Korean J. Math. **34** (2026), No. 1, pp. 1–11
https://dx.doi.org/10.11568/kjm.2026.34.1.1

# COEFFICIENT PROBLEMS ON Q-FRACTIONAL INTEGRAL OPERATOR DEFINED BY MODIFIED Q-OPOOLA DIFFERENTIAL OPERATOR

RISIKAT.A.BELLO, MASLINA DARUS, AND KHALID ALSHAMMARI ⋆

Abstract. In this paper, we study of a new $q$-fractional differential operator originated from the Srivastrava-Owa operator of fractional integration with modified $q$-Opoola derivative operator. The Fekete-Szego $H_2(1)$ functional and Second Hankel determinant $H_2(2)$ for normalized analytic function belonging to the family of $q$-starlike and $q$-convex functions in the open unit disk are investigated.

## 1. Introduction and definitions

We denote $U = \{z : z \in \mathbb{C} : |z| < 1\}$ the open unit disk in the complex plane and $A$ refers to the class of function $f(z)$ given by

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k, \qquad (z \in U) \tag{1.1}$$

which satisfies the usual normalization condition given by

$$f(0) = f'(0) - 1 = 0.$$

The study of function theory began in 1851. In 1916, this field became a good area for new research due to Bieberbach's coefficient conjecture [21]. The conjecture $|a_n|$ was proved in 1985 by De-Branges [22]. Between 1916 and 1985, many scholars tried to prove or disprove this conjecture. As a result, they identified several subfamilies of a class of univalent functions associated with different image domains. The families of starlike and convex functions are the most basic, most studied, and these subfamilies have various beautiful geometric representations.

Given that $f(z)$ and $g(z)$ are two analytic functions in $U$, then we say that the function $f(z)$ is a subordinate to the function $g(z)$, and written as

$$f(z) \prec g(z), \tag{1.2}$$

if there exist a Schwarz function $w(z)$ with the following conditions

$$w(0) = 0, |w(z)| < 1, \quad (z \in U),$$

Received June 21, 2024. Revised September 10, 2024. Accepted September 10, 2024.
2010 Mathematics Subject Classification: 30C45, 30C50.
Key words and phrases: Analytic function, univalent function, subordination, $q$-Opoola differential operator, Fekete-Szego functional, Hankel determinant.
∗ Corresponding author.

such that:

$$f(z) = g(w(z)). \tag{1.3}$$

We denote by $P$, the class of analytic functions $p(z)$ with the series form:

$$p(z) = 1 + c_1 z + c_2 z^2 + c_3 z^3 + \cdots , \tag{1.4}$$

and

$$\Re(p(z)) > 0, \quad z \in U.$$

Recently, Srivastava [18] and many others have studied the relationship between fractional calculus and q-calculus (quantum) in the Geometric Function Theory of Complex Analysis.
Srivastava et al. [8] showed in their article the importance of studying the concept by introducing a new q-fractional integral operator defined by the successful application of the Srivastava-Owa Operator [23] and the q-Ruscheweyh derivative of Aldweby and Darus [24].
Motivated by the work done in [8], we shall define an Operator associated with both Srivasatava and Owa [23] and Alatawi and Darus [1] of modified q-Opoola Differential Operator. We begin by stating the following definitions:

Definition 1.1. (Srivastava and Owa [23] The fractional integral of order $\alpha$ is defined for a function $f(z)$,

$$I_z^\alpha f(z) = \frac{1}{\Gamma(\alpha)} \int_0^z (z-\zeta)^{\alpha-1} f(\zeta) d\zeta. \quad (\alpha > 0) \tag{1.5}$$

where the function $f(z)$ is analytic in a simple-connected domain in the complex z-plane which has the origin and multiplicity $(z-\zeta)^{\alpha-1}$ is removed using $log(z-\zeta)$ to be real when $z - \zeta > 0$.

Definition 1.2. (See Alatawi and Darus [1]) For a function $f(z) \in A$, the linear operator $D_q^n(\mu, v, F, t) : A \mapsto A$ is given by

$$D_q^0(\mu, v, F, t) f(z) = f(z)$$

$$D_q^1(\mu, v, F, t) f(z) = zt D_q f(z) - zt(v-\mu) + (F + (v - \mu - F)t) f(z) = d_{q,t} f(z)$$

$$D_q^2(\mu, v, F, t) f(z) = d_{q,t}(D_q^1(\mu, v, F, t) f(z))$$

so that

$$D_q^n(\mu, v, F, t) f(z) = d_{q,t}(D_q^{n-1}(\mu, v, F, t) f(z))$$

which correspond to

$$D_q^n(\mu, v, F, t) f(z) = z + \sum_{k=2}^{\infty} (F + ([k]_q + v - \mu - F)t)^n a_k z^k, \quad z \in U. \tag{1.6}$$

If $F = 1$, (1.6) gives the class investigated by Lasode and Opoola [15].
Next, from Definition 1.1 and 1.2, we have the following:

Definition 1.3. The Srivastava-Owa operator $I_z^\alpha$ in Definition 1.1 and the modified $q$-Opoola differential operator $D_q^n(\mu, v, F, t)$ in Definition 1.2 applied together gives a presumably new $q$-fractional operator defined for a function $f \in A$ given by equation (1.1):

$$T_q^{\alpha,\mu,v,F,t} f(z) = I_z^\alpha D_q^n(\mu, v, F, t) = I_z^\alpha D_q^n(\mu, v, F, t) \left( z + \sum_{k=2}^{\infty} a_k z^k \right)$$

$$= z + \sum [F + ([k]_q + v - \mu - F)t]^n \frac{\Gamma(k+1)\Gamma(\alpha+2)}{\Gamma(k+1+\alpha)} a_k z^k.$$

Following the same approach of Srivastava *et.al*([see [8]) we derive the second Hankel determinant $H_2^2(f)$ for functions belonging to the classes $L^*_{q,\alpha,\mu,v,f,t}$ and $K^*_{q,\alpha,\mu,v,f,t}$

$$L^*_{q,\alpha,\mu,v,f,t} := \{f : f \in A \text{ and } \Re\left\{\frac{D_q T_q^{\alpha,\mu,v,f,t} f(z)}{T_q^{\alpha,\mu,v,f,t} f(z)}\right\} \prec 1 + sin\omega(z), (z \in U)\}$$

and

$$K^*_{q,\alpha,\mu,v,f,t} := \{f : f \in A \text{ and } \Re\left\{\frac{D_q\left(zD_q T_q^{\alpha,\mu,v,f,t} f(z)\right)}{D_q T_q^{\alpha,\mu,v,f,t} f(z)}\right\} \prec 1 + sin\omega(z), (z \in U)\}$$

respectively.
We study the same problems related to Fekete-Szego and Hankel Determinant as in [8]. The $q^{th}$ Hankel determinant represented by $H_q^n(f)$ whose elements are the coefficient function $f \in A$ given by (1.1) was defined by Pommerenke [4] as

$$H_q^n(f) = \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+q-1} \\ a_{n+1} & a_{n+2} & \cdots & a_{n+n} \\ \vdots & \vdots\ddots & \vdots & \vdots \\ a_{n+q-1} & a_{n+q} & \cdots & a_{n+2q-2} \end{vmatrix}.$$

This determinant has been considered by several authors for various classes and subclasses i.e for $H_2^1(f), H_2^2(f)$ and others, for instance see (Bello and Opoola [11], Riaz et.al [27])

$$H_2^1(f) = \begin{vmatrix} 1 & a_2 \\ a_2 & a_3 \end{vmatrix} = (a_3 - a_2^2), \quad a_1 = 1, and \quad H_2^2(f) = \begin{vmatrix} a_2 & a_3 \\ a_3 & a_4 \end{vmatrix} = (a_2 a_4 - a_3^2).$$

The study of upper bounds for the Fekete-Szego functional and the Hankel determinant for various subclasses of normalized analytic functions in the geometric function theory of complex analysis is still an interesting problem (see [3, 4, 8, 11, 16, 26]).

## 2. Preliminary Lemmas

In order to prove our desired results, we shall require the following lemmas:

Lemma 2.1. ( [10] ) If $p(z) \in P$ as in equation (1.4), then there exist some $|x| \leq 1$ such that

$$2c_2 = c_1^2 + \epsilon(4 - c_1^2),$$
$$4c_3 = c_1^3 + 2c_1\epsilon(4 - c_1^2) - c_1\epsilon^2(4 - c_1^2) + 2(4 - c_1^2)(1 - |\epsilon|^2)z.$$

Lemma 2.2. ( [10]) Let $p(z) \in P$, then

$$\left|c_2 - \frac{c_1^2}{2}\right| \leq 2 - \frac{|c_1^2|}{2}$$
$$|c_{n+k} - \mu c_n c_k| < 2, \qquad 0 < \mu < 1$$
$$|c_{n+2k} - \mu c_n c_k^2| \leq 2(1 + 2\mu).$$

Lemma 2.3. ( [11]) Let $p(z) \in P$, then for any $t \in \mathbb{R}$

$$|c_2 - tc_1^2| \leq \begin{cases} -t & if \quad t \leq -1, \\ 1 & if \quad -1 \leq t \leq 1, \\ t & if \quad t \geq 1. \end{cases}$$

Lemma 2.4. ( [6]) Let $p(z) \in P$. Then for any $\Re(p(z)) > 0$, $z \in U$, and $p(z) = 1 + c_1 z + c_2 z^2 + c_3 z^3 + \cdots$, then $|c_k| \leq 2, k = 1, 2, 3, \cdots$.

## 3. Main Results

In this section, we state and prove the main result of our investigation. The first results are the bounds for the first four initial coefficients for the new class $L^*_{q,\alpha,\mu,v,f,t}$.

Theorem 3.1. *Let $f(z)$ of the form (1.1) be in the class $L^*_{q,\alpha,\mu,v,f,t}$i.e $f(z) \in L^*_{q,\alpha,\mu,v,f,t}$, then*

$$|a_2| \leq \frac{1}{h_2([2]_q - 1)},$$

$$|a_3| \leq \frac{1}{4h_3([3]_q - 1)} \left[ \frac{[2]_q - 2}{[2]_q - 1} \right].$$

$$|a_4| \leq \frac{1}{2h_4([4]_q - 1)} \left[ \frac{6 + 11([2]_q - 1)([3]_q - 1)}{3([2]_q - 1)([3]_q - 1)} \right].$$

*where*

$$h_2 = [F + ([2]_q - v - \mu - F)t]^n \frac{\Gamma(3)\Gamma(2+\alpha)}{\Gamma(3+\alpha)},$$

$$h_3 = [F + ([3]_q - v - \mu - F)t]^n \frac{\Gamma(4)\Gamma(2+\alpha)}{\Gamma(4+\alpha)},$$

$$h_4 = [F + ([4]_q - v - \mu - F)t]^n \frac{\Gamma(5)\Gamma(2+\alpha)}{\Gamma(5+\alpha)}.$$

*Proof.* Since $f(z) \in L^*_{q,\alpha,\mu,v,f,t}$, we have

$$\frac{D_q T_q^{\alpha,\mu,v,f,t} f(z)}{T_q^{\alpha,\mu,v,f,t} f(z)} = 1 + \sin w(z). \tag{3.1}$$

Using the relationship between $p(z) \in P$ and Schwarz function, we can write:

$$p(z) = \frac{1 + w(z)}{1 - w(z)} = 1 + c_1 z + c_2 z^2 + c_3 z^3 + \cdots \tag{3.2}$$

such that,

$$w(z) = \frac{p(z) + 1}{p(z) - 1} = \frac{c_1 z + c_2 z^2 + c_3 z^3 + \cdots}{2 + c_1 z + c_2 z^2 + c_3 z^3 + \cdots}. \tag{3.3}$$

On the other hand,

$$1 + \sin w(z) = 1 + \frac{1}{2} c_1 z + \left( \frac{1}{2} c_2 - \frac{1}{4} c_1^2 \right) z^2 +$$

$$\left( -\frac{1}{2} c_1 c_2 + \frac{1}{2} c_3 + \frac{5}{48} c_1^3 \right) z^3 + \left( \frac{5}{16} c_2 c_1^2 - \frac{1}{32} c_1^4 - \frac{1}{4} c_2^2 - \frac{1}{2} c_1 c_3 + \frac{1}{2} c_4 \right) z^4 + \cdots . \tag{3.4}$$

We see that

$$D_qT_q^{\alpha,\mu,v,f,t}f(z) - T_q^{\alpha,\mu,v,f,t}f(z)$$

$$= ([2]_q - 1)h_2a_2z^2 + ([3]_q - 1)h_3a_3z^3 + ([4]_q - 1)h_4a_4z^4 + \cdots , \tag{3.5}$$

and

$$\sin w(z)T_q^{\alpha,\mu,v,f,t}f(z) = \frac{1}{2}c_1z + \left(\frac{1}{2}c_2 - \frac{1}{4}c_1^2\right)z^2 +$$

$$\left(-\frac{1}{2}c_1c_2 + \frac{1}{2}c_3 + \frac{5}{48}c_1^3\right)z^3 + \left(\frac{5}{16}c_2c_1^2 - \frac{1}{32}c_1^4 - \frac{1}{4}c_2^2 - \frac{1}{2}c_1c_3 + \frac{1}{2}c_4\right)z^4 + \cdots \tag{3.6}$$

Comparing the coefficient of $z, z^2, z^3, z^4$ in (3.5) and (3.6), we get

$$a_2 = \frac{c_1}{2h_2([2]_q - 1)}, \tag{3.7}$$

$$a_3 = \frac{1}{2h_3([3]_q - 1)}\left(c_2 - \frac{1}{2}c_1^2 + \frac{c_1^2}{2([2]_q - 1)}\right), \tag{3.8}$$

and

$$a_4 = \frac{1}{2h_4([4]_q - 1)}\left(c_3 - c_1c_3 + \frac{5}{24}c_1^3 + \frac{c_1c_2}{2([2]_q - 1)} - \frac{c_1^3}{4([2]_q - 1)} + \right.$$

$$\left. \frac{c_1c_2}{2([3]_q - 1)} - \frac{c_1^3}{4([3]_q - 1)} + \frac{c_1^3}{4([3]_q - 1)([3]_q - 1)}\right). \tag{3.9}$$

Applying Lemma 2.4 in (3.7) we obtain

$$|a_2| \leq \frac{1}{h_2([2]_q - 1)}. \tag{3.10}$$

Also by applying Lemma 2.3 into (3.8) for $a_3$, we get

$$|a_3| = \left|\frac{1}{2h_3([3]_q-1)}\left(c_2 - \frac{1}{2}c_1^2 + \frac{c_1^2}{2([2]_q-1)}\right)\right| = \frac{1}{2h_3([3]_q - 1)}\left|c_2 - \left(\frac{[2]_q-2}{2([2]_q-1)}\right)c_1^2\right|$$

and therefore

$$|a_3| \leq \frac{1}{4h_3([3]_q - 1)}\left[\frac{[2]_q - 2}{[2]_q - 1}\right]. \tag{3.11}$$

Also again by applying Lemma 2.2 into equation (3.9) for $a_4$, we get

$$|a_4| = \left|\frac{1}{2h_4([4]_q-1)}\left((c_3 - c_1c_2) + \frac{c_1}{2([2]_q-1)}(c_2 - \frac{c_1^2}{2}) + \frac{c_1}{2([3]_q-1)}(c_2 - \frac{c_1^2}{2}) + \frac{5c_1^3}{24} + \frac{1}{4([2]_q-1)([3]_q-1)}\right)\right|$$

$$|a_4| \leq \frac{1}{2h_4([4]_q - 1)}\left[\frac{6 + 11([2]_q - 1)([3]_q - 1)}{3([2]_q - 1)([3]_q - 1)}\right]. \tag{3.12}$$

□

Theorem 3.2. *Let $f(z)$ of the form (1.1) be in the class $L^*_{q,\alpha,\mu,v,f,t}$, then*

$$|a_3 - a_2^2| \le \frac{1}{[2]_{q-1}}\left(\frac{1}{h_3([2]_{q-1})} - \frac{1}{h_2^2([2]_{q-1})}\right)$$

*where*

$$h_2 = [F + ([2]_q + v - \mu - F)t]^n \frac{\Gamma(3)\Gamma(2+\alpha)}{\Gamma(3+\alpha)},$$

$$h_3 = [F + ([3]_q + v - \mu - F)t]^n \frac{\Gamma(4)\Gamma(2+\alpha)}{\Gamma(4+\alpha)}.$$

*Proof.* Since $f(z) \in L^*_{q,\alpha,\mu,v,f,t}$, from (3.7) and (3.8) we obtain

$$|a_3 - a_2^2| = \left|\frac{1}{2h_3([3]_q-1)}\left(c_2 - \frac{1}{2}c_1^2 + \frac{c_1^2}{2([2]_q-1)}\right) - \frac{c_1^2}{4h_2^2([2]_q-1)^2}\right|.$$

Without loss of generality, we assume $c_1 = c$ with $c \in [0,2]$, then applying Lemmas 2.1 and 2.4, we find that

$$|a_3 - a_2^2| = \left|\frac{1}{2h_3([3]_q-1)}\left(\frac{c^2+\epsilon(4-c^2)}{2} - \frac{c^2}{2} + \frac{c^2}{2([2]_q-1)} - \frac{c^2}{4h_2^2([2]_q-1)^2}\right)\right|.$$

By simplification of the above equation and using triangle inequality, we find for $|\epsilon| \le r = 1$ that.

$$|a_3 - a_2^2| \le \left|\frac{1}{2h_3([3]_q-1)}\left(\frac{(4-c^2)r}{2} + \frac{c^2}{2([2]_q-1)}\right) - \frac{c^2}{4h_2^2([2]_q-1)^2}\right)\right|.$$

Denoting the right hand side of the equation by $\gamma(c,r)$ and use the derivative test to minimize the function $\gamma(c,r)$ with $(c,r) \in [0,2] \times [0,1]$

$$\begin{aligned}|a_3 - a_2^2| &= \gamma(c,r)\\ &\le \max_{\{0\le r\le 1; 0\le c\le 2\}}\{\gamma(c,r)\} = \gamma(2,1).\end{aligned}$$

Then

$$|a_3 - a_2^2| \le \frac{1}{[2]_q - 1}\left(\frac{1}{h_3([3]_q-1)} - \frac{1}{h_2^2([2]_q-1)}\right)$$

which complete the proof of the Theorem 3.2. □

Theorem 3.3. *Let $f(z)$ be of the form (1.1) be in the class $L^*_{q,\alpha,\mu,v,f,t}$, then*

$$|a_2a_4 - a_3^2| \le \frac{6 - ([2]_q-1)([3]_q-1)}{6h_2h_4([2]_q-1)^2([3]_q-1)([4]_q-1)} - \frac{1}{h_2^2([2]_q-1)^2([3]_q-1)^2}$$

*where*

$$h_2 = [F + ([2]_q - v - \mu - F)t]^n \frac{\Gamma(3)\Gamma(2+\alpha)}{\Gamma(3+\alpha)},$$

$$h_3 = [F + ([3]_q - v - \mu - F)t]^n \frac{\Gamma(4)\Gamma(2+\alpha)}{\Gamma(4+\alpha)},$$

$$h_4 = [F + ([4]_q - v - \mu - F)t]^n \frac{\Gamma(5)\Gamma(2+\alpha)}{\Gamma(5+\alpha)}.$$

*Proof.* From equation (3.7) and (3.8), we have

$$\begin{aligned}|a_2a_4-a_3^2| &= \Bigg|\frac{c_1}{(2h_2([2]_q-1))(h_4([4]_q-1))}\Bigg(-\frac{c_1c_2}{2}+\frac{c_3}{2}+\frac{5c_1^3}{48}+\frac{c_1c_2}{4([2]_q-1)}\\ &\quad-\frac{c_1^3}{8([2]_q-1)}+\frac{c_1c_2}{4([3]_q-1)}-\frac{c_1^3}{8([3]_q-1)}+\frac{c_1^3}{8([2]_q-1)([3]_q-1)}\Bigg)\\ &\quad-\frac{1}{4h_3^2([3]_q-1)^2}\left(c_2-\frac{c_1^2}{2}+\frac{c_1^2}{2([2]_q-1)}\right)^2\Bigg|\\ &= \Bigg|\frac{1}{4h_2h_4([2]_q-1)([4]_q-1)}\Bigg(-c_1^2c_2+c_1c_3+\frac{5c_1^4}{24}+\frac{c_1^2c_2}{2([2]_q-1)}\\ &\quad-\frac{c_1^4}{4([2]_q-1)}+\frac{c_1^2c_2}{2([3]_q-1)}-\frac{c_1^4}{4([3]_q-1)}+\frac{c_1^4}{4([2]_q-1)([3]_q-1)}\Bigg)\\ &\quad-\frac{1}{4h_3^2([3]_q-1)^2}\left(c_2-\frac{c_1^2}{2}+\frac{c_1^2}{2([2]_q-1)}\right)^2\Bigg|.\end{aligned}$$

Without loss of generality, we can assume that $c_1=c$ with $c\in[0,2]$. Applying Lemma 2.1 and Lemma 2.3, we have

$$\begin{aligned}|a_2a_4-a_3^2| &= \Bigg|\frac{1}{4h_2h_4([2]_q-1)([4]_q-1)}\Bigg(\frac{-c^2(c^2+(4-c^2)\epsilon)}{2}\\ &\quad+\frac{c(c^3+2(4-c^2)\epsilon-c(4-c^2)\epsilon^2+2(4-c^2)(1-|\epsilon|^2)z)}{4}\\ &\quad+\frac{c^2(c^2+(4-c^2)\epsilon)}{4([2]_q-1)}-\frac{c^4}{4([2]_q-1)}+\frac{5c^2}{24}+\frac{c^2(c^2+(4-c^2)\epsilon)}{4([3]_q-1)}\\ &\quad-\frac{c^4}{4([3]_q-1)}+\frac{c^4}{8([2]_q-1)([3]_q-1)}\Bigg)-\frac{1}{4h_3^2([3]_q-1)^2}\Bigg(\frac{c^4+2c(4-c^2)\epsilon+(4-c^2)^2\epsilon^2}{4}\\ &\quad-\frac{c^2(c^2+(4-c^2))\epsilon}{2}+\frac{c^2(c^2+(4-c^2))\epsilon}{2([2]_q-1)}-\frac{c^4}{2([2]_q-1)}+\frac{c^4}{4}+\frac{c^4}{4([2]_q-1)}\Bigg)\Bigg|.\end{aligned}$$

By simplifying the right hand side of the equation above and using the triangle inequality, we find $|\epsilon|\le r=1$, such that

$$\begin{aligned}|a_2a_4-a_3^2| &\le \Bigg|\frac{1}{4h_2h_4([2]_q-1)([4]_q-1)}\Bigg(\frac{c^2(c^2+(4-c^2)r)}{2}\\ &\quad+\frac{c(c^3+2(4-c^2)r-c(4-c^2)r^2+2(4-c^2)(1-|r|^2)z)}{4}+\frac{5c^2}{24}\\ &\quad+\frac{c^2(c^2+(4-c^2)r)}{4([2]_q-1)}-\frac{c^4}{4([2]_q-1)}+\frac{c^2(c^2+(4-c^2)r)}{4([3]_q-1)}-\frac{c^4}{4([3]_q-1)}\\ &\quad+\frac{c^4}{8([2]_q-1)([3]_q-1)}\Bigg)-\frac{1}{4h_3^2([3]_q-1)^2}\Bigg(\frac{c^4+2c(4-c^2)r+(4-c^2)^2r^2}{4}\\ &\quad-\frac{c^2(c^2+(4-c^2))r}{2}+\frac{c^2(c^2+(4-c^2))r}{2([2]_q-1)}-\frac{c^4}{2([2]_q-1)}+\frac{c^4}{4}+\frac{c^4}{4([2]_q-1)}\Bigg)\Bigg|.\end{aligned}$$

We denote the right hand side of the last equation by $\gamma(\rho, r)$ and using the derivative test to maximize the function $\gamma(\rho, r)$ with $(\rho, r) \in [0,2] \times [0,1]$,

$$|a_2a_4 - a_3^2| = \gamma(\rho, r) \leq \max_{\{0\leq r\leq 1; 0\leq \rho\leq 2\}} \{\gamma(\rho, r)\} = \gamma(2,1),$$

and

$$|a_2a_4 - a_3^2| \leq \frac{6 - ([2]_q - 1)([3]_q - 1)}{(6h_2h_4([2]_q - 1)^2([3]_q - 1)([4]_q - 1)} - \frac{1}{h_3^2([2]_q - 1)^2([3]_q - 1)^2}.$$

This completes the proof. □

Theorem 3.4. *Let the function $f(z)$ in (1.11) be in the class $K^*_{q,\alpha,\mu,v,f,t}$ , then*

$$|a_3 - a_2^2| \leq \frac{1}{(h_3[3]_q([3]_{q-1})([2]_{q-1})} - \frac{1}{h_3^2([2]_q([2]_{q-1}))^2},$$

*where $h_2$ and $h_3$ follows as stated in Theorem 3.2.*

*Proof.* Since $f(z) \in K^*_{q,\alpha,\mu,v,f,t}$,then from the definition of subordination, there exist a Schwarz function $w(z)$

$$1 + \frac{qzD_q^z(T_q^{(\alpha,\mu,v,F,t)}f(z))}{D_q(T_q^{(\alpha,\mu,v,F,t)}f(z))} \prec 1 + \sin z$$

and can be written as follows

$$\frac{D_q(zD_q(T_q^{(\alpha,\mu,v,F,t)}f(z))}{D_q(T_q^{(\alpha,\mu,v,F,t)}f(z))} \prec 1 + \sin z.$$

Then

$$\frac{D_q(zD_q(T_q^{(\alpha,\mu,v,F,t)}f(z))}{D_q(T_q^{(\alpha,\mu,v,F,t)}f(z))} = 1 + \sin w(z),$$

equivalently, we have

$$D_q(zD_q(T_q^{(\alpha,\mu,v,F,t)}f(z)) - D_q(T_q^{(\alpha,\mu,v,F,t)}f(z)) = D_q(T_q^{(\alpha,\mu,v,F,t)}f(z)) \sin w(z)$$

which readily yields

$$[2]_q([2]_q - 1)h_2a_2z + [3]_q([3]_q - 1)h_3a_3z^2 + [4]_q([4]_q - 1)h_4a_4z^3 + \cdots$$

$$= \frac{c_1z}{2} + \left(\frac{c_2}{2} - \frac{c_1^2}{4} + \frac{[2]_qh_2a_2c_1}{2}\right)z^2 + \left(-\frac{c_1c_2}{2} + \frac{c_3}{2} + \frac{5c_1^3}{48} + \frac{[2]_qh_2a_2c_2}{2} - \right.$$

$$\left.\frac{[2]_qh_2a_2c_1^2}{4} + \frac{[3]_qh_3a_3c_1}{2}\right)z^3 + \cdots. \tag{3.13}$$

Comparing the coefficient on both sides of equation (3.13), we are led to the following equation:

$$a_2 = \frac{c_1}{2h_2[2]_q([2]_q - 1)}, \tag{3.14}$$

$$a_3 = \frac{1}{2h_3[3]_q([3]_q - 1)}\left(c_2 - \frac{1}{2}c_1^2 + \frac{c_1^2}{2([2]_q - 1)}\right), \tag{3.15}$$

$$a_4 = \frac{1}{2h_4[4]_q([4]_q - 1)}\left(c_3 - c_1c_3 + \frac{5}{24}c_1^3 + \frac{c_1c_2}{2([2]_q - 1)} - \frac{c_1^3}{4([2]_q - 1)} + \right.$$

$$\frac{c_1c_2}{2([3]_q-1)}-\frac{c_1^3}{4([3]_q-1)}+\frac{c_1^3}{4([3]_q-1)([3]_q-1)}\Bigg). \tag{3.16}$$

The proof of Theorem 3.4 follows the same pattern in the proof of Theorem 3.2. □

THEOREM 3.5. *Let $f(z)$ be of the form (1.1) be in the class $K^*_{q,\alpha,\mu,v,f,t}$, then*

$$|a_2a_4-a_3^2|\leq\frac{6-([2]_q-1)([3]_q-1)}{6h_2h_4[2]_q[4]_q([2]_q-1)^2([3]_q-1)([4]_q-1)}-\frac{1}{h_3^2[3]_q([2]_q-1)^2([3]_q-1)^2}$$

*Proof.* The proof follows the same pattern as the proof of Theorem 3.3. □

**Remark.** If $p_1=\frac{c_1}{2}$, $p_2=\frac{c_2}{2}-\frac{c_1^2}{4}$, and $p_3=-\frac{c_1c_2}{2}+\frac{c_3}{2}+\frac{5c_1^3}{48}$, then we deduce the coefficient estimate result for results of Srivastava *et al.* [8].

## 4. Conclusion

In our present study, the new q-fractional integral operator $T_q^{(\alpha,\mu,v,F,t)}$ which combines the modified q-Opoola differential operator $D_q^n(\mu,v,F,t)$ with the Srivastava-Owa fractional integral operator $I_z^\alpha$ was then applied to define two general classes $L^*_{q,\alpha,\mu,v,f,t}$ and $K^*_{q,\alpha,\mu,v,f,t}$ which are similar to the well known q-starlike and q-convex functions respectively. Coefficient estimates, Fekete-Szego functional and Second Hankel determinant for functions belonging to the newly defined classes $L^*_{q,\alpha,\mu,v,f,t}$ and $K^*_{q,\alpha,\mu,v,f,t}$ were derived. For further investigation, various other properties of these classes can also be studied.

## References


[1] A. Alatawi and M. A. Darus, *On a Certain subclass of analytic functions involving the modified q-Opoola derivative operator*, In. J. Nonlinear Anal. Appl. (2023), 1–8.

[2] A. Alsoboh and M. Darus, *On q-starlike functions with respect to k-symmetric points*, Acta Univ. Apulensis **60** (2019), 61–73.

[3] A. Alsoboh and M. A. Darus, *On Fekete-Szego problem associated with q-derivative operator*, J. Phys.: Conf. Ser. **1212** (2019), 012003.
https://doi.org/10.1088/1742-6596/1212/1/012003

[4] C. Pommerenke, *On the Hankel Determinant of Univalent functions*, Mathematika **14** (1967), 108–112.
https://doi.org/10.1112/S002557930000807X

[5] C. Pommerenke, *On the Coefficients and Hankel Determinant of Univalent Functions*, J. London Math. Soc. **41** (1966), 111–112.
https://doi.org/10.1112/jlms/s1-41.1.111

[6] C. Pommerenke, *Univalent functions*, Vandenhoeck & Ruprecht, Gottingen (1975).

[7] F. H. Jackson, *On q-functions and a certain difference operator*, Trans. Roy. Soc. Edinburgh **46** (2) (1909), 253–281.
https://doi.org/10.1017/S0080456800002751

[8] H. M. Srivastava, K. Alshammari, and M. A. Darus, *A New q-fractional Integral Operator and its applications to the coefficient problem involving the second Hankel determinant for q-starlike and q-convex functions*, J. Nonlinear Var. Anal. **7** (6) (2023), 985–994.
https://doi.org/10.23952/jnva.7.2023.6.07

[9] H. M. Srivastava, S. H. Hadi, and M. A. Darus, *Some subclass of p-valent $\gamma$-uniformly type q-starlike and q-convex functions defined by using a certain generalized q-Bernardi integral operator*, Rev. R. Acad. Cienc. Exactas Fis. Nat. Ser. A Mat. RACSAM (2022).
https://doi.org/10.1007/s13398-022-01378-3

[10] H. Tang et al., *Fourth Hankel Determinant Problem based on Certain Analytic Functions*, Symmetry **14** (4) (2022), 663.
https://doi.org/10.3390/sym14040663

[11] R. A. Bello and T. O. Opoola, *Upper Bounds for Fekete-Szego Functions and the Second Hankel Determinants for a Class of Starlike Functions*, IOSR J. Math. **13** (2) (2017), 34–39.

[12] R. M. Ali and R. N. Seenivasagan, *Coefficient bounds for p-valent functions*, Appl. Math. Comput. **187** (2007), 35–46.
https://doi.org/10.1016/j.amc.2006.08.100

[13] R. J. Libera and E. J. Zlotkiewicz, *Coefficients bounds for the inverse of a functions with derivative in P*, Proc. Amer. Math. Soc. **87** (2) (1983), 251–257.
https://doi.org/10.2307/2043698

[14] T. O. Opoola, *On a subclass of univalent functions defined by a generalised differential operator*, Int. J. Math. Anal. **11** (18) (2017), 869–876.
https://doi.org/10.12988/ijma.2017.7232

[15] O. A. Lasode and T. O. Opoola, *Some properties of a family of univalent functions defined by a generalized Opoola Differential Operator*, Gen. Math. **30** (1) (2022), 3–13.

[16] O. A. Lasode and T. O. Opoola, *Hankel Determinant of a Subclass of analytic and Bi-univalent functions defined by means of Subordination and q-differentiation*, Int. J. Nonlinear Anal. Appl. **13** (2) (2023), 3105–3114.

[17] V. Ravichandran and S. Verma, *Bound for the fifth coefficient of certain starlike functions*, C. R. Math. **353** (2015), 505–510.
https://doi.org/10.1016/j.crma.2015.03.003

[18] H. M. Srivastava, *Operators of basic (or q-)calculus and fractional q-calculus and their applications in geometric function theory of complex analysis*, Iran. J. Sci. Technol. Trans. A Sci. **44** (2020), 327–344.
https://doi.org/10.1007/s40995-019-00815-0

[19] H. M. Srivastava, *Some parametric and argument variations of the operators of fractional calculus and related special functions and integral transformations*, J. Nonlinear Convex Anal. **22** (2021), 1501–1520.

[20] H. M. Srivastava, *An introductory overview of fractional-calculus operators based upon Fox-Wright and related higher transcendental functions*, J. Adv. Engrg. Comput. **5** (2021), 135–166.
https://doi.org/10.55579/jaec.202153.340

[21] L. Bieberbach, *Über die koeffizienten derjenigen Potenzreihen, welche eine schlichte Abbildung des Einheitskreises vermitteln*, Sitzungsber. Preuss. Akad. Wiss. **138** (1916), 940–955.

[22] L. de Branges, *A proof of the Bieberbach conjecture*, Acta Math. **154** (1-2) (1985), 137–152.
https://doi.org/10.1007/BF02392821

[23] H. M. Srivastava and S. Owa (Eds.), *Univalent Functions, Fractional Calculus and Their Applications*, Halsted Press, John Wiley and Sons, New York (1989), 329–354.

[24] H. Aldweby and M. A. Darus, *Some Subordination Result on q-Analogue of Ruscheweyh Differential Operator*, Abstr. Appl. Anal. (2014), Art. ID 958563.
https://doi.org/10.1155/2014/958563

[25] M. A. Darus and D. K. Thomas, *3 Proofs of 2 well-known theorems on Starlike and Convex Functions*, J. Qual. Meas. Anal. (JQMA) **18** (1) (2022), 63–69.
https://www.ukm.my/jqma

[26] S. Afis and O. O. Semiu, *On Lemniscate of Bernoulli of q-Janowski type*, J. Niger. Soc. Phys. Sci. **4** (4) (2022), 961.
https://doi.org/10.46481/jnsps.2022.961

[27] R. Amina, R. Mohsan, B. A. Mohammad, and S. Afis, *The Second and Third Hankel determinants for starlike and convex functions associated with Three-Leaf function*, Heliyon **9** (1) (2023), e12748.
https://doi.org/10.1016/j.heliyon.2022.e12748

**Risikat Bello**
Department of Mathematics and Statistics,
Kwara State University, Malete Nigeria
*E-mail*: risikat.bello@kwasu.edu.ng

**Maslina Darus**
Department of Mathematical Sciences,
Faculty of Science and Technology,
Universiti Kebangsaan Malaysia, 43600 Bangi, Selango, Malaysia
*E-mail*: maslina@ukm.edu.my

**Khalid Alshammari**
Department of Mathematics, College of Science,
University of Hail, Hail 2440, Saudi Arabia
*E-mail*: khalid@jic.edu.sa