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ERROR ESTIMATES FOR THE FULLY DISCRETE STABILIZED GAUGE-UZAWA METHOD PART I: THE NAVIER-STOKES EQUATIONS

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Abstract. The stabilized Gauge-Uzawa method (SGUM), which is a second order projection type algorithm to solve the time-dependent Navier-Stokes equations, has been newly constructed in 2013 Pyo's paper. The accuracy of SGUM has been proved only for time discrete scheme in the same paper, but it is crucial to study for fully discrete scheme, because the numerical errors depend on discretizations for both space and time, and because discrete spaces between velocity and pressure can not be chosen arbitrary. In this paper, we find out properties of the fully discrete SGUM and estimate its errors and stability to solve the evolution Navier-Stokes equations. The main difficulty in this estimation arises from losing some cancellation laws due to failing divergence free condition of the discrete velocity function. This result will be extended to Boussinesq equations in the continuous research (part II) and is essential in the study of part II.

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1. Introduction

Given an open bounded polyhedral domain Ω in $\mathbb{R}^d$, with $d = 2$ or 3, we consider the time-dependent Navier-Stokes equations of incompressible fluids:

$$
\begin{aligned}
\mathbf{u}_t + (\mathbf{u}\cdot\nabla)\mathbf{u} + \nabla p - \mu\triangle\mathbf{u} &= \mathbf{f}, && \text{in } \Omega,\\
\nabla\cdot\mathbf{u} &= 0, && \text{in } \Omega,\\
\mathbf{u}(0,\mathbf{x}) &= \mathbf{u}^0, && \text{in } \Omega,
\end{aligned}
\tag{1.1}
$$

with vanishing Dirichlet boundary condition $\mathbf{u} = \mathbf{0}$ on $\partial\Omega$ and pressure mean-value $\int_\Omega p = 0$. The primitive variables are the (vector) velocity $\mathbf{u}$ and the (scalar) pressure p. The viscosity $\mu = Re^{-1}$ is the reciprocal of the Reynolds number Re.

The most popular solvers of (1.1) are the projection type methods which were introduced independently by Chorin [3] and Temam [17] in the late 60's to decouple $\mathbf{u}$ and p and thus reduce the computational cost. And the methods quickly gained popularity in the computational fluid dynamics community, and over the years, an enormous amount of efforts have been devoted to develop more accurate and efficient projection type schemes. One branch of the projection type methods is the Gauge-Uzawa method (GUM) which has been constructed in [11] to solve (1.1). GUM enhanced to solve more complicated problems which are the Boussinesq equations in [12] and the non-constant density fluid problems in [16]. However, GUM has been studied only for the first order backward Euler time scheme except normal mode error analysis for the Stokes equations in [15]. We construct second order GUM and analyze the superiority of the method in normal mode space in [15]. We also discover that the method is equivalent in continuous level to the consistent splitting method which is studied in [6]. But we could not obtain any theoretical proofs for them via energy estimate even for stability condition. Because we found out weak stability performance through numerical tests, we concentrate on overcoming the weak stability constraint without losing advantages of GUM and then we obtain the stabilized Gauge-Uzawa method (SGUM) which is based on the second order backward Euler time discrete scheme in [14]. And we estimated errors for the Navier Stokes equations via energy error estimate, but the error evaluation carried out only for time discrete scheme. The study for time is not enough to guarantee accuracy, because numerical errors depend on both time and space discretizations and because discrete spaces between velocity and pressure can not be chosen arbitrary (see Assumption 3 below). We can find an example in [13]: the numerical tests of the Van Kan method does not display second order convergence in [13], even though the optimal accuracy for the method had been proved for the time discrete scheme. Thus it is crucial to verify error decay for fully discrete algorithm and so we will estimate optimal convergence and stability for the fully discrete SGUM in this paper.

In the other direction, the rotational form of pressure-correction method has been constructed in [19]. The errors of the method have been estimated via energy estimate in [7] and via normal mode analysis in [15]. But both proofs had been carried out only for Stokes equations. In [14], they discover that SGUM is an equivalent method to the rotational form of pressure-correction method in the view of semi-discrete level. The main difference of two methods is pressure expression: the pressure in the rotational form of pressure-correction method is designed to be updated from previous step value ($p^{n+1} = p^n + \cdots$) in contrast to SGUM. This pressure expression acts upon main obstacle to treat convection term, and so the study in [7] had limited only for Stokes equations.

In this paper, we will prove stability and estimate errors for the fully discrete SGUM which is stated in Algorithm 1 to solve Navier Stokes equations. The main difficulty in this estimation arise from losing some cancellation laws due to failing divergence free condition of the discrete velocity function. In order to overcome this deficit, we introduce discontinuous velocity to hold (1.8) in the fully discrete Algorithm 1. The result in this paper will be extended to Boussinesq equations in the continuous research (part II) and is essential in the study of part II. We will denote τ as the time marching size. Also we will use δ as difference of two consecutive functions, for example, for any sequence function z^{n+1},

$$\delta z^{n+1} = z^{n+1} - z^n, \quad \delta\delta z^{n+1} = \delta(\delta z^{n+1}) = z^{n+1} - 2z^n + z^{n-1}, \quad \cdots .$$

In order to introduce the finite element discretizations we need further notations. Let $H^s(\Omega)$ be the Sobolev space with s derivatives in $L^2(\Omega)$, set $\mathbf{L}^2(\Omega) = \left(L^2(\Omega)\right)^d$ and $\mathbf{H}^s(\Omega) = \left(H^s(\Omega)\right)^d$, where $d = 2$ or 3, and denote by $L^2_0(\Omega)$ the subspace of $L^2(\Omega)$ of functions with vanishing meanvalue. We indicate with $\|\cdot\|_s$ the norm in $H^s(\Omega)$, and with $\langle \cdot\,, \cdot\rangle$ the inner product in $L^2(\Omega)$. Let $\mathfrak{T} = \{K\}$ be a shape-regular quasi-uniform partition of Ω of meshsize h into closed elements K [1, 2, 5]. The vector and scalar finite element spaces are:

$$\mathbb{W}_h := \{\mathbf{w}_h \in \mathbf{L}^2(\Omega) : \mathbf{w}_h|_K \in \mathcal{P}(K) \quad \forall K \in \mathfrak{T}\}, \quad \mathbb{V}_h := \mathbb{W}_h \cap \mathbf{H}_0^1(\Omega),$$
$$\mathbb{P}_h := \{q_h \in L_0^2(\Omega) \cap C^0(\Omega) : q_h|_K \in \mathcal{Q}(K) \quad \forall K \in \mathfrak{T}\},$$

where $\mathcal{P}(K)$ and $\mathcal{Q}(K)$ are spaces of polynomials with degree bounded uniformly with respect to $K \in \mathfrak{T}$ [2, 5]. We stress that the space $\mathbb{P}_h$ is composed of continuous functions to make sense. This implies the crucial equality

$$\langle \nabla\cdot \mathbf{w}_h \,,\, s_h\rangle = -\langle \mathbf{w}_h \,,\, \nabla s_h\rangle \,, \quad \forall \mathbf{w}_h \in \mathbb{V}_h, \forall s_h \in \mathbb{P}_h.$$

Using the following discrete counterpart of the form

$$\mathcal{N}(\mathbf{u}, \mathbf{v}, \mathbf{w}) := \langle (\mathbf{u}\cdot\nabla)\mathbf{v} \,,\, \mathbf{w}\rangle$$

$$\mathcal{N}(\mathbf{u}_h, \mathbf{v}_h, \mathbf{w}_h) := \frac{1}{2}\langle (\mathbf{u}_h\cdot\nabla)\mathbf{v}_h \,,\, \mathbf{w}_h\rangle - \frac{1}{2}\langle (\mathbf{u}_h\cdot\nabla)\mathbf{w}_h \,,\, \mathbf{v}_h\rangle \,, \tag{1.2}$$

We now introduce the fully discrete stabilized Gauge-Uzawa method (SGUM) which is studied for semi-discrete level in [14].

Algorithm 1 (The fully discrete stabilized Gauge-Uzawa method). *Compute $\mathbf{u}_h^1$ and p_h^1 via any first order projection method and set $\psi_h^1 = \frac{-2\tau}{3}p_h^1$ and $q_h^1 = 0$. Repeat for $1 \le n \le N = [\frac{T}{\tau} - 1]$.*

Step 1: *Set $\mathbf{u}_h^* = 2\mathbf{u}_h^n - \mathbf{u}_h^{n-1}$ and find $\widehat{\mathbf{u}}_h^{n+1} \in \mathbb{V}_h$ as the solution of*

$$\begin{aligned}\frac{1}{2\tau}\left\langle 3\widehat{\mathbf{u}}_h^{n+1} - 4\mathbf{u}_h^n + \mathbf{u}_h^{n-1} \,,\, \mathbf{w}_h\right\rangle + \langle \nabla p_h^n \,,\, \mathbf{w}_h\rangle + \mathcal{N}\left(\mathbf{u}_h^* \,,\, \widehat{\mathbf{u}}_h^{n+1} \,,\, \mathbf{w}_h\right)\\ +\mu\left\langle \nabla\widehat{\mathbf{u}}_h^{n+1} \,,\, \nabla\mathbf{w}_h\right\rangle = \left\langle \mathbf{f}(t^{n+1}) \,,\, \mathbf{w}_h\right\rangle, \quad \forall \mathbf{w}_h \in \mathbb{V}_h.\end{aligned} \tag{1.3}$$

Step 2: *Find $\psi_h^{n+1} \in \mathbb{P}_h$ as the solution of*

$$\left\langle \nabla\psi_h^{n+1} \,,\, \nabla\phi_h\right\rangle = \langle \nabla\psi_h^n \,,\, \nabla\phi_h\rangle + \left\langle \nabla\cdot\widehat{\mathbf{u}}_h^{n+1} \,,\, \phi_h\right\rangle, \quad \forall \phi_h \in \mathbb{P}_h. \tag{1.4}$$

Step 3: *Update $\mathbf{u}_h^{n+1}$ and $q_h^{n+1} \in \mathbb{P}_h$ according to*

$$\mathbf{u}_h^{n+1} = \widehat{\mathbf{u}}_h^{n+1} + \nabla\left(\psi_h^{n+1} - \psi_h^n\right), \tag{1.5}$$

$$\left\langle q_h^{n+1} \,,\, \phi_h\right\rangle = \langle q_h^n \,,\, \phi_h\rangle - \left\langle \nabla\cdot\widehat{\mathbf{u}}_h^{n+1} \,,\, \phi_h\right\rangle, \quad \forall \phi_h \in \mathbb{P}_h. \tag{1.6}$$

Step 4: *Update pressure p_h^{n+1} by*

$$p_h^{n+1} = -\frac{3\psi_h^{n+1}}{2\tau} + \mu q_h^{n+1}. \tag{1.7}$$

Remark 1.1 (Discontinuity of $\mathbf{u}_h^{n+1}$). We note that $\mathbf{u}_h^{n+1}$ is a discontinuous function across inter-element boundaries and that, in light of (1.4) and (1.5), $\mathbf{u}_h^{n+1}$ is discrete divergence free in the sense that

$$\left\langle \nabla \cdot \mathbf{u}_h^{n+1} \, , \, \phi_h \right\rangle = \left\langle \mathbf{u}_h^{n+1} \, , \, \nabla \phi_h \right\rangle = 0, \quad \forall \phi_h \in \mathbb{P}_h. \tag{1.8}$$

We now summarize the results of this paper along with organization. We introduce appropriate Assumptions 1-5 in §2 and introduce well known lemmas. In §3, we prove stability.

Theorem 1 (Stability). *The SGUM is unconditionally stable in the sense that, for all $\tau > 0$, the following a priori bound holds:*

$$\begin{aligned} &\left\|\mathbf{u}_h^{N+1}\right\|_0^2 + \left\|\widehat{\mathbf{u}}_h^{N+1}\right\|_0^2 + \left\|2\mathbf{u}_h^{N+1} - \mathbf{u}_h^N\right\|_0^2 + 3\left\|\nabla \psi_h^{N+1}\right\|_0^2 + 2\mu\tau\left\|q_h^{N+1}\right\|_0^2 \\ &+ \sum_{n=1}^N \left(\left\|\delta\delta\mathbf{u}_h^{n+1}\right\|_0^2 + 3\left\|\nabla\delta\psi_h^{n+1}\right\|_0^2\right) + \mu\tau\sum_{n=1}^N \left\|\nabla\widehat{\mathbf{u}}_h^{n+1}\right\|_0^2 \\ &\le \left\|\mathbf{u}_h^1\right\|_0^2 + \left\|2\mathbf{u}_h^1 - \mathbf{u}_h^0\right\|_0^2 + 3\left\|\nabla\psi_h^1\right\|_0^2 + 2\mu\tau\left\|q_h^1\right\|_0^2 + C\tau\sum_{n=1}^N \left\|\mathbf{f}(t^{n+1})\right\|_{-1}^2. \end{aligned} \tag{1.9}$$

We then prove the following accuracy results through several lemmas in §4.

Theorem 2 (Error estimates). *Suppose the exact solution of* (1.1) *is smooth enough and $\tau = Ch$. If Assumptions 1-5 below hold, then the errors of Algorithm 1 are bounded by*

$$\begin{aligned} &\tau\sum_{n=1}^N \left(\left\|\mathbf{u}(t^{n+1}) - \mathbf{u}_h^{n+1}\right\|_0^2 + \left\|\mathbf{u}(t^{n+1}) - \widehat{\mathbf{u}}_h^{n+1}\right\|_0^2\right) \le C\left(\tau^4 + h^4\right), \\ &\tau\sum_{n=1}^N \left(\left\|\mathbf{u}(t^{n+1}) - \widehat{\mathbf{u}}_h^{n+1}\right\|_1^2 + \left\|p(t^{n+1}) - p_h^{n+1}\right\|_0^2\right) \le C\left(\tau^2 + h^2\right). \end{aligned}$$

We finally conclude in §5 with numerical experiments.

2. Preliminaries

In this section, we introduce 5 assumptions and well known lemmas to use in proofs of main theorems. We resort to a duality argument via the following Stokes equations:

$$
\begin{aligned}
-\triangle \mathbf{v} + \nabla r &= \mathbf{g}, \quad \text{in } \Omega, \\
\nabla\cdot \mathbf{v} &= 0, \quad \text{in } \Omega,
\end{aligned}
\tag{2.1}
$$

with Dirichlet boundary condition $\mathbf{v} = 0$ on $\partial\Omega$. We now state a basic assumption about Ω.

Assumption 1 (Regularity of Ω). *The unique solution* $\{\mathbf{v}, r\}$ *of the steady Stokes equations* (2.1) *satisfies*

$$\|\mathbf{v}\|_2 + \|r\|_1 \le C\|\mathbf{g}\|_0.$$

We remark that the validity of Assumption 1 is known if $\partial\Omega$ is of class $\mathbf{C}^2$ [4, 8], or if $\partial\Omega$ is a two-dimensional convex polygon [10], and is generally believed for convex polyhedral [8].

In order to launch Algorithm 1, we need to set $(\mathbf{u}_h^1, p_h^1)$ via any first order projection method which holds the following condition.

Assumption 2 (Initial setting). *Let* $(\mathbf{u}(t^1), p(t^1))$ *be the exact solution of* (1.1) *at* $t = t^1$. *The initial value* $(\mathbf{u}_h^1, p_h^1)$ *satisfies*

$$
\begin{aligned}
&\left\|\mathbf{u}(t^1) - \mathbf{u}_h^1\right\|_0 \le C\left(\tau^2 + h^2\right), \\
&\left\|\mathbf{u}(t^1) - \mathbf{u}_h^1\right\|_1 + \left\|p(t^1) - p_h^1\right\|_1 \le C\left(\tau + h\right).
\end{aligned}
$$

We impose the following properties for relations between the spaces $\mathbb{V}_h$ and $\mathbb{P}_h$.

Assumption 3 (Discrete Inf-Sup condition). *There exists a constant* $\beta > 0$ *such that*

$$\inf_{s_h\in\mathbb{P}_h} \sup_{\mathbf{w}_h\in\mathbb{V}_h} \frac{\langle \nabla\cdot \mathbf{w}_h\, ,\, s_h\rangle}{\|\mathbf{w}_h\|_1 \|s_h\|_0} \ge \beta.$$

Assumption 4 (Shape regularity and quasiuniformity [1, 2, 5]). *There exists a constant* $C > 0$ *such that the ratio between the diameter* h_K *of an element* $K \in \mathfrak{T}$ *and the diameter of the largest ball contained in* K *is bounded uniformly by* C, *and* h_K *is comparable with the meshsize* h *for all* $K \in \mathfrak{T}$.

Assumption 5 (Approximability [1, 2, 5]). *For each* $(\mathbf{w}, s) \in \mathbf{H}^2(\Omega) \times H^1(\Omega)$, *there exist approximations* $(\mathbf{w}_h, s_h) \in \mathbb{V}_h \times \mathbb{P}_h$ *such that*

$$\|\mathbf{w} - \mathbf{w}_h\|_0 + h\|\mathbf{w} - \mathbf{w}_h\|_1 \le Ch^2\|\mathbf{w}\|_2 \quad \text{and} \quad \|s - s_h\|_0 \le Ch\|s\|_1.$$

The following elementary but crucial relations are derived in [18].

LEMMA 2.1 (Inverse inequality). *If I_h denotes the Clement interpolant, then*

$$\|I_h\mathbf{w}\|_{\mathbf{L}^3(\Omega)} \le Ch^{-d/6}\|I_h\mathbf{w}\|_0, \quad \textit{and} \quad \|\mathbf{w} - I_h\mathbf{w}\|_{\mathbf{L}^3(\Omega)} \le Ch^{2-d/6}\|\mathbf{w}\|_2.$$

LEMMA 2.2 (Div-grad relation). *If $\mathbf{w} \in \mathbf{H}_0^1(\Omega)$, then*

$$\|\nabla \cdot \mathbf{w}\|_0 \le \|\nabla\mathbf{w}\|_0.$$

Let now $(\mathbf{v}_h, r_h) \in \mathbb{V}_h \times \mathbb{P}_h$ indicate the finite element solution of (2.1), namely,

$$\begin{aligned} \langle\nabla\mathbf{v}_h\,,\,\nabla\mathbf{w}_h\rangle - \langle r_h\,,\,\nabla\cdot\mathbf{w}_h\rangle &= \langle\mathbf{g}\,,\,\mathbf{w}_h\rangle\,, \quad &\forall\mathbf{w}_h \in \mathbb{V}_h,\\ \langle\nabla\cdot\mathbf{w}_h\,,\,s_h\rangle &= 0, &\forall s_h \in \mathbb{P}_h. \end{aligned} \tag{2.2}$$

Then we can find the well known lemmas in [1, 2, 5]:

LEMMA 2.3 (Error estimates for mixed FEM). *Let $(\mathbf{v}, s) \in \mathbf{H}_0^1(\Omega) \times L_0^2(\Omega)$ be the solutions of (2.1) and $(\mathbf{v}_h, s_h) = \mathfrak{S}_h(\mathbf{v}, s) \in \mathbb{V}_h \times \mathbb{P}_h$ be the Stokes projections defined by (2.2), respectively. If Assumptions 1 and 3-5 hold, then*

$$\|\mathbf{v} - \mathbf{v}_h\|_0 + h\|\mathbf{v} - \mathbf{v}_h\|_1 + h\|r - r_h\|_0 \le Ch^2\left(\|\mathbf{v}\|_2 + \|r\|_1\right), \tag{2.3}$$

$$|||\mathbf{v} - \mathbf{v}_h||| := \|\mathbf{v} - \mathbf{v}_h\|_{\mathbf{L}^\infty(\Omega)} + \|\nabla(\mathbf{v} - \mathbf{v}_h)\|_{\mathbf{L}^3(\Omega)} \le C\|\mathbf{g}\|_0. \tag{2.4}$$

Proof. Inequality (2.3) is standard [1, 2, 5]. To establish (2.4), we just deal with the $\mathbf{L}^\infty$-norm since the other can be treated similarly. If I_h denotes the Clement interpolant, then $\|\mathbf{v} - I_h\mathbf{v}\|_{\mathbf{L}^\infty(\Omega)} \le C\|\mathbf{v}\|_2$ and

$$\|I_h\mathbf{v} - \mathbf{v}_h\|_{\mathbf{L}^\infty(\Omega)} \le Ch^{-d/2}\|I_h\mathbf{v} - \mathbf{v}_h\|_{\mathbf{L}^2(\Omega)} \le C\|\mathbf{v}\|_2$$

as a consequence of an inverse estimate and (2.3). This completes the proof. □

REMARK 2.4 (H^1 stability of r_h). The bound $\|\nabla r_h\|_0 \le C\left(\|\mathbf{v}\|_2 + \|r\|_1\right)$ is a simple by-product of (2.3). To see this, we add and subtract $I_h r$, use the stability of I_h in H^1, and observe that (2.3) implies $\|\nabla(r_h - I_h r)\|_0 \le Ch^{-1}\|r_h - I_h r\|_0 \le C$.

We finally state without proof several properties of the nonlinear form $\mathcal{N}$. In view of (1.2), we have a following properties of $\mathcal{N}$ for all $\mathbf{u}_h, \mathbf{v}_h, \mathbf{w}_h \in \mathbb{V}_h$:

$$\mathcal{N}(\mathbf{u}_h, \mathbf{v}_h, \mathbf{w}_h) = -\mathcal{N}(\mathbf{u}_h, \mathbf{w}_h, \mathbf{v}_h), \qquad \mathcal{N}(\mathbf{u}_h, \mathbf{v}_h, \mathbf{v}_h) = 0, \tag{2.5}$$

and

$$\nabla\cdot\,\mathbf{u}=0 \quad \Rightarrow \quad \mathcal{N}(\mathbf{u},\mathbf{v}_h,\mathbf{w}_h)=\langle(\mathbf{u}\cdot\nabla)\mathbf{v}_h\,,\,\mathbf{w}_h\rangle=-\langle(\mathbf{u}\cdot\nabla)\mathbf{w}_h\,,\,\mathbf{v}_h\rangle\,.$$

Applying Sobolev imbedding Lemma yields the following useful results.

Lemma 2.5 (Bounds on nonlinear convection [8, 11]). *Let* $\mathbf{u},\mathbf{v}\in\mathbf{H}^2(\Omega)$ *with* $\nabla\cdot\,\mathbf{u}=0$*, and let* $\mathbf{u}_h,\mathbf{v}_h,\mathbf{w}_h\in\mathbb{V}_h$*. Then*

$$\mathcal{N}(\mathbf{u},\mathbf{v}_h,\mathbf{w}_h)\le C\begin{cases}\|\mathbf{u}\|_1\|\mathbf{v}_h\|_1\|\mathbf{w}_h\|_1\\ \|\mathbf{u}\|_2\|\mathbf{v}_h\|_1\|\mathbf{w}_h\|_0\\ \|\mathbf{u}\|_2\|\mathbf{v}_h\|_0\|\mathbf{w}_h\|_1,\end{cases} \tag{2.6}$$

$$\mathcal{N}(\mathbf{u}_h,\mathbf{v},\mathbf{w}_h)\le C\|\mathbf{u}_h\|_0\|\mathbf{v}\|_2\|\mathbf{w}_h\|_1. \tag{2.7}$$

In addition

$$\mathcal{N}(\mathbf{u}_h,\mathbf{v}_h,\mathbf{w}_h)\le C\begin{cases}\|\mathbf{u}_h\|_0|\!|\!|\mathbf{v}_h|\!|\!|\|\mathbf{w}_h\|_1\\ \|\mathbf{u}_h\|_{\mathbf{L}^3(\Omega)}\|\mathbf{v}_h\|_1\|\mathbf{w}_h\|_1.\end{cases} \tag{2.8}$$

We will use the following algebraic identities frequently to treat time derivative terms.

Lemma 2.6 (Inner product of time derivative terms). *For any sequence* $\{z^n\}_{n=0}^N$*, we have*

$$\begin{aligned}&2\left\langle 3z^{n+1}-4z^n+z^{n-1}\,,\,z^{n+1}\right\rangle\\ &\qquad=\delta\left\|z^{n+1}\right\|_0^2+\delta\left\|2z^{n+1}-z^n\right\|_0^2+\left\|\delta\delta z^{n+1}\right\|_0^2,\end{aligned} \tag{2.9}$$

$$2\left\langle z^{n+1}-z^n\,,\,z^{n+1}\right\rangle=\left\|z^{n+1}\right\|_0^2-\|z^n\|_0^2+\left\|z^{n+1}-z^n\right\|_0^2, \tag{2.10}$$

and

$$2\left\langle z^{n+1}-z^n\,,\,z^{n}\right\rangle=\left\|z^{n+1}\right\|_0^2-\|z^n\|_0^2-\left\|z^{n+1}-z^n\right\|_0^2. \tag{2.11}$$

3. Proof of stability

In this section, we prove Theorem 1. We start the proof with rewriting the momentum equation (1.3) by using (1.5) and (1.7) as follows:

$$\begin{aligned}&\frac{1}{2\tau}\left\langle 3\mathbf{u}_h^{n+1}-4\mathbf{u}_h^n+\mathbf{u}_h^{n-1}\,,\,\mathbf{w}_h\right\rangle+\mathcal{N}\left(2\mathbf{u}_h^n-\mathbf{u}_h^{n-1}\,,\,\widehat{\mathbf{u}}_h^{n+1}\,,\,\mathbf{w}_h\right)\\ &-\left\langle\nabla\left(\frac{3}{2\tau}\psi_h^{n+1}-\mu q_h^n\right),\,\mathbf{w}_h\right\rangle+\mu\left\langle\nabla\widehat{\mathbf{u}}_h^{n+1}\,,\,\nabla\mathbf{w}_h\right\rangle=\left\langle\mathbf{f}(t^{n+1})\,,\,\mathbf{w}_h\right\rangle.\end{aligned}$$

We now choose $\mathbf{w}_h = 4\tau\widehat{\mathbf{u}}_h^{n+1}$ and use (2.9) to get

$$\begin{aligned}(3.1)\quad &\delta\big\|\mathbf{u}_h^{n+1}\big\|_0^2 + \delta\big\|2\mathbf{u}_h^{n+1} - \mathbf{u}_h^n\big\|_0^2 + \big\|\delta\delta\mathbf{u}_h^{n+1}\big\|_0^2 + 4\tau\mu\big\|\nabla\widehat{\mathbf{u}}_h^{n+1}\big\|_0^2 \\ &= 6\left\langle\nabla\psi_h^{n+1}, \widehat{\mathbf{u}}_h^{n+1}\right\rangle + 4\tau\mu\left\langle q_h^n, \nabla\cdot\widehat{\mathbf{u}}_h^{n+1}\right\rangle + 4\tau\left\langle\mathbf{f}(t^{n+1}), \widehat{\mathbf{u}}_h^{n+1}\right\rangle,\end{aligned}$$

and we denote by A_i, for $i = 1, 2, 3$, the three terms in the right hand side. We note here that convection term is vanished by (2.5). In conjunction with $\widehat{\mathbf{u}}_h^{n+1} = \mathbf{u}_h^{n+1} - \nabla\delta\psi_h^{n+1}$, (2.10) yields

$$\begin{aligned}A_1 &= -6\left\langle\nabla\psi_h^{n+1}, \nabla\delta\psi_h^{n+1}\right\rangle \\ &= -3\left(\big\|\nabla\psi_h^{n+1}\big\|_0^2 - \|\nabla\psi_h^n\|_0^2 + \big\|\nabla\delta\psi_h^{n+1}\big\|_0^2\right).\end{aligned}$$

Before we estimate A_2, we evaluate an inequality via choosing $\phi_h = \delta q_h^{n+1}$ in (1.6) to get

$$\big\|\delta q_h^{n+1}\big\|_0^2 = -\left\langle\nabla\cdot\widehat{\mathbf{u}}_h^{n+1}, \delta q_h^{n+1}\right\rangle \le \big\|\nabla\cdot\widehat{\mathbf{u}}_h^{n+1}\big\|_0\big\|\delta q_h^{n+1}\big\|_0.$$

In the view of Lemma 2.2, we conclude $\big\|\delta q_h^{n+1}\big\|_0^2 \le \big\|\nabla\cdot\widehat{\mathbf{u}}_h^{n+1}\big\|_0^2 \le \big\|\nabla\widehat{\mathbf{u}}_h^{n+1}\big\|_0^2$, whence

$$\begin{aligned}A_2 = &-4\mu\tau\left\langle q_h^n, \delta q_h^{n+1}\right\rangle = -2\mu\tau\left(\big\|q_h^{n+1}\big\|_0^2 - \|q_h^n\|_0^2 - \big\|\delta q_h^{n+1}\big\|_0^2\right) \\ &\le -2\mu\tau\left(\big\|q_h^{n+1}\big\|_0^2 - \|q_h^n\|_0^2\right) + 2\mu\tau\big\|\nabla\widehat{\mathbf{u}}_h^{n+1}\big\|_0^2.\end{aligned}$$

Clearly, we have

$$A_3 \le C\frac{\tau}{\mu}\big\|\mathbf{f}(t^{n+1})\big\|_{-1}^2 + \tau\mu\big\|\nabla\widehat{\mathbf{u}}_h^{n+1}\big\|_0^2.$$

Inserting A_1-A_3 back into (3.1) and summing over n from 1 to N lead (1.9) by help of $\big\|\widehat{\mathbf{u}}_h^{n+1}\big\|_0^2 = \big\|\mathbf{u}_h^{n+1}\big\|_0^2 + \big\|\nabla\delta\psi_h^{n+1}\big\|_0^2$ which comes form (1.8). □

4. Error estimates

In this section, we prove Theorem 2 which is error estimates for SGUM of Algorithm 1. This proof is carried out through several lemmas. We start the proof with defining $(\mathbf{U}_h^{n+1}, P_h^{n+1}) := \mathfrak{S}_h(\mathbf{u}(t^{n+1}), p(t^{n+1})) \in \mathbb{V}_h \times \mathbb{P}_h$ to be the Stokes projection of the true solution at time t^{n+1}. It means that $(\mathbf{U}_h^{n+1}, P_h^{n+1}) \in \mathbb{V}_h \times \mathbb{P}_h$ is the solution of, for all $\mathbf{w}_h \in \mathbb{V}_h$ and for all $s_h \in \mathbb{P}_h$,

$$\left\langle \nabla \mathbf{U}_h^{n+1}, \nabla \mathbf{w}_h \right\rangle + \left\langle \nabla P_h^{n+1}, \mathbf{w}_h \right\rangle = \left\langle \nabla \mathbf{u}(t^{n+1}), \nabla \mathbf{w}_h \right\rangle + \left\langle \nabla p(t^{n+1}), \mathbf{w}_h \right\rangle,$$
$$\left\langle \nabla \cdot \mathbf{U}_h^{n+1}, s_h \right\rangle = 0.$$

And we denote notations

$$\mathbf{G}^{n+1} := \mathbf{u}(t^{n+1}) - \mathbf{U}_h^{n+1}, \qquad g^{n+1} := p(t^{n+1}) - P_h^{n+1}.$$

From Lemma 2.3, we can deduce

$$\begin{aligned} &\left\| \mathbf{G}^{n+1} \right\|_0^2 + h^2 \left\| \mathbf{G}^{n+1} \right\|_1^2 + h^2 \left\| g^{n+1} \right\|_0^2 \\ &\qquad \leq C h^4 \left(\left\| \mathbf{u}(t^{n+1}) \right\|_2^2 + \left\| p(t^{n+1}) \right\|_1^2 \right), \end{aligned} \tag{4.1}$$

and

$$\begin{aligned} &\left\| \delta\mathbf{G}^{n+1} \right\|_0^2 + h^2 \left\| \delta\mathbf{G}^{n+1} \right\|_1^2 + h^2 \left\| \delta g^{n+1} \right\|_0^2 \\ &\qquad \leq C\tau h^4 \int_{t^n}^{t^{n+1}} \left(\left\| \mathbf{u}_t(t) \right\|_2^2 + \left\| p_t(t) \right\|_1^2 \right) dt. \end{aligned} \tag{4.2}$$

In conjunction with the definition $\left\| \cdot \right\|$ in (2.4), we can deduce

$$\left\| \mathbf{G}^{n+1} \right\| \leq C. \tag{4.3}$$

We now carry out error evaluate by comparing (4.10) below with (1.3)-(1.7). We derive strong estimates of order 1 and use the result to prove weak estimates of order 2 for the errors

$$\mathbf{E}_h^{n+1} := \mathbf{U}_h^{n+1} - \mathbf{u}_h^{n+1}, \quad \widehat{\mathbf{E}}_h^{n+1} := \mathbf{U}_h^{n+1} - \widehat{\mathbf{u}}_h^{n+1}, \quad e_h^{n+1} := P_h^{n+1} - p_h^{n+1}.$$

Then, in conjunction with (4.1), we can readily get of the same accuracy for the errors

$$\mathbf{E}^{n+1} := \mathbf{u}(t^{n+1}) - \mathbf{u}_h^{n+1}, \widehat{\mathbf{E}}^{n+1} := \mathbf{u}(t^{n+1}) - \widehat{\mathbf{u}}_h^{n+1}, e^{n+1} := p(t^{n+1}) - p_h^{n+1}.$$

In addition, we denote

$$\varepsilon_h^{n+1} := P_h^{n+1} + \frac{3\psi_h^{n+1}}{2\tau}.$$

Then we readily obtain the following crucial properties

$$\begin{aligned} &\mathbf{G}^{n+1} = \mathbf{E}^{n+1} - \mathbf{E}_h^{n+1} = \widehat{\mathbf{E}}^{n+1} - \widehat{\mathbf{E}}_h^{n+1} \text{ and } \widehat{\mathbf{E}}_h^{n+1} = \mathbf{E}_h^{n+1} + \nabla \delta \psi_h^{n+1}, \\ &\widehat{\mathbf{E}}^{n+1} = \widehat{\mathbf{E}}_h^{n+1} = \mathbf{G}^{n+1} = \mathbf{0}, \text{ on } \partial\Omega, \end{aligned} \tag{4.4}$$

as well as, from (1.8),

$$\left\langle \mathbf{E}_h^{n+1}\,,\, \nabla\phi_h \right\rangle = \left\langle \mathbf{E}^{n+1}\,,\, \nabla\phi_h \right\rangle = \left\langle \mathbf{G}^{n+1}\,,\, \nabla\phi_h \right\rangle = 0, \quad \forall\phi_h \in \mathbb{P}_h, \tag{4.5}$$

whence we deduce crucial orthogonality properties:

$$\|\widehat{\mathbf{E}}_h^{n+1}\|_0^2 = \|\mathbf{E}_h^{n+1}\|_0^2 + \|\nabla\delta\psi_h^{n+1}\|_0^2. \tag{4.6}$$

We also point out that, owing to Lemma 2.2, $q_h^{n+1} \in \mathbb{P}_h$ defined in (1.6) satisfies

$$\left\| q_h^{n+1} - q_h^n \right\|_0 \le \|\nabla\widehat{\mathbf{E}}^{n+1}\|_0. \tag{4.7}$$

We now estimate the first order accuracy for velocity in Lemma 4.1, and then the 2nd order accuracy for time-derivative of velocity in Lemma 4.3. The result of Lemma 4.1 is instrumental to treat convection term in proof of Lemma 4.3. We will use the results in Lemmas 4.1 and 4.3 to prove optimal error decay in Lemma 4.4. Finally, we will prove pressure error estimate in Lemma 4.5.

Lemma 4.1 (Reduced rate of convergence for velocity). *Suppose the exact solution of* (1.1) *is smooth enough. If Assumptions 2 and 4-5 hold, then the velocity error functions satisfy*

(4.8)

$$\left\|\widehat{\mathbf{E}}_h^{N+1}\right\|_0^2 + \left\|\mathbf{E}_h^{N+1}\right\|_0^2 + \left\|2\mathbf{E}_h^{N+1} - \mathbf{E}_h^N\right\|_0^2 + \frac{1}{2}\left\|\delta\delta\mathbf{E}_h^{N+1}\right\|_0^2 + 2\mu\tau\left\|q_h^{N+1}\right\|_0^2$$
$$+\sum_{n=1}^{N}\left\|\nabla\delta\psi_h^{n+1}\right\|_0^2 + \mu\tau\sum_{n=1}^{N}\left\|\nabla\widehat{\mathbf{E}}_h^{n+1}\right\|_0^2 + \frac{4\tau^2}{3}\left\|\nabla\varepsilon_h^{N+1}\right\|_0^2 \le C\left(\tau^2 + h^2\right).$$

Proof. By virtue of Taylor expansion for the exact velocity $\mathbf{u}(t)$, we get

$$\begin{aligned} &\frac{3\mathbf{u}(t^{n+1}) - 4\mathbf{u}(t^n) + \mathbf{u}(t^{n-1})}{2\tau} + \left(\mathbf{u}(t^{n+1})\cdot\nabla\right)\mathbf{u}(t^{n+1}) \\ &\qquad +\nabla p(t^{n+1}) - \mu\triangle\mathbf{u}(t^{n+1}) = \mathbf{R}^{n+1} + \mathbf{f}(t^{n+1}), \end{aligned} \tag{4.9}$$

where $\mathbf{R}^{n+1} := \frac{1}{\tau}\int_{t^n}^{t^{n+1}} \mathbf{u}_{ttt}(s)(s-t^n)^2 ds - \frac{1}{4\tau}\int_{t^{n-1}}^{t^{n+1}} \mathbf{u}_{ttt}(s)(t^{n-1}-s)^2 ds$ is the truncation error. In conjunction with the definition of Stokes projection $\{\mathbf{U}_h^{n+1}, P_h^{n+1}\}$, testing (4.9) with $\forall\mathbf{w}_h \in \mathbb{V}_h$ yields

$$\begin{aligned} &\frac{1}{2\tau}\left\langle 3\mathbf{u}(t^{n+1}) - 4\mathbf{u}(t^n) + \mathbf{u}(t^{n-1})\,,\, \mathbf{w}_h\right\rangle + \left\langle \nabla P_h^{n+1}\,,\, \mathbf{w}_h\right\rangle \\ &\quad +\mu\left\langle \nabla\mathbf{U}_h^{n+1}\,,\, \nabla\mathbf{w}_h\right\rangle + \mathcal{N}\left(\mathbf{u}(t^{n+1})\,,\, \mathbf{u}(t^{n+1})\,,\, \mathbf{w}_h\right) \\ &\qquad = \left\langle \mathbf{R}^{n+1}\,,\, \mathbf{w}_h\right\rangle + \left\langle \mathbf{f}(t^{n+1})\,,\, \mathbf{w}_h\right\rangle, \end{aligned} \tag{4.10}$$

We replace p_h^n term in (1.3) by (1.7) and then subtract from (4.10) to obtain

$$\begin{aligned} &\frac{1}{2\tau}\left\langle 3\widehat{\mathbf{E}}^{n+1} - 4\mathbf{E}^n + \mathbf{E}^{n-1}\,,\, \mathbf{w}_h\right\rangle + \mu\tau\left\langle \nabla\widehat{\mathbf{E}}_h^{n+1}\,,\,\nabla\mathbf{w}_h\right\rangle \\ &= -\left\langle \nabla\left(\delta P_h^{n+1} + \varepsilon_h^n\right),\,\mathbf{w}_h\right\rangle + \mu\left\langle \nabla q_h^n\,,\,\mathbf{w}_h\right\rangle + \left\langle \mathbf{R}^{n+1}\,,\,\mathbf{w}_h\right\rangle \\ &\quad + \mathcal{N}\left(2\mathbf{u}_h^n - \mathbf{u}_h^{n-1}\,,\,\widehat{\mathbf{u}}_h^{n+1}\,,\,\mathbf{w}_h\right) - \mathcal{N}\left(\mathbf{u}(t^{n+1})\,,\,\mathbf{u}(t^{n+1})\,,\,\mathbf{w}_h\right). \end{aligned} \tag{4.11}$$

Choosing $\mathbf{w}_h = 4\tau\widehat{\mathbf{E}}_h^{n+1} = 4\tau\left(\widehat{\mathbf{E}}^{n+1} - \mathbf{G}^{n+1}\right)$ in (4.11) and using (2.9), we easily get

$$\begin{aligned} &\left\|\mathbf{E}_h^{n+1}\right\|_0^2 + \left\|2\mathbf{E}_h^{n+1} - \mathbf{E}_h^n\right\|_0^2 + \left\|\delta\delta\mathbf{E}_h^{n+1}\right\|_0^2 + 6\left\|\nabla\delta\psi_h^{n+1}\right\|_0^2 \\ &-\left\|\mathbf{E}_h^n\right\|_0^2 - \left\|2\mathbf{E}_h^n - \mathbf{E}_h^{n-1}\right\|_0^2 + 4\mu\tau\left\|\nabla\widehat{\mathbf{E}}_h^{n+1}\right\|_0^2 = \sum_{n=1}^{6} A_i, \end{aligned} \tag{4.12}$$

where

$$\begin{aligned} A_1 &:= -4\tau\mathcal{N}\left(\mathbf{u}(t^{n+1})\,,\,\mathbf{u}(t^{n+1})\,,\,\widehat{\mathbf{E}}_h^{n+1}\right) + 4\tau\mathcal{N}\left(2\mathbf{u}_h^n - \mathbf{u}_h^{n-1}\,,\,\widehat{\mathbf{u}}_h^{n+1}\,,\,\widehat{\mathbf{E}}_h^{n+1}\right), \\ A_2 &:= -2\left\langle 3\mathbf{G}^{n+1} - 4\mathbf{G}^n + \mathbf{G}^{n-1}\,,\,\widehat{\mathbf{E}}_h^{n+1}\right\rangle, \quad A_3 := -4\tau\left\langle \nabla\delta P_h^{n+1}\,,\,\widehat{\mathbf{E}}_h^{n+1}\right\rangle, \\ A_4 &:= -4\tau\left\langle \nabla\varepsilon_h^n\,,\,\widehat{\mathbf{E}}_h^{n+1}\right\rangle, \qquad A_5 := 4\mu\tau\left\langle \nabla q_h^n\,,\,\widehat{\mathbf{E}}_h^{n+1}\right\rangle, \\ A_6 &:= 4\tau\left\langle \mathbf{R}^{n+1}\,,\,\widehat{\mathbf{E}}_h^{n+1}\right\rangle. \end{aligned}$$

We now estimate terms A_1 to A_6 separately. To tackle A_1, we first add and subtract $2\mathbf{u}(t^n) - \mathbf{u}(t^{n-1})$ to obtain

$$\begin{aligned} A_1 = &-4\tau\mathcal{N}\left(\delta\delta\mathbf{u}(t^{n+1})\,,\,\mathbf{u}(t^{n+1})\,,\,\widehat{\mathbf{E}}_h^{n+1}\right) - 4\tau\mathcal{N}\left(2\mathbf{u}_h^n - \mathbf{u}_h^{n-1}\,,\,\widehat{\mathbf{E}}^{n+1}\,,\,\widehat{\mathbf{E}}_h^{n+1}\right) \\ &- 4\tau\mathcal{N}\left(2\mathbf{E}^n - \mathbf{E}^{n-1}\,,\,\mathbf{u}(t^{n+1})\,,\,\widehat{\mathbf{E}}_h^{n+1}\right). \end{aligned}$$

Because of $\mathcal{N}\left(2\mathbf{u}_h^n - \mathbf{u}_h^{n-1}\,,\,\widehat{\mathbf{E}}_h^{n+1}\,,\,\widehat{\mathbf{E}}_h^{n+1}\right) = 0$, which comes from (2.5), the second term of A_1 can be replaced by

$$4\tau\mathcal{N}\left(2\mathbf{E}^n - \mathbf{E}^{n-1} - 2\mathbf{u}(t^n) + \mathbf{u}(t^{n-1})\,,\,\mathbf{G}^{n+1}\,,\,\widehat{\mathbf{E}}_h^{n+1}\right).$$

If we apply Lemma 2.5, then we can readily obtain

$$\begin{aligned} A_1 &\le C\tau\left(\left\|\delta\delta\mathbf{u}(t^{n+1})\right\|_0\left\|\mathbf{u}(t^{n+1})\right\|_2 + \left\|2\mathbf{E}^n - \mathbf{E}^{n-1}\right\|_0\left\|\mathbf{u}(t^{n+1})\right\|_2\right)\left\|\widehat{\mathbf{E}}_h^{n+1}\right\|_1 \\ &+C\tau\left(\left\|2\mathbf{E}^n - \mathbf{E}^{n-1}\right\|_0\left|\!\left|\!\left|\mathbf{G}^{n+1}\right|\!\right|\!\right| + \left\|2\mathbf{u}(t^n) - \mathbf{u}(t^{n-1})\right\|_2\left\|\mathbf{G}^{n+1}\right\|_0\right)\left\|\widehat{\mathbf{E}}_h^{n+1}\right\|_1. \end{aligned}$$

Since we have $\left\|\mathbf{u}(t^{n+1})\right\|_2 + \left|\!\left|\!\left|\mathbf{G}^{n+1}\right|\!\right|\!\right| \le M$ according to (4.3), we arrive at

$$\begin{aligned} A_1 \le & C\tau \left(\left\|2\mathbf{E}_h^n - \mathbf{E}_h^{n-1}\right\|_0^2 + \left\|2\mathbf{G}^n - \mathbf{G}^{n-1}\right\|_0^2 + \left\|\mathbf{G}^{n+1}\right\|_0^2 \right) \\ & + \mu\tau \left\|\nabla \widehat{\mathbf{E}}_h^{n+1}\right\|_0^2 + \frac{C\tau^4}{\mu} \int_{t^{n-1}}^{t^{n+1}} \|\mathbf{u}_{ttt}(t)\|_0^2 dt. \end{aligned}$$

In light of (4.2), A_2 becomes

$$A_2 \le C\tau \left\|\widehat{\mathbf{E}}_h^{n+1}\right\|_0^2 + Ch^4 \int_{t^{n-1}}^{t^{n+1}} \left(\|\mathbf{u}_t(t)\|_2^2 + \|p_t(t)\|_1^2 \right) dt.$$

In order to estimate A_3 and A_4, we note $\left\langle \nabla\varepsilon_h^n \, , \, \mathbf{E}_h^{n+1} \right\rangle = 0$ and $\widehat{\mathbf{E}}_h^{n+1} = \mathbf{E}_h^{n+1} + \nabla\delta\psi_h^{n+1}$ according to (4.5) and (4.4), respectively. The we have

$$\begin{aligned} A_3 &= -4\tau \left\langle \nabla\delta P_h^{n+1} \, , \, \nabla\delta\psi_h^{n+1} \right\rangle \\ &\le \left\|\nabla\delta\psi_h^{n+1}\right\|_0^2 + C\tau^3 \int_{t^n}^{t^{n+1}} \left(\|\mathbf{u}_t(t)\|_2^2 + \|p_t(t)\|_1^2 \right) dt. \end{aligned}$$

In conjunction with the definition $\varepsilon_h^{n+1} = P_h^{n+1} + \frac{3\psi_h^{n+1}}{2\tau}$, A_4 can be evaluated by

$$\begin{aligned} A_4 &= -4\tau \left\langle \nabla\varepsilon_h^n \, , \, \nabla\delta\psi_h^{n+1} \right\rangle = -\frac{8\tau^2}{3} \left\langle \nabla\varepsilon_h^n \, , \, \nabla \left(\delta\varepsilon_h^{n+1} - \delta P_h^{n+1} \right) \right\rangle \\ &\le -\frac{4\tau^2}{3} \left(\left\|\nabla\varepsilon_h^{n+1}\right\|_0^2 - \|\nabla\varepsilon_h^n\|_0^2 - \left\|\nabla\delta\varepsilon_h^{n+1}\right\|_0^2 \right) \\ &\qquad + C\tau^3 \|\nabla\varepsilon_h^n\|_0^2 + C\tau \left\|\nabla\delta P_h^{n+1}\right\|_0^2. \end{aligned}$$

If we now apply inequality $(a+b)^2 \le 4a^2 + \frac{4}{3}b^2$, then we can get

$$\begin{aligned} \frac{4\tau^2}{3} \left\|\nabla\delta\varepsilon_h^{n+1}\right\|_0^2 &= \frac{4\tau^2}{3} \left\| \nabla\delta P_h^{n+1} + \frac{3}{2\tau}\nabla\delta\psi_h^{n+1} \right\|_0^2 \\ &\le C\tau^2 \left\|\nabla\delta P_h^{n+1}\right\|_0^2 + 4\left\|\nabla\delta\psi_h^{n+1}\right\|_0^2. \end{aligned}$$

So we arrive at

$$\begin{aligned} A_4 \le & -\frac{4\tau^2}{3} \left(\left\|\nabla\varepsilon_h^{n+1}\right\|_0^2 - \|\nabla\varepsilon_h^n\|_0^2 \right) + C\tau^3 \|\nabla\varepsilon_h^n\|_0^2 \\ & + 4\left\|\nabla\delta\psi_h^{n+1}\right\|_0^2 + C\tau^2 \int_{t^{n-1}}^{t^{n+1}} \left(\|\mathbf{u}_t(t)\|_2^2 + \|p_t(t)\|_1^2 \right) dt. \end{aligned} \tag{4.13}$$

In light of $\nabla \cdot \mathbf{u}(t^{n+1}) = 0$ and (2.11), (1.6) and (4.7) yield

$$\begin{aligned} A_5 &= 4\mu\tau \left\langle q_h^n \, , \, \nabla \cdot \widehat{\mathbf{u}}_h^{n+1} \right\rangle = -4\mu\tau \left\langle q_h^n \, , \, \delta q_h^{n+1} \right\rangle \\ &= -2\mu\tau \left(\left\|q_h^{n+1}\right\|_0^2 - \|q_h^n\|_0^2 - \left\|\delta q_h^{n+1}\right\|_0^2 \right) \\ &\le -2\mu\tau \left(\left\|q_h^{n+1}\right\|_0^2 - \|q_h^n\|_0^2 \right) + 2\mu\tau \left\|\nabla\widehat{\mathbf{E}}_h^{n+1}\right\|_0^2. \end{aligned}$$

Also we readily get

$$A_6 \le C\tau\left\|\widehat{\mathbf{E}}_h^{n+1}\right\|_0^2 + C\tau^4\int_{t_{n-1}}^{t^{n+1}}\|\mathbf{u}_{ttt}(t)\|_0^2 dt.$$

Replacing A_1-A_6 back into (4.12) and summing over n from 1 to N imply

$$\begin{aligned}
&\left\|\mathbf{E}_h^{N+1}\right\|_0^2 + \left\|2\mathbf{E}_h^{N+1} - \mathbf{E}_h^N\right\|_0^2 + \sum_{n=1}^N\left(\left\|\delta\delta\mathbf{E}_h^{n+1}\right\|_0^2 + \left\|\nabla\delta\psi_h^{n+1}\right\|_0^2\right)\\
&+ \frac{4\tau^2}{3}\left\|\nabla\varepsilon_h^{N+1}\right\|_0^2 + 2\mu\tau\left\|q_h^{N+1}\right\|_0^2 + \mu\tau\sum_{n=1}^N\left\|\nabla\widehat{\mathbf{E}}_h^{n+1}\right\|_0^2\\
&\le 2\mu\tau\left\|q_h^1\right\|_0^2 + \frac{4\tau^2}{3}\left\|\nabla\varepsilon_h^1\right\|_0^2 + \left\|2\mathbf{E}^1 - \mathbf{E}^0\right\|_0^2 + \left\|\mathbf{E}^1\right\|_0^2 + C\tau^3\sum_{n=1}^N\|\nabla\varepsilon_h^n\|_0^2\\
&+ C\tau\sum_{n=1}^N\left(\left\|\widehat{\mathbf{E}}_h^{n+1}\right\|_0^2 + \left\|2\mathbf{E}^n - \mathbf{E}^{n-1}\right\|_0^2 + \left\|2\mathbf{G}^n - \mathbf{G}^{n-1}\right\|_0^2 + \left\|\mathbf{G}^{n+1}\right\|_0^2\right)\\
&+ \frac{C\tau^4}{\mu}\int_{t^{n-1}}^{t^{n+1}}\|\mathbf{u}_{ttt}(t)\|_0^2 dt + C\left(\tau^2 + h^4\right)\int_{t^{n-1}}^{t^{n+1}}\left(\|\mathbf{u}_t(t)\|_2^2 + \|p_t(t)\|_1^2\right)dt.
\end{aligned}$$

In light of $\varepsilon_h^1 = P_h^1 + \frac{3}{2\tau}\psi_h^1 = e_h^1$, we obtain $\left\|\nabla\varepsilon_h^1\right\|_0^2 \le C$ and the first four terms in the right hand side can be bounded by Assumption 2 and properties $\mathbf{E}^0 = \mathbf{0}$ and $q^1 = 0$ which are directly deduced from the conditions in Algorithm 1. And the next terms can be treated by the discrete Gronwall lemma. Finally, in conjunction with (4.1) and (4.6), we arrive at (4.8) and complete this proof. □

REMARK 4.2 (Suboptimal order). *The suboptimal accuracy result in Lemma 4.1 is due to terms of A_3 and A_4 which come from parts of pressure in the above estimate. To improve upon this, we must get rid of the terms and so we will use duality argument in Lemma 4.4. However, this suboptimal result is essential to control convection term in proofs of next lemmas to get optimal order.*

In order to use in the error estimate for time-derivative of velocity in Lemma 4.3, we need to evaluate optimal initial errors for the case $n = 1$. To do this, we have to compute again (4.13) and we rewrite A_4 as

$$\begin{aligned}
A_4 &= -\frac{8\tau^2}{3}\left\langle\nabla\varepsilon_h^1, \nabla\left(\delta\varepsilon_h^2 - \delta P_h^2\right)\right\rangle\\
&\le -\frac{4\tau^2}{3}\left(\left\|\nabla\varepsilon_h^2\right\|_0^2 - \left\|\nabla\varepsilon_h^1\right\|_0^2 - \left\|\nabla\delta\varepsilon_h^2\right\|_0^2\right)\\
&\qquad + C\tau^2\left\|\nabla\varepsilon_h^1\right\|_0^2 + C\tau^2\left\|\nabla\delta P_h^{n+1}\right\|_0^2
\end{aligned}$$

and so we conclude

$$\begin{aligned}
A_4 \le &-\frac{4\tau^2}{3}\left\|\nabla\varepsilon_h^2\right\|_0^2 + C\tau^2\left\|\nabla\varepsilon_h^1\right\|_0^2 + 4\left\|\nabla\delta\psi_h^2\right\|_0^2\\
&+ C\tau^3\int_{t^0}^{t^2}\left(\|p_{tt}(t)\|_1^2 + \|\mathbf{u}_{tt}(t)\|_2^2\right)dt
\end{aligned}$$

In light of Assumption 2, we arrive at

$$\left\|\widehat{\mathbf{E}}_h^2\right\|_0^2 + \left\|\mathbf{E}_h^2\right\|_0^2 + \left\|2\mathbf{E}_h^2 - \mathbf{E}_h^1\right\|_0^2 + \frac{1}{2}\left\|\delta\delta\mathbf{E}_h^2\right\|_0^2 + \left\|\nabla\delta\psi_h^2\right\|_0^2 + \mu\tau\left\|\nabla\widehat{\mathbf{E}}_h^2\right\|_0^2 + 2\mu\tau\left\|q_h^2\right\|_0^2 + \frac{4\tau^2}{3}\left\|\nabla\varepsilon_h^2\right\|_0^2 \leq C\left(\tau^4 + h^4\right). \tag{4.14}$$

We now start to estimate errors for time-derivative of velocity.

Lemma 4.3 (Error estimate for time-derivative of velocity). *Suppose the exact solution of* (1.1) *is smooth enough and* $\tau = Ch$. *If Assumptions 2 and 4-5 hold, then the time derivative velocity error functions satisfy*

$$\begin{aligned}\left\|\delta\mathbf{E}_h^{N+1}\right\|_0^2 + \left\|\delta\widehat{\mathbf{E}}_h^{N+1}\right\|_0^2 + \left\|2\delta\mathbf{E}_h^{N+1} - \delta\mathbf{E}_h^N\right\|_0^2 + \frac{4\tau^2}{3}\left\|\nabla\delta\varepsilon_h^{N+1}\right\|_0^2 \\ + \sum_{n=2}^{N}\left(\left\|\delta\delta\delta\mathbf{E}_h^{n+1}\right\|_0^2 + \left\|\nabla\delta\delta\psi_h^{n+1}\right\|_0 + \mu\tau\left\|\nabla\delta\widehat{\mathbf{E}}_h^{n+1}\right\|_0^2\right) \\ +2\mu\tau\left\|\delta q_h^{N+1}\right\|_0^2 \leq C\tau^2\left(\tau^2 + h^2\right).\end{aligned} \tag{4.15}$$

Proof. Subtracting two consecutive formulas (4.11) and choosing by $\mathbf{w}_h = 4\tau\delta\widehat{\mathbf{E}}_h^{n+1}$ yield

$$\begin{aligned}\left\|\delta\mathbf{E}_h^{n+1}\right\|_0^2 + \left\|2\delta\mathbf{E}_h^{n+1} - \delta\mathbf{E}_h^n\right\|_0^2 + \left\|\delta\delta\delta\mathbf{E}_h^{n+1}\right\|_0^2 + 6\left\|\nabla\delta\delta\psi_h^{n+1}\right\|_0^2 \\ -\left\|\delta\mathbf{E}_h^n\right\|_0^2 - \left\|2\delta\mathbf{E}_h^n - \delta\mathbf{E}_h^{n-1}\right\|_0^2 + 4\mu\tau\left\|\nabla\delta\widehat{\mathbf{E}}_h^{n+1}\right\|_0^2 = \sum_{i=1}^{6} A_i,\end{aligned} \tag{4.16}$$

where

$$\begin{aligned}A_1 &:= -4\tau\mathcal{N}\left(\mathbf{u}(t^{n+1}), \mathbf{u}(t^{n+1}), \delta\widehat{\mathbf{E}}_h^{n+1}\right) + 4\tau\mathcal{N}\left(\mathbf{u}(t^n), \mathbf{u}(t^n), \delta\widehat{\mathbf{E}}_h^{n+1}\right) \\ &\quad + 4\tau\mathcal{N}\left(2\mathbf{u}_h^n - \mathbf{u}_h^{n-1}, \widehat{\mathbf{u}}_h^{n+1}, \delta\widehat{\mathbf{E}}_h^{n+1}\right) - 4\tau\mathcal{N}\left(2\mathbf{u}_h^{n-1} - \mathbf{u}_h^{n-2}, \widehat{\mathbf{u}}_h^n, \delta\widehat{\mathbf{E}}_h^{n+1}\right), \\ A_2 &:= -2\left\langle 3\delta\mathbf{G}^{n+1} - 4\delta\mathbf{G}^n + \delta\mathbf{G}^{n-1}, \delta\widehat{\mathbf{E}}_h^{n+1}\right\rangle, \\ A_3 &:= -4\tau\left\langle\nabla\delta\delta P_h^{n+1}, \delta\widehat{\mathbf{E}}_h^{n+1}\right\rangle, \quad A_4 := -4\tau\left\langle\nabla\delta\varepsilon_h^n, \delta\widehat{\mathbf{E}}_h^{n+1}\right\rangle, \\ A_5 &:= 4\mu\tau\left\langle\nabla\delta q_h^n, \delta\widehat{\mathbf{E}}_h^{n+1}\right\rangle, \qquad A_6 := 4\tau\left\langle\delta\mathbf{R}^{n+1}, \delta\widehat{\mathbf{E}}_h^{n+1}\right\rangle.\end{aligned}$$

We now estimate each term A_1 to A_6 separately. We first rewrite A_1 as follows:

$$\begin{aligned}A_1 =& 4\tau\mathcal{N}\left(\delta\delta\mathbf{u}(t^n), \mathbf{u}(t^n), \delta\widehat{\mathbf{E}}_h^{n+1}\right) - 4\tau\mathcal{N}\left(\delta\delta\mathbf{u}(t^{n+1}), \mathbf{u}(t^{n+1}), \delta\widehat{\mathbf{E}}_h^{n+1}\right) \\ &+ 4\tau\mathcal{N}\left(2\mathbf{E}^{n-1} - \mathbf{E}^{n-2}, \mathbf{u}(t^n), \delta\widehat{\mathbf{E}}_h^{n+1}\right) \\ &- 4\tau\mathcal{N}\left(2\mathbf{E}^n - \mathbf{E}^{n-1}, \mathbf{u}(t^{n+1}), \delta\widehat{\mathbf{E}}_h^{n+1}\right) \\ &+ 4\tau\mathcal{N}\left(2\mathbf{u}_h^{n-1} - \mathbf{u}_h^{n-2}, \mathbf{G}^n + \widehat{\mathbf{E}}_h^n, \delta\widehat{\mathbf{E}}_h^{n+1}\right) \\ &- 4\tau\mathcal{N}\left(2\mathbf{u}_h^n - \mathbf{u}_h^{n-1}, \mathbf{G}^{n+1} + \widehat{\mathbf{E}}_h^{n+1}, \delta\widehat{\mathbf{E}}_h^{n+1}\right),\end{aligned}$$

and we denote by $A_{1,i}$, for $i = 1, 2, \cdots, 6$ the six terms in the right hand side. In estimating convection terms, we will use Lemma 2.5 frequently without notice. We recall $\|\mathbf{u}(t)\|_2 \le C$ to obtain

$$\begin{aligned} A_{1,1} + A_{1,2} \le & C\tau \left(\left\| \delta\delta\mathbf{u}(t^{n+1}) \right\|_0 \left\| \mathbf{u}(t^{n+1}) \right\|_2 + \|\delta\delta\mathbf{u}(t^n)\|_0 \|\mathbf{u}(t^n)\|_2 \right) \left\| \delta\widehat{\mathbf{E}}_h^{n+1} \right\|_1 \\ \le & \frac{\mu\tau}{6} \left\| \nabla\delta\widehat{\mathbf{E}}_h^{n+1} \right\|_0^2 + \frac{C\tau^4}{\mu} \int_{t^{n-2}}^{t^{n+1}} \|\mathbf{u}_{tt}(t)\|_0^2 dt. \end{aligned}$$

The result in Lemma 4.1, $\left\| 2\mathbf{E}^n - \mathbf{E}^{n-1} \right\|_0 \le C(\tau + h)$, is essential to treat next two convection terms. Invoking (2.7), we have

$$\begin{aligned} A_{1,3} + A_{1,4} \le & C\tau \left\| 2\mathbf{E}^n - \mathbf{E}^{n-1} \right\|_0 \left\| \delta\mathbf{u}(t^{n+1}) \right\|_2 \left\| \delta\widehat{\mathbf{E}}_h^{n+1} \right\|_1 \\ & + C\tau \left\| 2\delta\mathbf{E}^n - \delta\mathbf{E}^{n-1} \right\|_0 \|\mathbf{u}(t^n)\|_2 \left\| \delta\widehat{\mathbf{E}}_h^{n+1} \right\|_1 \\ \le & \frac{\mu\tau}{6} \left\| \nabla\delta\widehat{\mathbf{E}}_h^{n+1} \right\|_0^2 + \frac{C\tau}{\mu} \left\| 2\delta\mathbf{E}^n - \delta\mathbf{E}^{n-1} \right\|_0^2 + \frac{C\tau^2 \left(\tau^2 + h^2\right)}{\mu} \int_{t_{n-1}}^{n} \|\mathbf{u}_t(t)\|_2^2 dt. \end{aligned}$$

We note $\mathcal{N}\left(2\mathbf{u}_h^n - \mathbf{u}_h^{n-1}, \delta\widehat{\mathbf{E}}_h^{n+1}, \delta\widehat{\mathbf{E}}_h^{n+1}\right) = 0$ which comes from (2.5). Then we obtain

$$\begin{aligned} A_{1,5} + A_{1,6} = & -4\tau\mathcal{N}\left(2\delta\mathbf{u}_h^n - \delta\mathbf{u}_h^{n-1}, \mathbf{G}^{n+1} + \widehat{\mathbf{E}}_h^{n+1}, \delta\widehat{\mathbf{E}}_h^{n+1}\right) \\ & - 4\tau\mathcal{N}\left(2\mathbf{u}_h^{n-1} - \mathbf{u}_h^{n-2}, \delta\mathbf{G}^{n+1}, \delta\widehat{\mathbf{E}}_h^{n+1}\right) \\ = & 4\tau\mathcal{N}\left(2\delta\mathbf{E}^n - \delta\mathbf{E}^{n-1} - 2\delta\mathbf{u}(t^n) + \delta\mathbf{u}(t^{n-1}), \mathbf{G}^{n+1} + \widehat{\mathbf{E}}_h^{n+1}, \delta\widehat{\mathbf{E}}_h^{n+1}\right) \\ & + 4\tau\mathcal{N}\left(2\mathbf{E}^{n-1} - \mathbf{E}^{n-2} - 2\mathbf{u}(t^{n-1}) + \mathbf{u}(t^{n-2}), \delta\mathbf{G}^{n+1}, \delta\widehat{\mathbf{E}}_h^{n+1}\right) := B_1 + B_2. \end{aligned}$$

To attack B_1, we first note Lemma 2.1 which is, for any $\mathbf{w}_h \in \mathbb{V}_h$, $\|\mathbf{w}_h\|_{\mathbf{L}^3(\Omega)} \le Ch^{-d/6}\|\mathbf{w}_h\|_0$.
If we apply $\left\| \widehat{\mathbf{E}}^{n+1} + \mathbf{G}^{n+1} \right\|_0 + \sqrt{\tau + h} \left\| \widehat{\mathbf{E}}^{n+1} + \mathbf{G}^{n+1} \right\|_1 \le C(\tau + h)$ which is the result of Lemma 4.1, then we can conclude, in light of (2.8),

$$\begin{aligned} B_1 \le & C\tau \left\| 2\delta\mathbf{E}^n - \delta\mathbf{E}^{n-1} \right\|_{\mathbf{L}^3(\Omega)} \left\| \widehat{\mathbf{E}}_h^{n+1} + \mathbf{G}^{n+1} \right\|_1 \left\| \nabla\delta\widehat{\mathbf{E}}_h^{n+1} \right\|_0 \\ & + C\tau \left\| 2\delta\mathbf{u}(t^n) - \delta\mathbf{u}(t^{n-1}) \right\|_2 \left\| \widehat{\mathbf{E}}_h^{n+1} + \mathbf{G}^{n+1} \right\|_0 \left\| \nabla\delta\widehat{\mathbf{E}}_h^{n+1} \right\|_0 \\ \le & \frac{\mu\tau}{6} \left\| \nabla\delta\widehat{\mathbf{E}}_h^{n+1} \right\|_0^2 + \frac{C\tau}{\mu}\left(1 + \frac{\tau}{h}\right) \left\| 2\delta\mathbf{E}^n - \delta\mathbf{E}^{n-1} \right\|_0^2 \\ & + \frac{C\tau^2}{\mu}(\tau + h)^2 \int_{t^{n-2}}^{t^{n+1}} \|\mathbf{u}_t(t)\|_2^2 dt. \end{aligned}$$

We now estimate B_2 using $\|2\mathbf{E}^{n-1} - \mathbf{E}^{n-2}\|_{\mathbf{L}^3(\Omega)} \leq Ch^{-d/6}\|2\mathbf{E}^{n-1} - \mathbf{E}^{n-2}\|_0 \leq M$.

$$\begin{aligned}
B_2 \leq & C\tau\|2\mathbf{E}^{n-1} - \mathbf{E}^{n-2}\|_{\mathbf{L}^3(\Omega)}\|\delta\mathbf{G}^{n+1}\|_1\left\|\nabla\delta\widehat{\mathbf{E}}_h^{n+1}\right\|_0 \\
& + C\tau\|2\mathbf{u}(t^{n-1}) - \mathbf{u}(t^{n-2})\|_1\|\delta\mathbf{G}^{n+1}\|_1\left\|\nabla\delta\widehat{\mathbf{E}}_h^{n+1}\right\|_0 \\
\leq & C\tau\|\delta\mathbf{G}^{n+1}\|_1^2 + \frac{\mu\tau}{6}\left\|\nabla\delta\widehat{\mathbf{E}}_h^{n+1}\right\|_0^2 \\
\leq & \frac{\mu\tau}{6}\left\|\nabla\delta\widehat{\mathbf{E}}_h^{n+1}\right\|_0^2 + C\tau^2h^2\int_{t^{n-1}}^{t^{n+1}}\left(\|\mathbf{u}_t(t)\|_2^2 + \|p_t(t)\|_1^2\right)dt.
\end{aligned}$$

In light of Hölder inequality, (4.2) yields

$$\begin{aligned}
A_2 = & -2\left\langle 3\delta\mathbf{G}^{n+1} - 4\delta\mathbf{G}^n + \delta\mathbf{G}^{n-1}\,,\, \delta\widehat{\mathbf{E}}_h^{n+1}\right\rangle \\
\leq & \frac{\mu\tau}{6}\|\nabla\delta\mathbf{E}_h^{n+1}\|_0^2 + Ch^4\int_{t^{n-2}}^{t^{n+1}}\left(\|\mathbf{u}_t(t)\|_2^2 + \|p_t(t)\|_1^2\right)dt.
\end{aligned}$$

Integral by parts leads

$$\begin{aligned}
A_3 = & 4\tau\left\langle \delta\delta P_h^{n+1}\,,\, \nabla\cdot\delta\widehat{\mathbf{E}}_h^{n+1}\right\rangle \\
\leq & \frac{\mu\tau}{6}\left\|\nabla\delta\widehat{\mathbf{E}}_h^{n+1}\right\|_0^2 + \frac{C\tau^4}{\mu}\int_{t^{n-1}}^{t^{n+1}}\left(\|\mathbf{u}_{tt}(t)\|_2^2 + \|p_{tt}(t)\|_1^2\right)dt.
\end{aligned}$$

In order to tackle A_4, marking use of $\delta\widehat{\mathbf{E}}_h^{n+1} = \delta\mathbf{E}_h^{n+1} + \nabla\delta\delta\psi_h^{n+1}$. We readily get

$$\begin{aligned}
A_4 &= -4\tau\left\langle \nabla\delta\varepsilon_h^n\,,\, \delta\mathbf{E}_h^{n+1} + \nabla\delta\delta\psi_h^{n+1}\right\rangle \\
&= -\tfrac{8\tau^2}{3}\left\langle \nabla\delta\varepsilon_h^n\,,\, \nabla\left(\delta\delta\varepsilon_h^{n+1} - \delta\delta P_h^{n+1}\right)\right\rangle \\
&\leq -\tfrac{4\tau^2}{3}\left(\|\nabla\delta\varepsilon_h^{n+1}\|_0^2 - \|\nabla\delta\varepsilon_h^n\|_0^2 - \|\nabla\delta\delta\varepsilon_h^{n+1}\|_0^2\right) \\
&\quad + C\tau^3\|\nabla\delta\varepsilon_h^n\|_0^2 + C\tau\|\nabla\delta\delta P_h^{n+1}\|_0^2.
\end{aligned}$$

If we now apply inequality $(a+b)^2 \leq 4a^2 + \frac{4}{3}b^2$, then we can have

$$\begin{aligned}
\frac{4\tau^2}{3}\|\nabla\delta\delta\varepsilon_h^{n+1}\|_0^2 = & \frac{4\tau^2}{3}\left\|\nabla\delta\delta P_h^{n+1} + \frac{3}{2\tau}\nabla\delta\delta\psi_h^{n+1}\right\|_0^2 \\
& \leq C\tau^2\|\nabla\delta\delta P_h^{n+1}\|_0^2 + 4\|\nabla\delta\delta\psi_h^{n+1}\|_0^2.
\end{aligned}$$

So we arrive at

$$\begin{aligned}
A_4 \leq & -\frac{4\tau^2}{3}\left(\|\nabla\delta\varepsilon_h^{n+1}\|_0^2 - \|\nabla\delta\varepsilon_h^n\|_0^2\right) + 4\|\nabla\delta\delta\psi_h^{n+1}\|_0^2 \\
& + C\tau^3\|\nabla\delta\varepsilon_h^n\|_0^2 + C\tau^4\int_{t^{n-1}}^{t^{n+1}}\left(\|\mathbf{u}_{tt}(t)\|_2^2 + \|p_{tt}(t)\|_1^2\right)dt.
\end{aligned}$$

Invoking (4.7) and (2.11), (1.6) leads

$$\begin{aligned} A_5 &= -4\mu\tau \left\langle \delta q_h^n \,,\, \delta\delta q_h^{n+1} \right\rangle \\ &= -2\mu\tau \left(\left\| \delta q_h^{n+1} \right\|_0^2 - \left\| \delta q_h^n \right\|_0^2 - \left\| \delta\delta q_h^{n+1} \right\|_0^2 \right) \\ &\le -2\mu\tau \left(\left\| \delta q_h^{n+1} \right\|_0^2 - \left\| \delta q_h^n \right\|_0^2 \right) + 2\mu\tau \left\| \nabla \delta \widehat{\mathbf{E}}_h^{n+1} \right\|_0^2 . \end{aligned}$$

Finally, A_6 term becomes

$$\begin{aligned} A_6 &\le C\tau \left\| \delta \mathbf{R}^{n+1} \right\|_0 \left\| \delta \widehat{\mathbf{E}}_h^{n+1} \right\|_0 \\ &\le C\tau \left\| \delta \widehat{\mathbf{E}}_h^{n+1} \right\|_0^2 + C\tau^4 \int_{t^{n-1}}^{t^{n+1}} \left\| \mathbf{u}_{ttt}(t) \right\|_0^2 dt. \end{aligned}$$

Inserting above estimates into (4.16) and summing for n from 2 to N yield

$$\begin{aligned} &\left\| \delta \mathbf{E}_h^{N+1} \right\|_0^2 + \left\| 2\delta \mathbf{E}_h^{N+1} - \delta \mathbf{E}_h^N \right\|_0^2 + \sum_{n=2}^{N} \left(\left\| \delta\delta\delta \mathbf{E}_h^{N+1} \right\|_0^2 + \left\| \nabla \delta\delta \psi_h^{n+1} \right\|_0 \right) \\ &+ \mu\tau \sum_{n=2}^{N} \left\| \nabla \delta \widehat{\mathbf{E}}_h^{n+1} \right\|_0^2 + \frac{4\tau^2}{3} \left\| \nabla \delta \varepsilon_h^{N+1} \right\|_0^2 + 2\mu\tau \left\| \delta q_h^{N+1} \right\|_0^2 \le \left\| \delta \mathbf{E}_h^2 \right\|_0^2 \\ &+ \left\| 2\delta \mathbf{E}_h^2 - \delta \mathbf{E}_h^1 \right\|_0^2 + \frac{4\tau^2}{3} \left\| \nabla \delta \varepsilon_h^2 \right\|_0^2 + 2\mu\tau \left\| \delta q_h^2 \right\|_0^2 + C\tau^2 \sum_{n=2}^{N} \left\| \nabla \delta \widehat{\mathbf{E}}_h^n \right\|_0^2 \\ &+ \frac{C\tau}{\mu} \left(1 + \frac{\tau}{h} \right) \sum_{n=2}^{N} \left\| 2\delta \mathbf{E}_h^n - \delta \mathbf{E}_h^{n-1} \right\|_0^2 + C\tau^3 \sum_{n=2}^{N} \left\| \nabla \delta \varepsilon_h^n \right\|_0^2 \\ &+ C \frac{\left(\tau^2 + h^2 \right)^2}{\mu} \int_0^{t^{N+1}} \left(\left\| \mathbf{u}_{ttt} \right\|_0^2 + \left\| \mathbf{u}_{tt}(t) \right\|_2^2 + \left\| \mathbf{u}_t \right\|_2^2 + \left\| p_{tt}(t) \right\|_1^2 + \left\| p_t(t) \right\|_1^2 \right) dt. \end{aligned}$$

We note here $\tau^2 \sum_{n=2}^{N} \left\| \nabla \delta \widehat{\mathbf{E}}_h^n \right\|_0^2$ can be removed by cancellation with the term $\tau \sum_{n=2}^{N} \left\| \nabla \delta \widehat{\mathbf{E}}_h^n \right\|_0^2$ on the left hand side, provided τ is small enough. We need assumption $\tau = Ch$ to impose $\frac{\tau}{h} = C$. If we apply Gronwall inequality and then use (4.14), then we arrive at (4.15) and complete the proof. □

In order to extend to optimal accuracy, we use duality argument with the Stokes equations

$$\begin{aligned} -\triangle \mathbf{v}^{n+1} + \nabla r^{n+1} &= \mathbf{E}_h^{n+1}, \text{ in } \Omega, \\ \nabla \cdot \mathbf{v}^{n+1} &= 0, \qquad \text{on } \partial\Omega, \end{aligned} \tag{4.17}$$

with vanishing boundary condition $\mathbf{v} = \mathbf{0}$. And let $\{\mathbf{v}_h^{n+1}, r_h^{n+1}\} \in \mathbb{V}_h \times \mathbb{P}_h$ be the solution of the weak form of (4.17),

$$\begin{aligned} \left\langle \nabla \mathbf{v}_h^{n+1} \,,\, \nabla \mathbf{w}_h \right\rangle + \left\langle \nabla r_h^{n+1} \,,\, \mathbf{w}_h \right\rangle &= \left\langle \mathbf{E}_h^{n+1} \,,\, \mathbf{w}_h \right\rangle, \quad \forall \mathbf{w}_h \in \mathbb{V}_h \\ \left\langle \nabla \cdot \mathbf{v}_h^{n+1} \,,\, \phi_h \right\rangle &= 0, \qquad \forall \phi_h \in \mathbb{P}_h. \end{aligned} \tag{4.18}$$

According to Assumption 1, we have

$$\left\|\mathbf{v}^{n+1}\right\|_2 + \left\|r^{n+1}\right\|_1 \leq \left\|\mathbf{E}^{n+1}\right\|_0 \tag{4.19}$$

and so Lemma 2.3 yields

$$\begin{aligned} \left\|\mathbf{v}^{n+1} - \mathbf{v}_h^{n+1}\right\|_0 + h\left\|\mathbf{v}^{n+1} - \mathbf{v}_h^{n+1}\right\|_1 + h\left\|r^{n+1} - r_h^{n+1}\right\|_0 \leq h^2\left\|\mathbf{E}^{n+1}\right\|_0, \\ \left\|\left|\mathbf{v}^{n+1} - \mathbf{v}_h^{n+1}\right|\right\| \leq \left\|\mathbf{E}^{n+1}\right\|_0. \end{aligned} \tag{4.20}$$

In order to use in proof of next lemma, we derive 2 equations by considering for the case $\mathbf{w}_h = \mathbf{v}_h^{n+1}$ in (4.18)

$$\begin{aligned} \left\langle 3\widehat{\mathbf{E}}^{n+1} - 4\mathbf{E}^n + \mathbf{E}^{n-1}\,,\, \mathbf{v}_h^{n+1}\right\rangle &= \left\langle 3\mathbf{G}^{n+1} - 4\mathbf{G}^n + \mathbf{G}^{n-1}\,,\, \mathbf{v}_h^{n+1}\right\rangle \\ &\quad + \left\langle 3\nabla\mathbf{v}_h^{n+1} - 4\nabla\mathbf{v}_h^n + \nabla\mathbf{v}_h^{n-1}\,,\, \nabla\mathbf{v}_h^{n+1}\right\rangle \end{aligned} \tag{4.21}$$

and by choosing $\mathbf{w}_h = \widehat{\mathbf{E}}_h^{n+1}$ in (4.18)

$$\left\langle \nabla\widehat{\mathbf{E}}_h^{n+1}\,,\, \nabla\mathbf{v}_h^{n+1}\right\rangle = \left\langle \widehat{\mathbf{E}}_h^{n+1}\,,\, \mathbf{E}_h^{n+1}\right\rangle - \left\langle \nabla r_h^{n+1}\,,\, \widehat{\mathbf{E}}_h^{n+1}\right\rangle. \tag{4.22}$$

Lemma 4.4 (Full rate of convergence for velocity). *Let the pairs* $(\mathbf{v}^{n+1}, r^{n+1})$ *and* $(\mathbf{v}_h^{n+1}, r_h^{n+1})$ *be the solutions of* (4.17) *and* (4.18), *respectively. Let the exact solution of* (1.1) *is smooth enough and* $\tau = Ch$. *If Assumptions 1-2 and 4-5 hold, then we have*

$$\begin{aligned} \left\|\nabla\mathbf{v}_h^{N+1}\right\|_0^2 + \left\|\nabla\left(2\mathbf{v}_h^{N+1} - \mathbf{v}_h^N\right)\right\|_0^2 + \sum_{n=1}^{N}\left\|\nabla\delta\delta\mathbf{v}_h^{n+1}\right\|_0^2 \\ + 2\mu\tau\sum_{n=1}^{N}\left\|\mathbf{E}_h^{n+1}\right\|_0^2 \leq C\left(\tau^4 + h^4\right). \end{aligned} \tag{4.23}$$

Proof. We choose $\mathbf{w}_h = 4\tau\mathbf{v}_h^{n+1} \in \mathbb{V}_h$ in (4.11) and then we apply (4.21) and (4.22) to obtain

$$\begin{aligned} \left\|\nabla\mathbf{v}_h^{n+1}\right\|_0^2 + \left\|\nabla\left(2\mathbf{v}_h^{n+1} - \mathbf{v}_h^n\right)\right\|_0^2 + \left\|\nabla\delta\delta\mathbf{v}_h^{n+1}\right\|_0^2 - \left\|\nabla\mathbf{v}_h^n\right\|_0^2 \\ - \left\|\nabla\left(2\mathbf{v}_h^n - \mathbf{v}_h^{n-1}\right)\right\|_0^2 + 4\mu\tau\left\|\mathbf{E}_h^{n+1}\right\|_0^2 = \sum_{i=1}^{4} A_i, \end{aligned} \tag{4.24}$$

where

$$A_1 := 4\tau\mathcal{N}\left(2\mathbf{u}_h^n - \mathbf{u}_h^{n-1}\,,\, \widehat{\mathbf{u}}_h^{n+1}\,,\, \mathbf{v}_h^{n+1}\right) - 4\tau\mathcal{N}\left(\mathbf{u}(t^{n+1})\,,\, \mathbf{u}(t^{n+1})\,,\, \mathbf{v}_h^{n+1}\right),$$

$$A_2 := -2\left\langle 3\mathbf{G}^{n+1} - 4\mathbf{G}^n + \mathbf{G}^{n-1}\,,\, \mathbf{v}_h^{n+1}\right\rangle, \qquad A_3 := 4\mu\tau\left\langle \widehat{\mathbf{E}}_h^{n+1}\,,\, \nabla r_h^{n+1}\right\rangle,$$

$$A_4 := 4\tau\left\langle \mathbf{R}^{n+1}\,,\, \mathbf{v}_h^{n+1}\right\rangle,$$

We now estimate A_1 to A_4 separately. The convection term A_1 can be rewritten as follows:

$$A_1 = -4\tau\mathcal{N}\left(\delta\delta\mathbf{u}(t^{n+1}),\, \mathbf{u}(t^{n+1}),\, \mathbf{v}_h^{n+1}\right) + 4\tau\mathcal{N}\left(2\mathbf{E}^n - \mathbf{E}^{n-1},\, \widehat{\mathbf{E}}^{n+1},\, \mathbf{v}_h^{n+1}\right)$$
$$- 4\tau\mathcal{N}\left(2\mathbf{u}(t^n) - \mathbf{u}(t^{n-1}),\, \widehat{\mathbf{E}}^{n+1},\, \mathbf{v}_h^{n+1}\right)$$
$$- 4\tau\mathcal{N}\left(2\mathbf{E}^n - \mathbf{E}^{n-1},\, \mathbf{u}(t^{n+1}),\, \mathbf{v}_h^{n+1}\right) = \sum_{i=1}^{4} A_{1,i}.$$

To estimate convection terms, we will use frequently Lemma 2.5 without notice. Using $\left\|\mathbf{u}(t^{n+1})\right\|_2 \le M$, we can readily get

$$\begin{aligned} A_{1,1} \le& C\tau\left\|\delta\delta\mathbf{u}(t^{n+1})\right\|_0\left\|\mathbf{u}(t^{n+1})\right\|_2\left\|\mathbf{v}_h^{n+1}\right\|_1 \\ \le& C\tau\left\|\nabla\mathbf{v}_h^{n+1}\right\|_0^2 + C\mu\tau^4\int_{t^{n-1}}^{t^{n+1}} \|\mathbf{u}_{tt}(t)\|_0^2 dt \end{aligned}$$

and

$$\begin{aligned} A_{1,4} \le& C\tau\left(\|\mathbf{E}^n\|_0 + \|\delta\mathbf{E}^n\|_0\right)\left\|\mathbf{u}(t^{n+1})\right\|_2\left\|\mathbf{v}_h^{n+1}\right\|_1 \\ \le& \frac{\mu\tau}{2}\left(\|\mathbf{E}_h^n\|_0^2 + \|\mathbf{G}^n\|_0^2 + \|\delta\mathbf{E}_h^n\|_0^2 + \|\delta\mathbf{G}^n\|_0^2\right) + \frac{C\tau}{\mu}\left\|\nabla\mathbf{v}_h^{n+1}\right\|_0^2. \end{aligned}$$

Because $\nabla\cdot\left(2\mathbf{u}(t^n) - \mathbf{u}(t^{n-1})\right) = 0$ and $2\mathbf{u}(t^n) - \mathbf{u}(t^{n-1}) = \mathbf{0}$ on boundary, we can use (2.6) and so we get

$$\begin{aligned} A_{1,3} \le& C\tau\left\|2\mathbf{u}(t^n) - \mathbf{u}(t^{n-1})\right\|_2\left\|\widehat{\mathbf{E}}^{n+1}\right\|_0\left\|\mathbf{v}_h^{n+1}\right\|_1 \\ \le& \frac{\mu\tau}{2}\left(\left\|\mathbf{E}_h^{n+1}\right\|_0^2 + \left\|\mathbf{G}^{n+1}\right\|_0^2 + \left\|\nabla\delta\psi_h^{n+1}\right\|_0^2\right) + \frac{C\tau}{\mu}\left\|\nabla\mathbf{v}_h^{n+1}\right\|_0^2. \end{aligned}$$

We now apply

$$\begin{aligned} \left\|\nabla\delta\psi_h^{n+1}\right\|_0^2 =& \frac{4\tau^2}{9}\left\|\nabla\left(\delta\varepsilon_h^{n+1} - \delta P_h^{n+1}\right)\right\|_0^2 \\ \le& C\tau^2\left\|\nabla\delta\varepsilon_h^{n+1}\right\|_0^2 + C\tau^3\int_{t^n}^{t^{n+1}}\left(\|\mathbf{u}_t(t)\|_2^2 + \|p_t(t)\|_1^2\right)dt, \end{aligned} \tag{4.25}$$

to derive

$$\begin{aligned} A_{1,3} \le& \frac{\mu\tau}{2}\left(\left\|\mathbf{E}_h^{n+1}\right\|_0^2 + \left\|\mathbf{G}^{n+1}\right\|_0^2\right) + C\mu\tau^3\left\|\nabla\delta\varepsilon_h^{n+1}\right\|_0^2 \\ &+ C\mu\tau^4\int_{t^n}^{t^{n+1}}\left(\|\mathbf{u}_t(t)\|_2^2 + \|p_t(t)\|_1^2\right)dt + \frac{C\tau}{\mu}\left\|\nabla\mathbf{v}_h^{n+1}\right\|_0^2. \end{aligned}$$

If we apply $\left\|2\mathbf{E}^n - \mathbf{E}^{n-1}\right\|_{\mathbf{L}^3(\Omega)} \le Ch^{-d/6}\left\|2\mathbf{E}^n - \mathbf{E}^{n-1}\right\|_0 \le Ch^{-d/6}\left(\tau + h\right)$ which derives from Lemmas 2.1 and 4.1, then we can derive, by using Lemma 2.5 and

(4.19)-(4.20),

$$
\begin{aligned}
A_{1,2} \leq & C\tau\left\|2\mathbf{E}^n - \mathbf{E}^{n-1}\right\|_0 \left\|\widehat{\mathbf{E}}^{n+1}\right\|_1 \left\|\mathbf{v}^{n+1}\right\|_2 \\
& + C\tau\left\|2\mathbf{E}^n - \mathbf{E}^{n-1}\right\|_{\mathbf{L}^3(\Omega)} \left\|\widehat{\mathbf{E}}^{n+1}\right\|_1 \left\|\mathbf{v}^{n+1} - \mathbf{v}_h^{n+1}\right\|_1 \\
\leq & C\tau\left(\tau + h\right)\left\|\widehat{\mathbf{E}}^{n+1}\right\|_1 \left\|\mathbf{v}^{n+1}\right\|_2 + C\tau h\left\|\widehat{\mathbf{E}}^{n+1}\right\|_1 \left\|\mathbf{v}^{n+1}\right\|_2 \\
\leq & \frac{C}{\mu}\tau\left(\tau + h\right)^2\left(\left\|\nabla\widehat{\mathbf{E}}_h^{n+1}\right\|_0^2 + \left\|\nabla\mathbf{G}^{n+1}\right\|_0^2\right) + \frac{\mu\tau}{2}\left\|\mathbf{E}_h^{n+1}\right\|_0^2.
\end{aligned}
$$

In conjunction with (4.2), we can have

$$
\begin{aligned}
A_2 & \leq \frac{C}{\tau}\left(\left\|\delta\mathbf{G}^{n+1}\right\|_0^2 + \left\|\delta\mathbf{G}^n\right\|_0^2\right) + C\tau\left\|\mathbf{v}_h^{n+1}\right\|_0^2 \\
& \leq C\tau\left\|\mathbf{v}_h^{n+1}\right\|_0^2 + Ch^4\int_{t^{n-1}}^{t^{n+1}}\left(\left\|\mathbf{u}_t(t)\right\|_2^2 + \left\|p_t(t)\right\|_1^2\right)dt.
\end{aligned}
$$

The definition of $\widehat{\mathbf{E}}_h^{n+1} = \mathbf{E}_h^{n+1} + \nabla\delta\psi_h^{n+1}$ and (4.19) gives us

$$
A_3 = 4\mu\tau\left\langle\nabla\delta\psi_h^{n+1}, \nabla r_h^{n+1}\right\rangle \leq \mu\tau\left\|\mathbf{E}_h^{n+1}\right\|_0^2 + \frac{C\tau}{\mu}\left\|\nabla\delta\psi_h^{n+1}\right\|_0^2.
$$

If we apply (4.25) again, then we arrive at

$$
A_3 \leq \mu\tau\left\|\mathbf{E}_h^{n+1}\right\|_0^2 + \frac{C\tau^3}{\mu}\left\|\nabla\delta\varepsilon_h^{n+1}\right\|_0^2 + \frac{C\tau^4}{\mu}\int_{t^n}^{t^{n+1}}\left\|\nabla p_t(t)\right\|_0^2 dt.
$$

On the other hand, the truncation error term becomes

$$
A_4 = 4\tau\left\langle\mathbf{R}^{n+1}, \mathbf{v}_h^{n+1}\right\rangle \leq C\tau\left\|\nabla\mathbf{v}_h^{n+1}\right\|_0^2 + C\tau^4\int_{t^{n-1}}^{t^{n+1}}\left\|\mathbf{u}_{ttt}(s)\right\|_0^2 dt.
$$

Invoking $\mathbf{v}^0 = \mathbf{0}$, inserting above estimates from A_1 and A_4 into (4.24) and summing over n from 1 to N give us

$$
\begin{aligned}
& \left\|\nabla\mathbf{v}_h^{N+1}\right\|_0^2 + \left\|\nabla\left(2\mathbf{v}_h^{N+1} - \mathbf{v}_h^N\right)\right\|_0^2 + \sum_{n=1}^{N}\left\|\nabla\delta\delta\mathbf{v}_h^{n+1}\right\|_0^2 + \mu\tau\sum_{n=1}^{N}\left\|\mathbf{E}_h^{n+1}\right\|_0^2 \\
& \leq 5\left\|\nabla\mathbf{v}_h^1\right\|_0^2 + C\tau\left(\tau + h\right)^2\sum_{n=1}^{N}\left(\left\|\nabla\widehat{\mathbf{E}}_h^{n+1}\right\|_0^2 + \left\|\nabla\mathbf{G}^{n+1}\right\|_0^2 + \left\|\nabla\delta\varepsilon_h^{n+1}\right\|_0^2\right) \\
& + C\tau\sum_{n=1}^{N}\left\|\nabla\mathbf{v}_h^{n+1}\right\|_0^2 \\
& + C\mu\tau\sum_{n=1}^{N}\left(\left\|\mathbf{E}_h^{n+1}\right\|_0^2 + \left\|\delta\mathbf{E}_h^{n+1}\right\|_0^2 + \left\|\mathbf{G}^{n+1}\right\|_0^2 + \left\|\delta\mathbf{G}^n\right\|_0^2\right) \\
& + C\mu\left(\tau^4 + h^4\right)\int_0^{t^{N+1}}\left(\left\|\mathbf{u}_{ttt}(t)\right\|_0^2 + \left\|\mathbf{u}_{tt}(t)\right\|_0^2 + \left\|\mathbf{u}_t(t)\right\|_2^2 + \left\|p_t(t)\right\|_1^2\right)dt.
\end{aligned}
$$

Applying the discrete Gronwall inequality and Lemmas 4.1 and 4.3, we obtain (4.23). □

We now estimate the pressure error in $L^2(0,T;L^2(\Omega))$. This hinges on the error estimate for the time derivative of velocity of Lemma 4.3.

LEMMA 4.5 (Pressure error estimate). *Let the exact solution of* (1.1) *is smooth enough and* $\tau = Ch$. *If Assumptions 1-5 hold, then we have*

$$\tau \sum_{n=1}^{N} \left\| e_h^{n+1} \right\|_0^2 \leq C\left(\tau^2 + h^2\right). \tag{4.26}$$

PROOF. We first recall again inf-sup condition in Assumption 3. Consequently, it suffices to estimate $\left\langle e^{n+1}, \nabla\cdot \mathbf{w}\right\rangle$ in terms of $\|\nabla \mathbf{w}\|_0$. In conjunction with (1.7), we can rewrite (4.11) as

$$\begin{aligned}
&\left\langle e_h^{n+1}, \nabla\cdot \mathbf{w}_h\right\rangle = \frac{1}{2\tau}\left\langle 3\mathbf{E}^{n+1} - 4\mathbf{E}^n + \mathbf{E}^{n-1}, \mathbf{w}_h\right\rangle \\
&+ \mu\left\langle \nabla\widehat{\mathbf{E}}_h^{n+1}, \nabla \mathbf{w}_h\right\rangle + \mathcal{N}\left(\delta\delta\mathbf{u}(t^{n+1}), \mathbf{u}(t^{n+1}), \mathbf{w}_h\right) \\
&+ \mathcal{N}\left(2\mathbf{u}(t^{n+1}) - \mathbf{u}(t^n), \widehat{\mathbf{E}}^{n+1}, \mathbf{w}_h\right) + \mathcal{N}\left(2\mathbf{E}^n - \mathbf{E}^{n-1}, \widehat{\mathbf{u}}_h^{n+1}, \mathbf{w}_h\right) \\
&- \mu\left\langle \nabla\delta q_h^{n+1}, \mathbf{w}_h\right\rangle - \left\langle \mathbf{R}^{n+1}, \mathbf{w}_h\right\rangle = \sum_{i=1}^{7} A_i.
\end{aligned} \tag{4.27}$$

We now proceed to estimate each term A_1 to A_7 separately. We readily obtain

$$A_1 \leq \frac{C}{\tau}\left(\left\|\delta\mathbf{E}^{n+1}\right\|_0 + \|\delta\mathbf{E}^n\|_0\right)\|\mathbf{w}_h\|_0 \leq \frac{C}{\tau}\left(\left\|\delta\mathbf{E}^{n+1}\right\|_0 + \|\delta\mathbf{E}^n\|_0\right)\|\nabla\mathbf{w}_h\|_0$$

and

$$A_2 \leq C\left\|\nabla\widehat{\mathbf{E}}_h^{n+1}\right\|_0\|\nabla\mathbf{w}_h\|_0.$$

Term A_3 and A_4 can be dealt with the aid of Lemma 2.5 and $\left\|\mathbf{u}(t^{n+1})\right\|_2 \leq M$ as follows:

$$A_3 \leq C\left\|\delta\delta\mathbf{u}(t^{n+1})\right\|_0\left\|\mathbf{u}(t^{n+1})\right\|_2\|\mathbf{w}_h\|_1 \leq C\left\|\delta\delta\mathbf{u}(t^{n+1})\right\|_1\|\nabla\mathbf{w}_h\|_1$$

and

$$A_4 \leq C\left\|2\mathbf{u}(t^{n+1}) - \mathbf{u}(t^n)\right\|_2\left\|\widehat{\mathbf{E}}^{n+1}\right\|_0\|\mathbf{w}_h\|_1 \leq C\left\|\widehat{\mathbf{E}}^{n+1}\right\|_0\|\nabla\mathbf{w}_h\|_0.$$

In light of $\left\|\widehat{\mathbf{u}}_h^{n+1}\right\|_1 = \left\|\widehat{\mathbf{E}}^{n+1} - \mathbf{u}(t^{n+1})\right\|_1 \leq C$ from Lemma 4.1, we can have

$$\begin{aligned}
A_5 &\leq C\left\|2\mathbf{E}^n - \mathbf{E}^{n-1}\right\|_{\mathbf{L}^3(\Omega)}\left\|\widehat{\mathbf{u}}_h^{n+1}\right\|_1\|\mathbf{w}_h\|_1 \\
&\leq \frac{C}{\sqrt{h}}\left(\left\|\mathbf{E}^{n+1}\right\|_0 + \|\mathbf{E}^n\|_0\right)\|\nabla\mathbf{w}_h\|_0.
\end{aligned}$$

Integrate by parts and Hölder inequality yield

$$A_6 \leq C\left\|\delta q_h^{n+1}\right\|_0\|\nabla\mathbf{w}_h\|_0.$$

On the other hand, we have

$$A_7 \leq \left\|\mathbf{R}^{n+1}\right\|_{-1}\|\nabla\mathbf{w}\|_0.$$

Inserting the estimates for A_1 to A_7 back into (4.27), and employing discrete inf-sup condition in Assumption 3, we obtain

$$C\|e^{n+1}\|_0 \leq \frac{1}{\tau}\left(\|\delta\mathbf{E}^{n+1}\|_0 + \|\delta\mathbf{E}^n\|_0\right) + \frac{C}{\sqrt{h}}\left(\|\mathbf{E}^{n+1}\|_0 + \|\mathbf{E}^n\|_0\right) + \|\delta q_h^{n+1}\|_0$$
$$+ \left\|\nabla\widehat{\mathbf{E}}_h^{n+1}\right\|_0 + \|\delta\delta\mathbf{u}(t^{n+1})\|_1 + \left\|\widehat{\mathbf{E}}^{n+1}\right\|_0 + \|\mathbf{R}^{n+1}\|_{-1}.$$

If we now square, multiply by τ, and sum over n from 1 to N, then Lemmas 4.1 and 4.3-4.4 derives (4.26). □

5. Numerical experiments

In this section, we perform two numerical experiments: the first is to check accuracy and the second is to test stability. In the first experiments, we choose square domain $[0,1] \times [0,1]$ and impose forcing term the exact solution to become

$$u = \exp(t)\sin^2(\pi x)\sin(2\pi y),$$
$$v = -\exp(t)\sin(2\pi x)\sin^2(\pi y),$$
$$p = \exp(t)\cos(\pi x)\cos(\pi y).$$

$\tau = h$	1/16	1/32	1/64	1/128	1/256
$\|E\|_0$	0.00384017	0.00130831	0.000391824	0.000107996	2.8413e-05
	Order	1.553466	1.739427	1.859228	1.926355
$\|E\|_{L^\infty}$	0.0112291	0.00381103	0.0011382	0.00031345	8.24393e-05
	Order	1.558989	1.743427	1.860447	1.926831
$\|E\|_1$	0.0798332	0.0239901	0.00680859	0.00183099	0.000476245
	Order	1.734550	1.817011	1.894732	1.942848
$\|e\|_0$	0.0986215	0.0332739	0.0099548	0.00267395	0.000704462
	Order	1.567511	1.740927	1.896420	1.924379
$\|e\|_{L^\infty}$	0.53446	0.214847	0.0759942	0.0236211	0.0070079
	Order	1.314772	1.499348	1.685813	1.753022

Table 1. Error decay for Algorithm 1

Table 1 is the error decay for Algorithm 1. In this computation, we use Taylor-Hood (P2-P1) finite element on the uniform mesh. We impose $\tau = h$ and $\mu = 1$. These error decay is optimal and consists to Theorem 2.

In [9], a rectangular driven cavity of low viscosity flow is performed in a domain $[0, 0.75] \times [0, 1]$ with initial and boundary conditions as described in Figure 1. In this paper, we carry out the experiment with $\mu = 1/10,000$ and $h = 1/256$, and then we hire $\tau = 0.5$ as big as possible. Most of numerical algorithms have upper bound for the size of τ to make hold the stability constraint, like $\tau \le Ch$, where C depends on the Reynolds numbers. So, the smaller τ has to be imposed for the bigger Reynolds numbers problem. But Algorithm 1 becomes released from the limitation of the time marching size τ by Theorem 1.

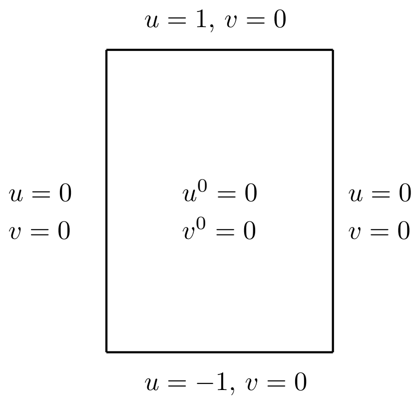

FIGURE 1. Initial and boundary conditions for rectangular driven cavity flow in the domain $[0, 0.75] \times [0, 1]$.

Figure 2 is the numerical result of the rectangular driven cavity flow at time $T = 100$ of Algorithm 1 and displays still stable even for high viscosity flow with $\mu = 1/10,000$ under extremely strong unstable conditions $h = 1/256$ and $\tau = 0.5$. The $\tau = 0.5$ is extremely big size for this case, but it is still stable. We thus can conclude that Algorithm 1 is unconditionally stable and consists to Theorem 1. We note here that this experiment is to verify only stability for any τ, not to check accuracy. So we impose very big $\tau = 0.5$ and the big τ is the main reason of the oscillations in Figure 2. We need to use a reasonable τ, if we want to obtain more accurate results, because stability and accuracy do not depend each other.

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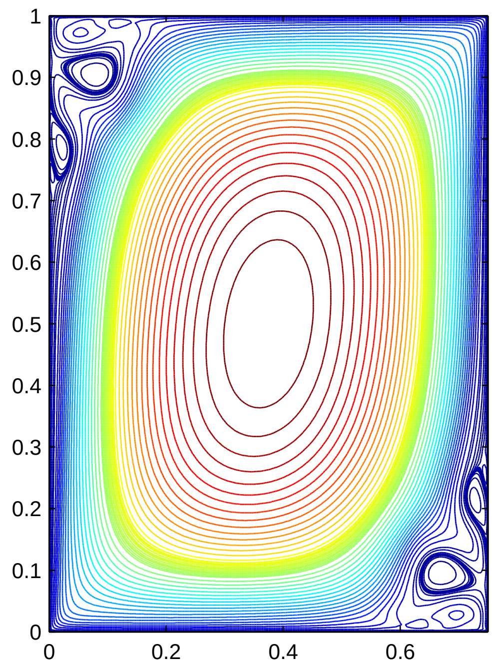

FIGURE 2. Rectangular driven cavity streamline at $T = 100$ for Algorithm 1 with $\mu = 1/10,000$, $h = 1/256$, $\tau = 0.5$

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