l B)). \end{aligned}$$

($\Leftarrow$) Put $B = (e_Y)^y$. Then $g(y) = g(\sqcup(e_Y)^y) = \sqcap g^{\rightarrow}((e_Y)^y)$. By the definition of $\sqcap g^{\rightarrow}((e_Y)^y)$,

$$e_X(g(y), g(w)) \geq g^{\rightarrow}((e_Y)^y)(g(w)) = \bigvee_{g(z)=g(w)}(e_Y)^y(z) \geq e_Y(w, y).$$

Thus, g is order-reversing.

Define $f : X \to Y$ as $f(x) = \sqcup g^{\leftarrow}((e_X)_x)$. By the definition of $f(x_1) = \sqcup g^{\leftarrow}((e_X)_{x_1})$, we have

$$\begin{aligned} e_Y(f(x_1), f(x_2)) &= \bigwedge_{z\in Y}(g^{\leftarrow}((e_X)_{x_1})(z) \to e_Y(z, f(x_2))) \\ &\geq \bigwedge_{z\in Y}(g^{\leftarrow}((e_X)_{x_1})(z) \to g^{\leftarrow}((e_X)_{x_2})(z)) \\ &= \bigwedge_{z\in Y}(e_X(x_1, g(z)) \to e_X(x_2, g(z))) \\ &\geq e_X(x_2, x_1). \end{aligned}$$

Thus, f is order-reversing. Since

$$g(f(x)) = g(\sqcup g^{\leftarrow}((e_X)_x)) = \sqcap g^{\rightarrow}(g^{\leftarrow}((e_X)_x))$$

$$\begin{aligned} e_X(x, g(f(x))) &= \bigwedge_{z\in X}(g^{\rightarrow}(g^{\leftarrow}((e_X)_x))(z) \to e_X(x, z)) \\ &= \bigwedge_{z\in X}(\bigvee_{g(w)=z}(g^{\leftarrow}((e_X)_x)(w) \to e_X(x, z)) \\ &= \bigwedge_{z\in X}(\bigvee_{g(w)=z} e_X(x, g(w)) \to e_X(x, z)) \\ &= \bigwedge_{w\in Y}(e_X(x, g(w))) \to e_X(x, g(w))) = 1. \end{aligned}$$

Since $f(g(y)) = \sqcup g^{\leftarrow}((e_X)_{g(y)})$,

$$\begin{aligned} e_Y(y, f(g(y)) &\geq g^{\leftarrow}((e_X)_{g(y)})(y) = (e_X)_{g(y)}(g(y)) = 1, \\ e_X(x, g(y)) &\leq e_Y(f(g(y)), f(x)) = e_Y(f(g(y)), f(x)) \\ &\odot\ e_Y(y, f(g(y))) \leq e_Y(y, f(x)) \\ e_Y(y, f(x)) &\leq e_X(g(f(x)), g(y)) = e_X(g(f(x)), g(y)) \\ &\odot\ e_Y(x, g(f(x))) \leq e_X(x, g(y)). \end{aligned}$$

Thus $e_X(x, g(y)) = e_Y(y, f(x))$.

(2) and (3) are similarly proved in (1) and Theorem 3.5 in [11], respectively.

(4) First, (e_X, f, g, e_Y) is a dual residuated connection iff $f(\sqcap A) = \sqcap f^{\rightarrow}(A)$ for all $A \in L^X$.

($\Rightarrow$) Put $y_1 = \sqcap f^{\rightarrow}(A)$. Then

$$\begin{aligned} e_Y(y, y_1) &= \bigwedge_{z\in Y}(f^{\rightarrow}(A)(z) \rightarrow e_Y(y,z)) \\ &= \bigwedge_{x\in X}(\bigvee_{f(x)=z} A(x) \rightarrow e_Y(y, f(x))) \\ &= \bigwedge_{x\in X}\bigwedge_{f(x)=z}(A(x) \rightarrow e_X(g(y), x)) \\ &= \bigwedge_{x\in X}(A(x) \rightarrow e_X(g(y), x)) \\ &= e_X(g(y), \sqcap A) = e_Y(y, f(\sqcap A)). \end{aligned}$$

Hence $y_1 = f(\sqcap A) = \sqcap f^{\rightarrow}(A)$.

($\Leftarrow$) Put $A = (e_X)_x$. Since $\sqcap(e_X)_x = x$, then $f(x) = f(\sqcap(e_X)_x) = \sqcap f^{\rightarrow}((e_X)_x)$. By the definition of $\sqcap f^{\rightarrow}((e_X)_x)$,

$$e_Y(f(x), f(z)) \geq f^{\rightarrow}((e_X)_x)(f(z)) = \bigvee_{f(d)=f(z)} (e_X)_x(d) \geq e_X(x, z).$$

Thus, f is an order preserving map.

Define $g : Y \rightarrow X$ as $g(y) = \sqcap f^{\leftarrow}((e_Y)_y)$. By the definition of $g(y_2) = \sqcap f^{\leftarrow}((e_Y)_{y_2})$, we have

$$\begin{aligned} e_X(g(y_1), g(y_2)) &= \bigwedge_{z\in X}(f^{\leftarrow}((e_Y)_{y_2})(z) \rightarrow e_X(g(y_1), z)) \\ &\geq \bigwedge_{z\in X}(f^{\leftarrow}((e_Y)_{y_2})(z) \rightarrow f^{\leftarrow}((e_Y)_{y_1})(z)) \\ &= \bigwedge_{z\in X}(e_Y(y_2, f(z)) \rightarrow e_Y(y_1, f(z))) \\ &\geq e_Y(y_1, y_2). \end{aligned}$$

Thus, g is an order preserving map. Since

$$f(g(y)) = f(\sqcap f^{\leftarrow}((e_Y)_y)) = \sqcap f^{\rightarrow}(f^{\leftarrow}((e_Y)_y))$$

$$\begin{array}{rcl} e_Y(y, f(g(y))) & = & \bigwedge_{z\in X}(f^{\rightarrow}(f^{\leftarrow}((e_Y)_y))(z) \rightarrow e_Y(y,z)) \\ & = & \bigwedge_{z\in X}(\bigvee_{f(x)=z} (f^{\leftarrow}((e_Y)_y)(x) \rightarrow e_Y(y,z)) \\ & = & \bigwedge_{z\in X}(\bigvee_{f(x)=z} e_Y(y,f(x)) \rightarrow e_Y(y,f(x))) = 1. \end{array}$$

Since $g(f(x)) = \sqcap f^{\leftarrow}((e_Y)_{f(x)})$,

$$e_X(g(f(x)),x) \geq f^{\leftarrow}((e_Y)_{f(x)})(x) = (e_X)_{f(x)}(f(x)) = 1.$$

$$\begin{array}{rcl} e_X(g(y),x) & \leq & e_Y(f(g(y)),f(x)) = e_Y(f(g(y)),f(x)) \\ & \odot & e_Y(y,f(g(y))) \leq e_Y(y,f(x)) \\ e_Y(y,f(x)) & \leq & e_X(g(y),g(f(x))) = e_X(g(y),g(f(x))) \\ & \odot & e_Y(g(f(x)),x) \leq e_X(g(y),x). \end{array}$$

Thus $e_X(g(y),x) = e_Y(y,f(x))$.

Second, (e_X, f, g, e_Y) is a dual residuated connection iff $g(\sqcup B) = \sqcup g^{\rightarrow}(B)$ for all $B \in L^Y$.

($\Rightarrow$) Put $x_0 = \sqcup g^{\rightarrow}(B)$. Then $g(\sqcup B) = \sqcup g^{\rightarrow}(B)$ from:

$$\begin{array}{rcl} e_X(x_0,x) & = & \bigwedge_{z\in X}(g^{\rightarrow}(B)(z) \rightarrow e_X(z,x)) \\ & = & \bigwedge_{z\in X}((\bigvee_{g(y)=z} B(y) \rightarrow e_X(g(y),x)) \\ & = & \bigwedge_{z\in X} \bigwedge_{g(y)=z} (B(y) \rightarrow e_Y(y,f(x))) \\ & = & \bigwedge_{y\in Y}(B(y) \rightarrow e_Y(y,f(x))) \\ & = & e_Y(\sqcup B, f(x)) = e_X(g(\sqcup B),x). \end{array}$$

Thus, $x_0 = g(\sqcup B) = \sqcup g^{\rightarrow}(B)$.

($\Leftarrow$) Put $B = (e_Y)^y$. Since $\sqcup(e_Y)^y = y$, we have $g(y) = g(\sqcup(e_Y)^y) = \sqcup g^{\to}((e_Y)^y)$. By the definition of $\sqcup g^{\to}((e_Y)^y)$,

$$\begin{aligned}
e_X(g(y), g(z)) &= \bigwedge_{p\in X}(g^{\to}((e_Y)^y)(p) \to e_X(p, g(z))) \\
&= \bigwedge_{p\in X}(\bigvee_{g(w)=p} (e_Y)^y(w) \to e_X(g(w), g(z))) \\
&= \bigwedge_{p\in X}\bigwedge_{g(w)=p} (e_Y(w, y) \to e_X(g(w), g(z))) \\
&= \bigwedge_{w\in Y}(e_Y(w, y) \to e_X(g(w), g(z))) \\
e_Y(w, y) &\leq \bigwedge_{z\in Y}(e_X(g(y), g(z)) \to e_X(g(w), g(z))) \\
&= e_X(g(w), g(y)).
\end{aligned}$$

Thus, g is order-preserving.

Define $f : X \to Y$ as $f(x) = \sqcup g^{\leftarrow}((e_X)^x)$. Since $e_Y(f(x), w) \leq g^{\leftarrow}((e_X)^x)(z) \to e_Y(z, w)$,

$$g^{\leftarrow}((e_X)^x)(z) \leq \bigwedge_{w\in Y}(e_X(g(w), x) \to e_Y(z, w)) = e_Y(z, g(y)).$$

Thus, g is order-preserving, by the definition of $f(x_1) = \sqcup g^{\leftarrow}((e_X)^{x_1})$,

$$\begin{aligned}
e_Y(f(x_1), f(x_2)) &= \bigwedge_{z\in X}(g^{\leftarrow}((e_X)^{x_1})(z) \to e_Y(z, f(x_2))) \\
&\geq \bigwedge_{z\in X}(g^{\leftarrow}((e_X)^{x_1})(z) \to g^{\leftarrow}((e_X)^{x_2})(z)) \\
&= \bigwedge_{z\in X}(e_X(g(z), x_1) \to e_X(g(z), x_2)) \\
&\geq e_X(x_1, x_2).
\end{aligned}$$

Since $g(f(x)) = g(\sqcup g^{\leftarrow}((e_X)^x)) = \sqcup g^{\rightarrow}(g^{\leftarrow}((e_X)^x))$, we have

$$\begin{aligned}
e_X(g(f(x)), x) &= \bigwedge_{z\in X} (g^{\rightarrow}(g^{\leftarrow}((e_X)^x))(z) \rightarrow e_X(z, x)) \\
&= \bigwedge_{z\in X} (\bigvee_{g(y)=z} (g^{\leftarrow}((e_X)^x))(z) \rightarrow e_X(z, x)) \\
&= \bigwedge_{z\in X} (\bigvee_{g(y)=z} e_X(g(y), x) \rightarrow e_Y(g(y), x)) = 1.
\end{aligned}$$

Since $f(g(y)) = \sqcup g^{\leftarrow}((e_X)^{g(y)})$,

$$\begin{aligned}
e_Y(y, f(g(y)) &\geq g^{\leftarrow}((e_X)^{g(y)})(y) = (e_X)^{g(y)}(g(y)) = 1, \\
e_X(g(y), x) &\leq e_Y(f(g(y)), f(x)) = e_Y(f(g(y)), f(x)) \\
&\odot\; e_Y(y, f(g(y))) \leq e_Y(y, f(x)) \\
e_Y(y, f(x)) &\leq e_X(g(y), g(f(x))) = e_X(g(y), g(f(x))) \\
&\odot\; e_Y(g(f(x)), x) \leq e_X(g(y), x).
\end{aligned}$$

Thus $e_X(g(y), x) = e_Y(y, f(x))$.

□

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