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CONE $\mathfrak{C}$-CLASS FUNCTIONS USING $(CLR_{\Gamma\mathfrak{C}})$-PROPERTY ON CONE b-NORMED SPACES WITH APPLICATION

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Abstract. In this article, we demonstrate the conditions for the existence of common fixed points (CFP) theorems for four self-maps satisfying the common limit range (CLR)-property on cone b-normed spaces $(CbNS)$ via $\mathfrak{C}$-class functions. Furthermore, we have a unique common fixed point for two weakly compatible (WC) pairings. Towards the end, the existence and uniqueness of common solutions for systems of functional equations arising in dynamic programming are discussed as an application of our main result.

1. Introduction

In 1989, Bakhtin [6] introduced b-metric space an extension of metric space. He established the renowned contraction principle in metric spaces extension in b-metric spaces. Huang and Zhang [11] gave the notion of cone metric space, replacing the set of real numbers by ordered Banach Space and introduced some fixed point theorems for function satisfying contractive conditions in Banach Spaces. Rezapour and Hamalbarani [24] were generalized result of [11] by omitting the normality condition, which is milestone in developing fixed point theory in cone metric space; for more information see [2, 14]. Sintunavarat and Kumam [25] recently proposed a novel idea of the (CLR)-property (common limit range property) that does not impose such constraints. The importance of the (CLR)-property ensures that range subspace closeness is not required. In 2011, Sintunavarat and Kumam [25] prove some common fixed point theorems under weakly compatible mappings using new property that is (CLR)-property. Recently many mathematicians use this property (CLR)-Property in different spaces to prove fixed point theorem. In 2013, Jitendra [15] put his efforts to proved common fixed point theorem in Metric Space by Property $(E.A.)$ as well as (CLR)-property and given an example and prove common fixed point theorem from both methods and Pathak [22] prove their main theorem using (CLR)-property for two pairs of self-mappings in complex-valued metric space under weak compatibility.

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In 2016, Rao [23] proved common coupled fixed point theorem in partial metric space by (CLR)-property. Khan et al. [18] used a new technique to prove fixed point theorems on metric space by altering distances between the points employing suitably equipped continuous control functions. Ansari [3] introduced the notion of $\mathfrak{C}$-class function as a major generalization of Banach contraction principle and obtained some fixed point results.

We provide the sufficient conditions for the existence of (CFP) theorems for four self-maps satisfying the (CLR) property on $(CbNS)$ via $\mathfrak{C}$-class functions. We explore the existence and uniqueness of common solutions for a system of functional equations that arise in dynamic programming are discussed as an application of our main result.

2. Preliminaries

Definition 2.1. [11] Let $(E, \|.\|)$ be the real Banach space. A subset P of E is called a cone if and only if:

(b_1) P is closed, non empty and $P \neq \{0\}$
(b_2) $a\mathfrak{h} + b\mathfrak{g} \in P$ for all $\mathfrak{h}, \mathfrak{g} \in P$ and non negative real numbers a, b
(b_3) $P \cap (-P) = \{0\}$.

Given a cone $P \subset E$, we define a partial ordering $\preceq$ with respect to P by $\mathfrak{h} \preceq \mathfrak{g}$ if and only if $\mathfrak{g} - \mathfrak{h} \in P$. We will write $\mathfrak{h} \prec \mathfrak{g}$ to indicate that $\mathfrak{h} \preceq \mathfrak{g}$ but $\mathfrak{h} \neq \mathfrak{g}$, while $\mathfrak{h}, \mathfrak{g}$ will stand for $\mathfrak{g} - \mathfrak{h} \in intP$, where $intP$ denotes the interior of P. The cone P is called normal if there is a number $k > 0$ such that $0 \leq \mathfrak{h} \leq \mathfrak{g}$ implies $\|\mathfrak{h}\| \leq k\|\mathfrak{g}\|$ for all $\mathfrak{h}, \mathfrak{g} \in E$. The least positive number satisfying the above is called the normal constant.

Definition 2.2. [13, 27] Let M be a nonempty set and $b \geq 1$ be a given real number. A mapping $D_b^c : M \times M \to E$ is said to be cone b-metric if and only if, for all $\mathfrak{h}, \mathfrak{g}, \upsilon \in M$, the following conditions are satisfied:

(cb_1) $D_b^c(\mathfrak{h}, \mathfrak{g}) = 0$ if and only if $\mathfrak{h} = \mathfrak{g}$,
(cb_2) $D_b^c(\mathfrak{h}, \mathfrak{g}) = D_b^c(\mathfrak{g}, \mathfrak{h})$,
(cb_3) $D_b^c(\mathfrak{h}, \mathfrak{g}) \leq b[D_b^c(\mathfrak{h}, \upsilon) + D_b^c(\upsilon, \mathfrak{g})]$.

If $b = 1$, then the mapping (D_b^c) is cone metric space (for more details, we refer to [11]). Then (M, D_b^c) is called a cone b-metric space.

The following example shows that cone b-metric need not be a cone metric.

Example 2.3. [12] Let $E = R^2$

$$P = \{(\mathfrak{h}, \mathfrak{g}) : \mathfrak{h}, \mathfrak{g} \geq 0\}$$

$M = R$ and $D_b^c : M \times M \to E$ such that

$$D_b^c(\mathfrak{h}, \mathfrak{g}) = (|\mathfrak{h} - \mathfrak{g}|^p, \alpha|\mathfrak{h} - \mathfrak{g}|^p)$$

where $\alpha \geq 0$ and $p > 1$ are two real constants. Then (M, D_b^c) is a cone b-metric space with the coefficient $b = 2^p > 1$, but not a cone metric space. In fact, we only need to prove (iii) in Definition (2.2) as follows:
Let $\mathfrak{h}, \mathfrak{g}, \upsilon \in M$. Set $\mu = \mathfrak{h} - \upsilon, \nu = \upsilon - \mathfrak{g}$, so $\mathfrak{h} - \mathfrak{g} = \mu + \nu$. From the inequality

$$(a + b)^p \leq (2 \max\{a, b\})^p \leq 2^p(a^p + b^p) \text{ for all} a, b \geq 0.$$

we have

$$|\mathfrak{h}-\mathfrak{g}|^p = |\mu+\nu|^p \leq (|\mu|+|\nu|)^p \leq 2^p(|\mu|^p+|\nu|^p) = 2^p(|\mathfrak{h}-\upsilon|^p+|\upsilon-\mathfrak{g}|^p)$$

which implies that $D_b^c(\mathfrak{h},\mathfrak{g}) \leq b[D_b^c(\mathfrak{h},\upsilon)+D_b^c(\upsilon,\mathfrak{g})]$ with $b=2^p>1$. But

$$|\mathfrak{h}-\mathfrak{g}|^p \leq |\mathfrak{h}-\mathfrak{g}|^p + |\upsilon-\mathfrak{g}|^p$$

is impossible for all $\mathfrak{h}>\upsilon>\mathfrak{g}$. Indeed, taking account of the inequality

$$(a+b)^p > a^p+b^p \text{ for all} a,b\geq 0,$$

$$|\mathfrak{h}-\mathfrak{g}|^p = |\mu+\nu|^p = (\mu+\nu)^p > \mu^p+\nu^p = |\mathfrak{h}-\upsilon|^p+|\upsilon-\mathfrak{g}|^p$$

for all $\mathfrak{h}>\upsilon>\mathfrak{g}$. Thus, $D_b^c(\mathfrak{h},\mathfrak{g}) \leq D_b^c(\mathfrak{h},\mathfrak{g})+D_b^c(\mathfrak{g},\upsilon)$ is not satisfied, i.e., (M,D_b^c) is not a cone metric space.

Definition 2.4. [26] Let M be a vector space over R. Suppose the mapping $\|.\|: M\to E$ satisfies

(i) $\|\mathfrak{h}\|\geq 0$ for all$\mathfrak{h}\in M$
(ii) $\|\mathfrak{h}\|=0$ if and only if $\mathfrak{h}=0$
(iii) $\|\mathfrak{h}+\mathfrak{g}\|\leq\|\mathfrak{h}\|+\|\mathfrak{g}\|$ for all $\mathfrak{h},\mathfrak{g}\in M$
(iv) $\|k\mathfrak{h}\|=|k|\|\mathfrak{h}\|$ for all $k\in R$

Then $\|.\|$ is called a norm on M, and $(M,\|.\|)$ is called a cone normed space. Clearly each cone normed space is a cone metric space with metric defined by $D_b^c(\mathfrak{h},\mathfrak{g}) = \|\mathfrak{h}-\mathfrak{g}\|$

Example 2.5. [27] Let $M=R^2, E=R^2$, and $P=\{(\mathfrak{h},\mathfrak{g})\in E|\mathfrak{h},\mathfrak{g}\geq 0\}$, we define $\|(\mathfrak{h},\mathfrak{g})\| = (|\mathfrak{h}|^2,|\mathfrak{g}|^2)$. Then $(M,\|.\|)$ is a $(CbNS)$.

Proof. First, according to the definition 2.4 of $\|(\mathfrak{h},\mathfrak{g})\|$, we obtain

$$\|(\mathfrak{h},\mathfrak{g})\| = (|\mathfrak{h}|^2,|\mathfrak{g}|^2)\geq 0,$$

for all $\mu=(\mathfrak{h},\mathfrak{g})\in M$. It is obvious that $\|(\mathfrak{h},\mathfrak{g})\|=0 \; iff \; (\mathfrak{h},\mathfrak{g})=0$. Then, we have

$$(\mathfrak{h}+\mathfrak{g})^p \leq \mathfrak{h}^p+\mathfrak{g}^p, \text{for all } \mathfrak{h},\mathfrak{g}\geq 0, 0<p\leq 1,$$

$$(\mathfrak{h}+\mathfrak{g})^p \leq 2^{p-1}(\mathfrak{h}^p+\mathfrak{g}^p), \text{for all } \mathfrak{h},\mathfrak{g}\geq 0, p\geq 1.$$

We check for any $(\mathfrak{h}_1,\mathfrak{g}_1),(\mathfrak{h}_2,\mathfrak{g}_2)\in M$, thus, it follows

$$\begin{aligned}\|(\mathfrak{h}_1,\mathfrak{g}_1)+(\mathfrak{h}_2,\mathfrak{g}_2,\| &= \|(\mathfrak{h}_1+\mathfrak{h}_2,\ \mathfrak{g}_1+\mathfrak{g}_2)\| \\ &= (|\mathfrak{h}_1+\mathfrak{h}_2|^2,\ |\mathfrak{g}_1+\mathfrak{g}_2|^2) \\ &\leq 2(|\mathfrak{h}_1|^2+|\mathfrak{h}_2|^2,\ |\mathfrak{g}_1|^2+|\mathfrak{g}_2|^2) \\ &= 2(|\mathfrak{h}_1|^2,|\mathfrak{h}_2|^2,)+2(\ |\mathfrak{g}_1|^2,|\mathfrak{g}_2|^2) \\ &= 2\,\|(\mathfrak{h}_1,\mathfrak{g}_1)\| + 2\,\|(\mathfrak{h}_2,\mathfrak{g}_2)\|.\end{aligned}$$

If any $\mu=(\mathfrak{h},\mathfrak{g})\in M, k\in R$, there are

$$\|k\mu\| = \|k(\mathfrak{h},\mathfrak{g})\| = \|(k\mathfrak{h},k\mathfrak{g})\| = |k\mathfrak{h}|^2,|k\mathfrak{g}|^2 = |k|^2(|\mathfrak{h}|^2,|\mathfrak{g}|^2) = |k|^2\,\|\mu\|.$$

According, by definition (2.4), $(M,\|.\|)$ is a $(CbNS)$ with coefficient $b=2>1$. □

Definition 2.6. [12,27] Let $(M,\|.\|)$ be a $(CbNS)$, $\mathfrak{h}\in M$ and $\{\mathfrak{h}_n\}$ be a sequence in M. Then

(i) $\{\mathfrak{h}_n\}$ converges to $\mathfrak{h}$ whenever, for every $c\in E$ with $0\ll c$, there is a natural number N such that $\|(\mathfrak{h}_n,\mathfrak{h})\|\ll c$ for all $n\geq N$. We denote this by $\lim_{n\to+\infty}\mathfrak{h}_n=\mathfrak{h}$ or $\mathfrak{h}_n\to\mathfrak{h}(n\to+\infty)$.

(ii) $\{\mathfrak{h}_n\}$ is a Cauchy sequence whenever, for every $c \in E$ with $0 \ll c$, there is a natural number N such that $\|(\mathfrak{h}_n, \mathfrak{h}_m)\| \ll c$ for all $n, m \geq N$

(iii) $(M, \|.\|)$ is a complete cone normed b-metric space if every Cauchy sequence is convergent. A complete cone normed space is a cone Banach space.

Lemma 2.7. [27] *Let $(M, \|.\|)$ be a $(CbNS)$. P be a normal cone with constant K. Let $\{\mu_p\}$ be a sequence in M. Then*

(i) $\{\mu_p\}$ b-converges to $\mathfrak{h}$ if and only if $\|\mu_p - \mu\| \to 0$ as $p \to +\infty$.

(ii) $\{\mu_p\}$ is a b-Cauchy sequence if and only if $\|\mu_p - \mu_m\| \to 0$ as $p, m \to +\infty$.

(iii) If the $\{\mu_p\}$ b-converges to μ and $\{\mathfrak{g}_n\}$ b-converges to ν then $\|\mu_p - \nu_p\| \to \|\mu - \nu\|$.

Lemma 2.8. [5] *Let $(M, \|.\|)$ be a $(CbNS)$ with $b \geq 1$. Suppose that $\{\mathfrak{h}_n\}$ and $\{\mathfrak{g}_n\}$ are b-convergent to $\mathfrak{h}$ and $\mathfrak{g}$, respectively. Then we have,*

$$\frac{1}{b^2}\|\mathfrak{h} - \mathfrak{g}\| \leq \lim_{n \to +\infty} \inf \|\mathfrak{h}_n - \mathfrak{g}_n\| \leq \lim_{n \to +\infty} sup\|\mathfrak{h}_n - \mathfrak{g}_n\| \leq b^2\|\mathfrak{h} - \mathfrak{g}\|.$$

In particular, if $M = L$, then we have $\lim_{n \to +\infty} \|(\mathfrak{h}_n - \mathfrak{g}_n)\| = 0$, moreover for each $\upsilon \in M$, we have

$$\frac{1}{b}\|\mathfrak{h} - \upsilon\| \leq \lim_{n \to +\infty} \inf \|\mathfrak{h}_n - \upsilon\| \leq \lim_{n \to +\infty} \sup \|\mathfrak{h}_n - \upsilon\| \leq b\|\mathfrak{h} - \upsilon\|.$$

Definition 2.9. [1] Let $\aleph$ and $\beth$ be two self maps defined on a $(CbNS)$ $(M, \|.\|)$ maps $\aleph$ and $\beth$ are said to be (WC) if they commute at coincidence points, that is if $\aleph\mathfrak{h} = \beth\mathfrak{h}$ for all $\mathfrak{h} \in M$ then $\aleph\beth\mathfrak{h} = \beth\aleph\mathfrak{h}$.

Example 2.10. Define $\aleph, \beth : R \to R$ by $\aleph(\mathfrak{h}) = \frac{\mathfrak{h}}{4}$, for all $\mathfrak{h} \in \mathcal{R}$ and $\beth(\mathfrak{h}) = \mathfrak{h}^2$, for all $\mathfrak{h} \in \mathcal{R}$. Here, 0 and $\frac{1}{4}$ are two coincidence points for the maps $\aleph$ and $\beth$. Note that $\aleph$ and $\beth$ commute at $0, i.e. \aleph\beth(0) = \beth\aleph(0) = 0$, but $\aleph\beth(\frac{1}{4}) = \aleph(\frac{1}{16}) = \frac{1}{64}$ and $\beth\aleph(\frac{1}{4}) = \beth(\frac{1}{16}) = \frac{1}{256}$ so $\aleph$ and $\beth$ are not (WC) on R.

Lemma 2.11. [1] *Let $\aleph$ and $\beth$ be (WC) self-maps of a $(CbNS)$ $(M, \|.\|)$. If $\aleph$ and $\beth$ have a unique point of coincidence, that is $x = \aleph\mathfrak{h} = \beth\mathfrak{h}$ then x is the unique (CFP) of $\aleph$ and $\beth$.*

Lemma 2.12. *Let $(M, \|.\|)$ is a $(CbNS)$. If there exits two sequences $\{\mathfrak{h}_n\}$ and $\{\mathfrak{g}_n\}$ such that $\lim_{n \to +\infty} \|\mathfrak{h}_n - \mathfrak{g}_n\| = 0$, whenever $\{\mathfrak{h}_n\}$ is a sequence in M such that $\lim_{n \to +\infty} \mathfrak{h}_n = \mathcal{T}$ for some $\mathcal{T} \in M$ then $\lim_{n \to +\infty} \mathfrak{g}_n = \mathcal{T}$.*

Proof. By a triangle inequality in $(CbNS)$, we have $\|(\mathfrak{g}_n - \mathcal{T})\| \leq b(\|\mathfrak{g}_n - \mathfrak{h}_n\| + \|\mathfrak{h}_n - \mathcal{T}\|)$. Now by taking the upper limit when $n \to +\infty$ in the above inequality we get

$$\lim_{n \to +\infty} \sup \|\mathfrak{g}_n - \mathcal{T}\| \leq b(\lim_{n \to +\infty} \sup \|\mathfrak{h}_n - \mathfrak{g}_n\| + \lim_{n \to +\infty} \sup \|\mathfrak{h}_n - \mathcal{T}\|) = 0.$$

□

Definition 2.13. [16] Let $(M, \|.\|)$ be a $(CbNS)$. A pair $\{\aleph, \beth\}$ is said to be compatible if $\lim_{n \to +\infty} \|\aleph\beth\mathfrak{h}_n - \beth\aleph\mathfrak{h}_n\| = 0$, for every sequence $\{\mathfrak{h}_n\}$ in M with $\lim_{n \to +\infty} \aleph\mathfrak{h}_n = \lim_{n \to +\infty} \beth\mathfrak{h}_n = \mathcal{T}$ for some $\mathcal{T} \in M$.

DEFINITION 2.14. [25] Let $(M, \|.\|)$ be a $(CbNS)$ and $\aleph, \beth : M \to M$, two mapping $\aleph$ and $\beth$ are said to satisfy the (CLR)-property of $\aleph$ if

$$\lim_{n\to+\infty} \beth\{\mathfrak{h}_n\} = \lim_{n\to+\infty} \aleph\{\mathfrak{h}_n\} = \aleph(\mathfrak{h}) \text{ for some } \mathfrak{h} \in M$$

EXAMPLE 2.15. Let $M = [0, +\infty)$ and define $D_b^c(\mathfrak{h}, \mathfrak{g}) = \|\mathfrak{h} - \mathfrak{g}\|$ for all $\mathfrak{h}, \mathfrak{g} \in M$ and $\aleph, \beth : M \to M$, defined by $\beth(\mathfrak{h}) = \mathfrak{h} + 2$ and $\aleph(\mathfrak{h}) = 3\mathfrak{h}$, for all $\mathfrak{h} \in M$. Consider the sequence $\{\mathfrak{h}_n\} = \{1 + \frac{1}{n}\}$. Then

$$\lim_{n\to+\infty} \beth\{\mathfrak{h}_n\} = \lim_{n\to+\infty} \beth\{1 + \frac{1}{n}\} = \lim_{n\to+\infty} \{3 + \frac{1}{n}\} = 3;$$

$$\lim_{n\to+\infty} \aleph\{\mathfrak{h}_n\} = \lim_{n\to+\infty} \aleph\{1 + \frac{1}{n}\} = \lim_{n\to+\infty} \{3 + \frac{3}{n}\} = 3;$$

Thus,

$$\lim_{n\to+\infty} \beth\{\mathfrak{h}_n\} = \lim_{n\to+\infty} \aleph\{\mathfrak{h}_n\} = 3 = \aleph(1).$$

So $\aleph$ and $\beth$ are satisfy the $(CLR_\aleph)$-property.

DEFINITION 2.16. [4] A mapping $\mathfrak{F} : P^2 \to P$ is called $\mathfrak{C}$-class function if it is continuous and satisfies following axioms:

1 $\mathfrak{F}(s, t) \leq s$for all $(s, t) \in P$;
2 $\mathfrak{F}(s, t) = s$ implies that either $s = 0$ or $t = 0$.

Let us denote $\mathcal{C}$ the family of $\mathfrak{C}$-class functions.

REMARK 2.17. [4] Clearly, for some $\mathfrak{F}$ we have $\mathfrak{F}(0, 0) = 0$.

EXAMPLE 2.18. [4] The following functions $\mathfrak{F} : P^2 \to P$ are elements of $\mathcal{C}$, for all $s, t \in [0, +\infty)$:

1 $\mathfrak{F}(s, t) = s - t, \mathfrak{F}(s, t) = s \Rightarrow t = 0$;
2 $\mathfrak{F}(s, t) = ms, 0 < m < 1, \mathfrak{F}(s, t) = s \Rightarrow s = 0$;
3 $\mathfrak{F}(s, t) = s\beta(s), \beta : [0, +\infty) \to [0, 1)$ is continuous, $\mathfrak{F}(s, t) = s \Rightarrow s = 0$;
4 $\mathfrak{F}(s, t) = \Psi(s)$, where $\Psi : P \to P, \Psi(0) = 0, \Psi(s) > 0$ for all $s \in P$ with $s$such that 0 and $\Psi(S) \leq s$for all $s \in P$.
5 $\mathfrak{F}(s, t) = s - \phi(s), \mathfrak{F}(s, t) = s \Rightarrow s = 0$, here $\phi : [0, +\infty) \to [0, +\infty)$ is a continuous function such that $\phi(t) = 0 \Leftrightarrow t = 0$.

DEFINITION 2.19. [18] A function $\psi : R_0^+ \to R_0^+$ is called an altering distance function if the condition hold:

(i) ψ is continuous and non decreasing.
(ii) $\psi(t) = 0$ iif and only if $t = 0$.

DEFINITION 2.20. [3] An ultra altering distance function is a non decreasing and continuous mapping $\phi : R_0^+ \to R_0^+$ such that $\phi(t) > 0, t > 0$ and $\phi(0) = 0$.

LEMMA 2.21. [3] *Let ψ and ϕ are altering and ultra alterng distance functions respectively, $\mathfrak{F} \in \mathcal{C}$ and $\{q_n\}$ a decreasing sequence in P such that*

$$\psi(q_{n+1}) \leq \mathfrak{F}(\psi(q_n), \phi(q_n)) \text{ for all } n \geq 1.$$

Then $\lim_{n\to+\infty} q_n = 0$.

3. Main Results

Throughout this article, we denote $\psi \in \Psi$ is an altering distance function, $\phi \in \Phi$ is an ultra altering distance function. Now, we prove our main result.

LEMMA 3.1. *Let* $(M, \|.\|)$ *be a complete* $(CbNS)$ *with the norm* $D_b^c(\mathfrak{h}, 0) = \|\mathfrak{h}\|$ *and the co-efficient* $b \geq 1$. *Suppose that the mappings* $\aleph, \beth, \Gamma$, *and* $\mathfrak{L}$ *are four self-mappings of a* $(CbNS)$ $(M, \|.\|)$ *satisfies the following conditions:*

$$\psi(\|(\aleph\mathfrak{h} - \beth\mathfrak{g})\| \leq \mathfrak{F}\left(\psi(\daleth(\mathfrak{h}, \mathfrak{g})), \phi(\daleth(\mathfrak{h}, \mathfrak{g}))\right) \tag{3.1}$$

for all $\mathfrak{h}, \mathfrak{g} \in M$, *where*

$$\daleth(\mathfrak{h}, \mathfrak{g}) = \max\left\{\|\Gamma\mathfrak{h} - \mathfrak{L}\mathfrak{g}\|, \frac{\|\aleph\mathfrak{h}-\Gamma\mathfrak{h}\|}{1+\|\aleph\mathfrak{h}-\Gamma\mathfrak{h}\|}, \frac{\|\beth\mathfrak{g}-\mathfrak{L}\mathfrak{g}\|}{1+\|\beth\mathfrak{g}-\mathfrak{L}\mathfrak{g}\|}, \frac{\|\aleph\mathfrak{h}-\mathfrak{L}\mathfrak{g}\|+\|\Gamma\mathfrak{h}-\beth\mathfrak{g}\|}{2b}, \frac{\|\aleph\mathfrak{h}-\Gamma\mathfrak{h}\|+\|\beth\mathfrak{g}-\mathfrak{L}\mathfrak{g}\|}{2b}\right\}.$$

Further assume that

1. The pairs $(\aleph, \Gamma)$ and $(\beth, \mathfrak{L})$ satisfied the (CLR_Γ) and $(CLR_\mathfrak{L})$ properties respectively.
2. $\aleph(M) \subseteq \mathfrak{L}(M)$ and $\beth(M) \subseteq \Gamma(M)$.
3. $\Gamma(M)$ and $\mathfrak{L}(M)$ are closed in M.
4. $\{\beth\mathfrak{g}_n\}$ converges for each sequence $\{\mathfrak{g}_n\}$ in M whenever $\{\mathfrak{L}\mathfrak{g}_n\}$ converges (respectively $\{\aleph\mathfrak{h}_n\}$ converges for every sequence $\{\mathfrak{h}_n\}$ in M whenever $\{\Gamma\mathfrak{h}_n\}$ converges).

Then the pair $(\aleph, \Gamma)$ and $(\beth, \mathfrak{L})$ share the (CLR)-property of Γ.

Proof. If the pair $(\aleph, \Gamma)$ satisfy the (CLR_Γ), there is sequence $\{\mathfrak{h}_n\}$ in M such that

$$\lim_{n\to+\infty} \aleph\mathfrak{h}_n = \lim_{n\to+\infty} \Gamma\mathfrak{h}_n = \mathcal{T}$$

where $\mathcal{T} \in \Gamma(M)$. Now, since $\aleph(M) \subseteq \mathfrak{L}(M)$, every sequence $\{\mathfrak{h}_n\}$, there is a sequence $\{\mathfrak{g}_n\}$ in M such that $\aleph\mathfrak{h}_n = \mathfrak{L}\mathfrak{g}_n$. As $\mathfrak{L}(M)$ is closed, so

$$\lim_{n\to+\infty} \mathfrak{L}\mathfrak{g}_n = \lim_{n\to+\infty} \aleph\mathfrak{h}_n = \mathcal{T}.$$

So that $\mathcal{T} \in \mathfrak{L}(M)$ and in all $\mathcal{T} \in \Gamma(M) \cap \mathfrak{L}(M)$. Thus, we get $\aleph\mathfrak{h}_n \to \mathcal{T}, \Gamma\mathfrak{h}_n \to \mathcal{T}$, $\mathfrak{L}\mathfrak{g}_n \to \mathcal{T}$, as $n \to +\infty$.

Let us show that $\beth\mathfrak{g}_n \to \mathcal{T}$ as $n \to +\infty$. On the contrary suppose that $\beth\mathfrak{g}_n \to \int(\neq \mathcal{T})$ as $n \to +\infty$. Putting $\mathfrak{h} = \mathfrak{h}_n$ and $\mathfrak{g} = \mathfrak{g}_n$ in (3.1), we get

$$\psi(\|\aleph\mathfrak{h}_n - \beth\mathfrak{g}_n\|) \leq \mathfrak{F}\left(\psi(\daleth(\mathfrak{h}_n, \mathfrak{g}_n)), \phi(\daleth(\mathfrak{h}_n, \mathfrak{g}_n))\right) \tag{3.2}$$

$$
\begin{aligned}
&= \mathfrak{F}(\psi(\max\{\|\Gamma\mathfrak{h}_n-\mathfrak{L}\mathfrak{g}_n\|,\frac{\|\aleph\mathfrak{h}_n-\Gamma\mathfrak{h}_n\|}{1+\|\aleph\mathfrak{h}_n-\Gamma\mathfrak{h}_n\|},\frac{\|\beth\mathfrak{g}_n-\mathfrak{L}\mathfrak{g}_n\|}{1+\|\beth\mathfrak{g}_n-\mathfrak{L}\mathfrak{g}_n\|},\\
&\quad \frac{\|\aleph\mathfrak{h}_n-\mathfrak{L}\mathfrak{g}_n\|+\|\Gamma\mathfrak{h}_n-\beth\mathfrak{g}_n\|}{2b},\frac{\|\aleph\mathfrak{h}_n-\Gamma\mathfrak{h}_n\|+\|\beth\mathfrak{g}_n-\mathfrak{L}\mathfrak{g}_n\|}{2b}\}),\\
&\quad \phi(\max\{\|\Gamma\mathfrak{h}_n-\mathfrak{L}\mathfrak{g}_n\|,\frac{\|\aleph\mathfrak{h}_n-\Gamma\mathfrak{h}_n\|}{1+\|\aleph\mathfrak{h}_n-\Gamma\mathfrak{h}_n\|},\frac{\|\beth\mathfrak{g}_n-\mathfrak{L}\mathfrak{g}_n\|}{1+\|\beth\mathfrak{g}_n-\mathfrak{L}\mathfrak{g}_n\|},\\
&\quad \frac{\|\aleph\mathfrak{h}_n-\mathfrak{L}\mathfrak{g}_n\|+\|\Gamma\mathfrak{h}_n-\beth\mathfrak{g}_n\|}{2b},\frac{\|\aleph\mathfrak{h}_n-\Gamma\mathfrak{h}_n\|+\|\beth\mathfrak{g}_n-\mathfrak{L}\mathfrak{g}_n\|}{2b}\}))\\
&= \mathfrak{F}(\psi(\max\{\|\Gamma\mathfrak{h}_n-\aleph\mathfrak{h}_n\|,\frac{\|\aleph\mathfrak{h}_n-\Gamma\mathfrak{h}_n\|}{1+\|\aleph\mathfrak{h}_n-\Gamma\mathfrak{h}_n\|},\frac{\|\beth\mathfrak{g}_n-\aleph\mathfrak{h}_n\|}{1+\|\beth\mathfrak{g}_n-\aleph\mathfrak{h}_n\|},\\
&\quad \frac{\|\aleph\mathfrak{h}_n-\aleph\mathfrak{h}_n\|+\|\Gamma\mathfrak{h}_n-\beth\mathfrak{g}_n\|}{2b},\frac{\|\aleph\mathfrak{h}_n-\Gamma\mathfrak{h}_n\|+\|\beth\mathfrak{g}_n-\aleph\mathfrak{h}_n\|}{2b}\}),\\
&\quad \phi(\max\{\|\Gamma\mathfrak{h}_n-\aleph\mathfrak{h}_n\|,\frac{\|\aleph\mathfrak{h}_n-\Gamma\mathfrak{h}_n\|}{1+\|\aleph\mathfrak{h}_n-\Gamma\mathfrak{h}_n\|},\frac{\|\beth\mathfrak{g}_n-\aleph\mathfrak{h}_n\|}{1+\|\beth\mathfrak{g}_n-\aleph\mathfrak{h}_n\|},\\
&\quad \frac{\|\aleph\mathfrak{h}_n-\aleph\mathfrak{h}_n\|+\|\Gamma\mathfrak{h}_n-\beth\mathfrak{g}_n\|}{2b},\frac{\|\aleph\mathfrak{h}_n-\Gamma\mathfrak{h}_n\|+\|\beth\mathfrak{g}_n-\aleph\mathfrak{h}_n\|}{2b}\}))\\
&\leq \mathfrak{F}(\psi(\max\{\|\Gamma\mathfrak{h}_n-\aleph\mathfrak{h}_n\|,\|\aleph\mathfrak{h}_n-\Gamma\mathfrak{h}_n\|,\|\beth\mathfrak{g}_n-\aleph\mathfrak{h}_n\|,\\
&\quad \frac{\|\Gamma\mathfrak{h}_n-\beth\mathfrak{g}_n\|}{2b},\frac{\|\aleph\mathfrak{h}_n-\Gamma\mathfrak{h}_n\|+\|\beth\mathfrak{g}_n-\aleph\mathfrak{h}_n\|}{2b}\}),\\
&\quad \phi(\max\{\|\Gamma\mathfrak{h}_n-\aleph\mathfrak{h}_n\|,\|\aleph\mathfrak{h}_n-\Gamma\mathfrak{h}_n\|,\|\beth\mathfrak{g}_n-\aleph\mathfrak{h}_n\|,\\
&\quad \frac{\|\Gamma\mathfrak{h}_n-\beth\mathfrak{g}_n\|}{2b},\frac{\|\aleph\mathfrak{h}_n-\Gamma\mathfrak{h}_n\|+\|\beth\mathfrak{g}_n-\aleph\mathfrak{h}_n\|}{2b}\}))\\
&\leq \mathfrak{F}(\psi(\max\{\|\Gamma\mathfrak{h}_n-\aleph\mathfrak{h}_n\|,\|\aleph\mathfrak{h}_n-\Gamma\mathfrak{h}_n\|,\|\beth\mathfrak{g}_n-\aleph\mathfrak{h}_n\|,\\
&\quad \frac{b[\|\Gamma\mathfrak{h}_n-\aleph\mathfrak{h}_n\|+\|\aleph\mathfrak{h}_n-\beth\mathfrak{g}_n\|]}{2b},\frac{\|\aleph\mathfrak{h}_n-\Gamma\mathfrak{h}_n\|+\|\beth\mathfrak{g}_n-\aleph\mathfrak{h}_n\|}{2b}\}),\\
&\quad \phi(\max\{\|\Gamma\mathfrak{h}_n-\aleph\mathfrak{h}_n\|,\|\aleph\mathfrak{h}_n-\Gamma\mathfrak{h}_n\|,\|\beth\mathfrak{g}_n-\aleph\mathfrak{h}_n\|,\\
&\quad \frac{b[\|\Gamma\mathfrak{h}_n-\aleph\mathfrak{h}_n\|+\|\aleph\mathfrak{h}_n-\beth\mathfrak{g}_n\|]}{2b},\frac{\|\aleph\mathfrak{h}_n-\Gamma\mathfrak{h}_n\|+\|\beth\mathfrak{g}_n-\aleph\mathfrak{h}_n\|}{2b}\}))\\
&= \mathfrak{F}(\psi(\max\{\|\Gamma\mathfrak{h}_n-\aleph\mathfrak{h}_n\|,\|\aleph\mathfrak{h}_n-\Gamma\mathfrak{h}_n\|,\|\beth\mathfrak{g}_n-\aleph\mathfrak{h}_n\|,\\
&\quad \frac{\|\Gamma\mathfrak{h}_n-\aleph\mathfrak{h}_n\|+\|\aleph\mathfrak{h}_n-\beth\mathfrak{g}_n\|}{2},\frac{\|\aleph\mathfrak{h}_n-\Gamma\mathfrak{h}_n\|+\|\beth\mathfrak{g}_n-\aleph\mathfrak{h}_n\|}{2b}\}),\\
&\quad \phi(\max\{\|\Gamma\mathfrak{h}_n-\aleph\mathfrak{h}_n\|,\|\aleph\mathfrak{h}_n-\Gamma\mathfrak{h}_n\|,\|\beth\mathfrak{g}_n-\aleph\mathfrak{h}_n\|,\\
&\quad \frac{\|\Gamma\mathfrak{h}_n-\aleph\mathfrak{h}_n\|+\|\aleph\mathfrak{h}_n-\beth\mathfrak{g}_n\|}{2},\frac{\|\aleph\mathfrak{h}_n-\Gamma\mathfrak{h}_n\|+\|\beth\mathfrak{g}_n-\aleph\mathfrak{h}_n\|}{2b}\}))\\
&= \mathfrak{F}\left(\psi(\max\{\|\mathcal{T}-\int\|,\frac{\|\aleph\mathfrak{h}_n-\beth\mathfrak{g}_n\|}{2}\}),\phi(\max\{\|\mathcal{T}-\int\|,\frac{\|\aleph\mathfrak{h}_n-\beth\mathfrak{g}_n\|}{2}\})\right)
\end{aligned}
$$

implies that

$$\psi(\|\mathcal{T}-\int\|) \leq \mathfrak{F}(\psi(\|\mathcal{T}-\int\|),\phi(\|\mathcal{T}-\int\|)) \tag{3.3}$$

Taking $n \to +\infty$, we get

$$\psi(\|\mathcal{T}-\int\|) \leq \mathfrak{F}(\psi(\lim_{n\to+\infty}\|\mathcal{T}-\int\|),\phi(\lim_{n\to+\infty}\|\mathcal{T}-\int\|))$$

So either, $\psi(\lim_{n\to+\infty}\|\mathcal{T}-\int\|)=0$ or $\phi(\lim_{n\to+\infty}\|\mathcal{T}-\int\|)=0$. Thus $\lim_{n\to+\infty}\|\mathcal{T}-\int\|=0$ implies that $\mathcal{T}=\int$, a contradiction. Hence, $\int\to\mathcal{T}$ which shows that the pairs $(\aleph,\Gamma)$ and $(\beth,\mathfrak{L})$ share the(CLR)-property of $\Gamma\mathfrak{L}$ which completes the proof. Using the above lemma, in the following theorem, we show the existence of unique (CFP). □

THEOREM 3.2. *Let $(M,\|.\|)$ be a complete $(CbNS)$ with the norm $D_b^c(\mathfrak{h},0)=\|\mathfrak{h}\|$ and the co-efficient $b\geq 1$. suppose that the mappings $\aleph,\beth,\Gamma$ and $\mathfrak{L}$ are four self-mappings of a cone normed b-metric space $(M,\|.\|)$ satisfying (3.1). If the pairs $(\aleph,\Gamma)$ and $(\beth,\mathfrak{L})$ have a point of coincidence. Moreover if $(\aleph,\Gamma)$ and $(\beth,\mathfrak{L})$ are (WC) the $\aleph,\beth,\Gamma$ and $\mathfrak{L}$ have a unique (CFP) .*

Proof. Since the pairs $(\aleph,\Gamma)$ and $(\beth,\mathfrak{L})$ satisfying the (CLR)-property of $\Gamma\mathfrak{L}$, so there exists sequence $\{\mathfrak{h}_n\}$ and $\{\mathfrak{g}_n\}$ in M such that

$$\lim_{n\to+\infty}\aleph\mathfrak{h}_n=\lim_{n\to+\infty}\Gamma\mathfrak{h}_n=\lim_{n\to+\infty}\beth\mathfrak{g}_n=\lim_{n\to+\infty}\mathfrak{L}\mathfrak{g}_n=\mathcal{T}.$$

where $\mathcal{T}\in\Gamma(M)\cap\mathfrak{L}(M)$ since $\Gamma(M)\subseteq\mathfrak{L}(M)$ so a point $\int\in M$ such that $\Gamma\int=\mathcal{T}$ we show that $\aleph\int=\Gamma\int$. Putting $\mathfrak{h}=\int,\mathfrak{g}=\mathfrak{g}_n$ in (3.1), we get

$$\psi(\|\aleph\int-\beth\mathfrak{g}_n\|)\leq\mathfrak{F}(\psi(\urcorner(\int,\mathfrak{g}_n)),\phi(\urcorner(\int,\mathfrak{g}_n)))\tag{3.4}$$

$$\begin{aligned}
&= \mathfrak{F}(\psi(\max\{\|\Gamma\int-\mathfrak{L}\mathfrak{g}_n\|,\frac{\|\aleph\int-\Gamma\int\|}{1+\|\aleph\int-\Gamma\int\|},\frac{\|\beth\mathfrak{g}_n-\mathfrak{L}\mathfrak{g}_n\|}{1+\|\beth\mathfrak{g}_n-\mathfrak{L}\mathfrak{g}_n\|},\\
&\quad \frac{\|\aleph\int-\mathfrak{L}\mathfrak{g}_n\|+\|\Gamma\int-\beth\mathfrak{g}_n\|}{2b},\frac{\|\aleph\int-\Gamma\int\|+\|\beth\mathfrak{g}_n-\mathfrak{L}\mathfrak{g}_n\|}{2b}\}),\\
&\quad \phi(\max\{\|\Gamma\int-\mathfrak{L}\mathfrak{g}_n\|,\frac{\|\aleph\int-\Gamma\int\|}{1+\|\aleph\int-\Gamma\int\|},\frac{\|\beth\mathfrak{g}_n-\mathfrak{L}\mathfrak{g}_n\|}{1+\|\beth\mathfrak{g}_n-\mathfrak{L}\mathfrak{g}_n\|},\\
&\quad \frac{\|\aleph\int-\mathfrak{L}\mathfrak{g}_n\|+\|\Gamma\int-\beth\mathfrak{g}_n\|}{2b},\frac{\|\aleph\int-\Gamma\int\|+\|\beth\mathfrak{g}_n-\mathfrak{L}\mathfrak{g}_n\|}{2b}\}))\\
&= \mathfrak{F}(\psi(\max\{\|\mathcal{T}-\mathfrak{L}\mathfrak{g}_n\|,\frac{\|\aleph\int-\mathcal{T}\|}{1+\|\aleph\int-\mathcal{T}\|},\frac{\|\beth\mathfrak{g}_n-\mathfrak{L}\mathfrak{g}_n\|}{1+\|\beth\mathfrak{g}_n-\mathfrak{L}\mathfrak{g}_n\|},\\
&\quad \frac{\|\aleph\int-\mathfrak{L}\mathfrak{g}_n\|+\|\mathcal{T}-\beth\mathfrak{g}_n\|}{2b},\frac{\|\aleph\int-\mathcal{T}\|+\|\beth\mathfrak{g}_n-\mathfrak{L}\mathfrak{g}_n\|}{2b}\}),\\
&\quad \phi(\max\{\|\mathcal{T}-\mathfrak{L}\mathfrak{g}_n\|,\frac{\|\aleph\int-\mathcal{T}\|}{1+\|\aleph\int-\mathcal{T}\|},\frac{\|\beth\mathfrak{g}_n-\mathfrak{L}\mathfrak{g}_n\|}{1+\|\beth\mathfrak{g}_n-\mathfrak{L}\mathfrak{g}_n\|},\\
&\quad \frac{\|\aleph\int-\mathfrak{L}\mathfrak{g}_n\|+\|\mathcal{T}-\beth\mathfrak{g}_n\|}{2b},\frac{\|\aleph\int-\mathcal{T}\|+\|\beth\mathfrak{g}_n-\mathfrak{L}\mathfrak{g}_n\|}{2b}\}))\\
&\leq \mathfrak{F}(\psi(\max\{\|\mathcal{T}-\mathfrak{L}\mathfrak{g}_n\|,\|\aleph\int-\mathcal{T}\|,\|\beth\mathfrak{g}_n-\mathfrak{L}\mathfrak{g}_n\|,\\
&\quad \frac{\|\aleph\int-\mathfrak{L}\mathfrak{g}_n\|+\|\mathcal{T}-\beth\mathfrak{g}_n\|}{2b},\frac{\|\aleph\int-\mathcal{T}\|+\|\beth\mathfrak{g}_n-\mathfrak{L}\mathfrak{g}_n\|}{2b}\}),\\
&\quad \phi(\max\{\|\mathcal{T}-\mathfrak{L}\mathfrak{g}_n\|,\|\aleph\int-\mathcal{T}\|,\|\beth\mathfrak{g}_n-\mathfrak{L}\mathfrak{g}_n\|,\\
&\quad \frac{\|\aleph\int-\mathfrak{L}\mathfrak{g}_n\|+\|\mathcal{T}-\beth\mathfrak{g}_n\|}{2b},\frac{\|\aleph\int-\mathcal{T}\|+\|\beth\mathfrak{g}_n-\mathfrak{L}\mathfrak{g}_n\|}{2b}\}))
\end{aligned}$$

Making $n\to+\infty$, we get

$$\begin{aligned}
\psi(\|\aleph\int - \beth\mathfrak{g}_n\|) \leq\ & \mathfrak{F}\left(\psi(\lim_{n\to+\infty}\daleth(\int,\mathfrak{g}_n)),\phi(\lim_{n\to+\infty}\daleth(\int,\mathfrak{g}_n))\right)\\
=\ & \mathfrak{F}(\psi(\max\{\|\mathcal{T}-\mathcal{T}\|,\|\aleph\int-\mathcal{T}\|,\|\mathcal{T}-\mathcal{T}\|,\frac{\|\aleph\int-\mathcal{T}\|+\|\mathcal{T}-\mathcal{T}\|}{2b},\\
& \frac{\|\aleph\int-\mathcal{T}\|+\|\mathcal{T}-\mathcal{T}\|}{2b}\}),\\
& \phi(\max\{\|\mathcal{T}-\mathcal{T}\|,\|\aleph\int-\mathcal{T}\|,\|\mathcal{T}-\mathcal{T}\|,\frac{\|\aleph\int-\mathcal{T}\|+\|\mathcal{T}-\mathcal{T}\|}{2b},\\
& \frac{\|\aleph\int-\mathcal{T}\|+\|\mathcal{T}-\mathcal{T}\|}{2b}\}))\\
=\ & \mathfrak{F}\left(\psi(\max\{\|\aleph\int-\mathcal{T}\|,\frac{\|\aleph\int-\mathcal{T}\|}{2b}\}),\phi(\max\{\|\aleph\int-\mathcal{T}\|,\frac{\|\aleph\int-\mathcal{T}\|}{2b}\})\right)
\end{aligned}$$

implies that

$$\psi(\|\aleph\int-\mathcal{T}\|)\leq\mathfrak{F}(\psi(\|\aleph\int-\mathcal{T}\|)),\phi(\|\aleph\int-\mathcal{T}\|))$$

thus either $\psi(\|\aleph\int-\mathcal{T}\|)=0$ or $\phi(\|\aleph\int-\mathcal{T}\|)=0$. Hence $\aleph\int=\mathcal{T}=\Gamma\int$. Therefore, $\int$ is the point of confidence of the pair $(\aleph,\Gamma)$. As $\mathcal{T}\in\mathfrak{L}(M)$, there exist a point $\gamma\in M$ such that $\mathfrak{L}\gamma=\mathcal{T}$. We assert that $\beth\gamma=\mathfrak{L}\gamma$. Putting $\mathfrak{h}=\int$ and $\mathfrak{g}=\gamma$ in equation (3.1) we get

$$\psi(\|\aleph\int-\beth\gamma\|)\leq\mathfrak{F}(\psi(\daleth(\int,\gamma)),\phi(\daleth(\int,\gamma))) \tag{3.5}$$

$$\begin{aligned}
=\ & \mathfrak{F}(\psi(\max\{\|\Gamma\int-\mathfrak{L}\gamma\|,\frac{\|\aleph\int-\Gamma\int\|}{1+\|\aleph\int-\Gamma\int\|},\frac{\|\beth\gamma-\mathfrak{L}\gamma\|}{1+\|\beth\gamma-\mathfrak{L}\gamma\|},\\
& \frac{\|\aleph\int-\mathfrak{L}\gamma\|+\|\Gamma\int-\beth\gamma\|}{2b},\frac{\|\aleph\int-\Gamma\int\|+\|\beth\gamma-\mathfrak{L}\gamma\|}{2b}\}),\\
& \phi(\max\{\|\Gamma\int-\mathfrak{L}\gamma\|,\frac{\|\aleph\int-\Gamma\int\|}{1+\|\aleph\int-\Gamma\int\|},\frac{\|\beth\gamma-\mathfrak{L}\gamma\|}{1+\|\beth\gamma-\mathfrak{L}\gamma\|},\\
& \frac{\|\aleph\int-\mathfrak{L}\gamma\|+\|\Gamma\int-\beth\gamma\|}{2b},\frac{\|\aleph\int-\Gamma\int\|+\|\beth\gamma-\mathfrak{L}\gamma\|}{2b}\}))\\
=\ & \mathfrak{F}(\psi(\max\{\|\Gamma\int-\mathcal{T}\|,\frac{\|\aleph\int-\Gamma\int\|}{1+\|\aleph\int-\Gamma\int\|},\frac{\|\beth\gamma-\mathcal{T}\|}{1+\|\beth\gamma-\mathcal{T}\|},\\
& \frac{\|\aleph\int-\mathcal{T}\|+\|\Gamma\int-\beth\gamma\|}{2b},\frac{\|\aleph\int-\Gamma\int\|+\|\beth\gamma-\mathcal{T}\|}{2b}\}),\\
& \phi(\max\{\|\Gamma\int-\mathcal{T}\|,\frac{\|\aleph\int-\Gamma\int\|}{1+\|\aleph\int-\Gamma\int\|},\frac{\|\beth\gamma-\mathcal{T}\|}{1+\|\beth\gamma-\mathcal{T}\|},\\
& \frac{\|\aleph\int-\mathcal{T}\|+\|\Gamma\int-\beth\gamma\|}{2b},\frac{\|\aleph\int-\Gamma\int\|+\|\beth\gamma-\mathcal{T}\|}{2b}\}))\\
\leq\ & \mathfrak{F}(\psi(\max\{\|\Gamma\int-\mathcal{T}\|,\|\aleph\int-\Gamma\int\|,\|\beth\gamma-\mathcal{T}\|,\\
& \frac{\|\aleph\int-\mathcal{T}\|+\|\Gamma\int-\beth\gamma\|}{2b},\frac{\|\aleph\int-\Gamma\int\|+\|\beth\gamma-\mathcal{T}\|}{2b}\}),\\
& \phi(\max\{\|\Gamma\int-\mathcal{T}\|,\|\aleph\int-\Gamma\int\|,\|\beth\gamma-\mathcal{T}\|,\\
& \frac{\|\aleph\int-\mathcal{T}\|+\|\Gamma\int-\beth\gamma\|}{2b},\frac{\|\aleph\int-\Gamma\int\|+\|\beth\gamma-\mathcal{T}\|}{2b}\}))
\end{aligned}$$

$$
\begin{aligned}
&= \mathfrak{F}(\psi(\max\{\|\mathcal{T}-\mathcal{T}\|, \|\mathcal{T}-\mathcal{T}\|, \|\gimel\gamma-\mathcal{T}\|, \frac{\|\mathcal{T}-\mathcal{T}\|+\|\mathcal{T}-\gimel\gamma\|}{2b},\\
&\quad \frac{\|\mathcal{T}-\mathcal{T}\|+\|\gimel\gamma-\mathcal{T}\|}{2b}\}),\\
&\quad \phi(\max\{\|\mathcal{T}-\mathcal{T}\|, \|\mathcal{T}-\mathcal{T}\|, \|\gimel\gamma-\mathcal{T}\|, \frac{\|\mathcal{T}-\mathcal{T}\|+\|\mathcal{T}-\gimel\gamma\|}{2b},\\
&\quad \frac{\|\mathcal{T}-\mathcal{T}\|+\|\gimel\gamma-\mathcal{T}\|}{2b}\}))\\
&= \mathfrak{F}(\psi(\max\{\|\gimel\gamma-\mathcal{T}\|, \frac{\|\mathcal{T}-\gimel\gamma\|}{2b}\}), \phi(\max\{\|\gimel\gamma-\mathcal{T}\|, \frac{\|\mathcal{T}-\gimel\gamma\|}{2b}\}))
\end{aligned}
$$

which implies that

$$\psi(\|\gimel\gamma-\mathcal{T}\|) \leq \mathfrak{F}\left(\psi(\|\gimel\gamma-\mathcal{T}\|), \phi(\|\gimel\gamma-\mathcal{T}\|)\right).$$

Thus either $\psi(\|\gimel\gamma-\mathcal{T}\|)=0$ or $\phi(\|\gimel\gamma-\mathcal{T}\|)=0$. Hence, $\gimel\gamma=\mathcal{T}$ *i.e.* $\gimel\gamma=\mathcal{T}=\mathfrak{L}\gamma$. Hence γ is a point of coincidence of pair $(\gimel,\mathfrak{L})$.

Since the pair $(\aleph,\Gamma)$ is weakly compatible and $\aleph\mathcal{T}=\Gamma\mathcal{T}$. Therefore, $\aleph\mathcal{T}=\aleph\Gamma\int=\Gamma\aleph\int=\Gamma\mathcal{T}$. Now we show that $\mathcal{T}$ is a (CFP) of the pair $(\aleph,\Gamma)$. Putting $\mathfrak{h}=\mathcal{T}, \mathfrak{g}=\gamma$ in (3.1), we get

$$\psi(\|\aleph\mathcal{T}-\gimel\gamma\|) \leq \mathfrak{F}(\psi(\daleth(\mathcal{T},\gamma)), \phi(\daleth(\mathcal{T},\gamma))) \tag{3.6}$$

$$
\begin{aligned}
&= \mathfrak{F}(\psi(\max\{\|\Gamma\mathcal{T}-\mathfrak{L}\gamma\|, \frac{\|\aleph\mathcal{T}-\Gamma\mathcal{T}\|}{1+\|\aleph\mathcal{T}-\Gamma\mathcal{T}\|}, \frac{\|\gimel\gamma-\mathfrak{L}\gamma\|}{1+\|\gimel\gamma-\mathfrak{L}\gamma\|},\\
&\quad \frac{\|\aleph\mathcal{T}-\mathfrak{L}\gamma\|+\|\Gamma\mathcal{T}-\gimel\gamma\|}{2b}, \frac{\|\aleph\mathcal{T}-\Gamma\mathcal{T}\|+\|\gimel\gamma-\mathfrak{L}\gamma\|}{2b}\}),\\
&\quad \phi(\max\{\|\Gamma\mathcal{T}-\mathfrak{L}\gamma\|, \frac{\|\aleph\mathcal{T}-\Gamma\mathcal{T}\|}{1+\|\aleph\mathcal{T}-\Gamma\mathcal{T}\|}, \frac{\|\gimel\gamma-\mathfrak{L}\gamma\|}{1+\|\gimel\gamma-\mathfrak{L}\gamma\|},\\
&\quad \frac{\|\aleph\mathcal{T}-\mathfrak{L}\gamma\|+\|\Gamma\mathcal{T}-\gimel\gamma\|}{2b}, \frac{\|\aleph\mathcal{T}-\Gamma\mathcal{T}\|+\|\gimel\gamma-\mathfrak{L}\gamma\|}{2b}\}))\\
&= \mathfrak{F}(\psi(\max\{\|\aleph\mathcal{T}-\mathfrak{L}\gamma\|, \frac{\|\aleph\mathcal{T}-\aleph\mathcal{T}\|}{1+\|\aleph\mathcal{T}-\aleph\mathcal{T}\|}, \frac{\|\gimel\gamma-\mathfrak{L}\gamma\|}{1+\|\gimel\gamma-\mathfrak{L}\gamma\|},\\
&\quad \frac{\|\aleph\mathcal{T}-\mathfrak{L}\gamma\|+\|\aleph\mathcal{T}-\gimel\gamma\|}{2b}, \frac{\|\aleph\mathcal{T}-\aleph\mathcal{T}\|+\|\gimel\gamma-\mathcal{T}\|}{2b}\}),\\
&\quad \phi(\max\{\|\aleph\mathcal{T}-\mathfrak{L}\gamma\|, \frac{\|\aleph\mathcal{T}-\aleph\mathcal{T}\|}{1+\|\aleph\mathcal{T}-\aleph\mathcal{T}\|}, \frac{\|\gimel\gamma-\mathfrak{L}\gamma\|}{1+\|\gimel\gamma-\mathfrak{L}\gamma\|},\\
&\quad \frac{\|\aleph\mathcal{T}-\mathfrak{L}\gamma\|+\|\aleph\mathcal{T}-\gimel\gamma\|}{2b}, \frac{\|\aleph\mathcal{T}-\aleph\mathcal{T}\|+\|\gimel\gamma-\mathcal{T}\|}{2b}\}))\\
&\leq \mathfrak{F}(\psi(\max\{\|\aleph\mathcal{T}-\mathfrak{L}\gamma\|, \|\aleph\mathcal{T}-\aleph\mathcal{T}\|, \|\gimel\gamma-\mathfrak{L}\gamma\|,\\
&\quad \frac{\|\aleph\mathcal{T}-\mathfrak{L}\gamma\|+\|\aleph\mathcal{T}-\gimel\gamma\|}{2b}, \frac{\|\aleph\mathcal{T}-\aleph\mathcal{T}\|+\|\gimel\gamma-\mathfrak{L}\gamma\|}{2b}\}),\\
&\quad \phi(\max\{\|\aleph\mathcal{T}-\mathfrak{L}\gamma\|, \|\aleph\mathcal{T}-\aleph\mathcal{T}\|, \|\gimel\gamma-\mathfrak{L}\gamma\|,\\
&\quad \frac{\|\aleph\mathcal{T}-\mathfrak{L}\gamma\|+\|\aleph\mathcal{T}-\gimel\gamma\|}{2b}, \frac{\|\aleph\mathcal{T}-\aleph\mathcal{T}\|+\|\gimel\gamma-\mathfrak{L}\gamma\|}{2b}\}))
\end{aligned}
$$

$$
\begin{aligned}
&= \mathfrak{F}(\psi(\max\{\|\aleph\mathcal{T}-\mathcal{T}\|,\|\aleph\mathcal{T}-\aleph\mathcal{T}\|,\|\mathcal{T}-\mathcal{T}\|,\frac{\|\aleph\mathcal{T}-\mathcal{T}\|+\|\aleph\mathcal{T}-\mathcal{T}\|}{2b},\\
&\quad \frac{\|\aleph\mathcal{T}-\aleph\mathcal{T}\|+\|\mathcal{T}-\mathcal{T}\|}{2b}\}).\\
&\quad \phi(\max\{\|\aleph\mathcal{T}-\mathcal{T}\|,\|\aleph\mathcal{T}-\aleph\mathcal{T}\|,\|\mathcal{T}-\mathcal{T}\|,\frac{\|\aleph\mathcal{T}-\mathcal{T}\|+\|\aleph\mathcal{T}-\mathcal{T}\|}{2b},\\
&\quad \frac{\|\aleph\mathcal{T}-\aleph\mathcal{T}\|+\|\mathcal{T}-\mathcal{T}\|}{2b}\}))\\
&= \mathfrak{F}(\psi(\max\{\|\aleph\mathcal{T}-\mathcal{T}\|,\frac{\|\aleph\mathcal{T}-\mathcal{T}\|}{b}\}),\phi(\max\{\|\aleph\mathcal{T}-\mathcal{T}\|,\frac{\|\aleph\mathcal{T}-\mathcal{T}\|}{b}\})).
\end{aligned}
$$

Implies that,

$$\psi(\|\aleph\mathcal{T}-\mathcal{T}\|)\leq\mathfrak{F}(\psi(\|\aleph\mathcal{T}-\mathcal{T}\|),\phi(\|\aleph\mathcal{T}-\mathcal{T}\|)).$$

Thus either, $\psi(\|\aleph\mathcal{T}-\mathcal{T}\|)=0$ or $\phi(\|\aleph\mathcal{T}-\mathcal{T}\|)=0$, which gives that, $\aleph\mathcal{T}=\mathcal{T}, i.e, \aleph\mathcal{T}=\mathcal{T}=\Gamma\mathcal{T}$. Which show that $\mathcal{T}$ is a (CFP) of the pair $(\aleph,\Gamma)$.

Similarly, Putting $\mathfrak{h}=\int, \mathfrak{g}=\mathcal{T}$ in (3.1), we get

$$\psi(\|\aleph\int-\beth\mathcal{T}\|)\leq\mathfrak{F}(\psi(\daleth(\int,\mathcal{T})),\phi(\daleth(\int,\mathcal{T}))) \tag{3.7}$$

$$
\begin{aligned}
&= \mathfrak{F}(\psi(\max\{\|\Gamma\int-\mathfrak{L}\mathcal{T}\|,\frac{\|\aleph\int-\Gamma\int\|}{1+\|\aleph\int-\Gamma\int\|},\frac{\|\beth\mathcal{T}-\mathfrak{L}\mathcal{T}\|}{1+\|\beth\mathcal{T}-\mathfrak{L}\mathcal{T}\|},\\
&\quad \frac{\|\aleph\int-\mathfrak{L}\mathcal{T}\|+\|\Gamma\int-\beth\mathcal{T}\|}{2b},\frac{\|\aleph\int-\Gamma\int\|+\|\beth\mathcal{T}-\mathfrak{L}\mathcal{T}\|}{2b}\}),\\
&\quad \phi(\max\{\|\Gamma\int-\mathfrak{L}\mathcal{T}\|,\frac{\|\aleph\int-\Gamma\int\|}{1+\|\aleph\int-\Gamma\int\|},\frac{\|\beth\mathcal{T}-\mathfrak{L}\mathcal{T}\|}{1+\|\beth\mathcal{T}-\mathfrak{L}\mathcal{T}\|},\\
&\quad \frac{\|\aleph\int-\mathfrak{L}\mathcal{T}\|+\|\Gamma\int-\beth\mathcal{T}\|}{2b},\frac{\|\aleph\int-\Gamma\int\|+\|\beth\mathcal{T}-\mathfrak{L}\mathcal{T}\|}{2b}\}))\\
&= \mathfrak{F}(\psi(\max\{\|\Gamma\int-\mathfrak{L}\mathcal{T}\|,\frac{\|\aleph\int-\Gamma\int\|}{1+\|\aleph\int-\Gamma\int\|},\frac{\|\mathfrak{L}\mathcal{T}-\mathfrak{L}\mathcal{T}\|}{1+\|\mathfrak{L}\mathcal{T}-\mathfrak{L}\mathcal{T}\|},\\
&\quad \frac{\|\aleph\int-\mathfrak{L}\mathcal{T}\|+\|\Gamma\int-\mathfrak{L}\mathcal{T}\|}{2b},\frac{\|\aleph\int-\Gamma\int\|+\|\mathfrak{L}\mathcal{T}-\mathfrak{L}\mathcal{T}\|}{2b}\}),\\
&\quad \phi(\max\{\|\Gamma\int-\mathfrak{L}\mathcal{T}\|,\frac{\|\aleph\int-\Gamma\int\|}{1+\|\aleph\int-\Gamma\int\|},\frac{\|\mathfrak{L}\mathcal{T}-\mathfrak{L}\mathcal{T}\|}{1+\|\mathfrak{L}\mathcal{T}-\mathfrak{L}\mathcal{T}\|},\\
&\quad \frac{\|\aleph\int-\mathfrak{L}\mathcal{T}\|+\|\Gamma\int-\mathfrak{L}\mathcal{T}\|}{2b},\frac{\|\aleph\int-\Gamma\int\|+\|\mathfrak{L}\mathcal{T}-\mathfrak{L}\mathcal{T}\|}{2b}\}))\\
&\leq \mathfrak{F}(\psi(\max\{\|\Gamma\int-\mathfrak{L}\mathcal{T}\|,\|\aleph\int-\Gamma\int\|,\|\mathfrak{L}\mathcal{T}-\mathfrak{L}\mathcal{T}\|,\\
&\quad \frac{\|\aleph\int-\mathfrak{L}\mathcal{T}\|+\|\Gamma\int-\mathfrak{L}\mathcal{T}\|}{2b},\frac{\|\aleph\int-\Gamma\int\|+\|\mathfrak{L}\mathcal{T}-\mathfrak{L}\mathcal{T}\|}{2b}\}),\\
&\quad \phi(\max\{\|\Gamma\int-\mathfrak{L}\mathcal{T}\|,\|\aleph\int-\Gamma\int\|,\|\mathfrak{L}\mathcal{T}-\mathfrak{L}\mathcal{T}\|,\\
&\quad \frac{\|\aleph\int-\mathfrak{L}\mathcal{T}\|+\|\Gamma\int-\mathfrak{L}\mathcal{T}\|}{2b},\frac{\|\aleph\int-\Gamma\int\|+\|\mathfrak{L}\mathcal{T}-\mathfrak{L}\mathcal{T}\|}{2b}\}))
\end{aligned}
$$

$$= \mathfrak{F}(\psi(\max\{\|\mathcal{T}-\mathfrak{L}\mathcal{T}\|, \|\mathcal{T}-\mathcal{T}\|, \|\mathfrak{L}\mathcal{T}-\mathfrak{L}\mathcal{T}\|, \frac{\|\mathcal{T}-\mathfrak{L}\mathcal{T}\|+\|\mathcal{T}-\mathfrak{L}\mathcal{T}\|}{2b}, \frac{\|\mathcal{T}-\mathcal{T}\|+\|\mathfrak{L}\mathcal{T}-\mathfrak{L}\mathcal{T}\|}{2b}\}). \phi(\max\{\|\mathcal{T}-\mathfrak{L}\mathcal{T}\|, \|\mathcal{T}-\mathcal{T}\|, \|\mathfrak{L}\mathcal{T}-\mathfrak{L}\mathcal{T}\|, \frac{\|\mathcal{T}-\mathfrak{L}\mathcal{T}\|+\|\mathcal{T}-\mathfrak{L}\mathcal{T}\|}{2b}, \frac{\|\mathcal{T}-\mathcal{T}\|+\|\mathfrak{L}\mathcal{T}-\mathfrak{L}\mathcal{T}\|}{2b}\}))$$

$$= \mathfrak{F}(\psi(\max\{\|\mathcal{T}-\mathfrak{L}\mathcal{T}\|, \frac{\|\mathcal{T}-\mathfrak{L}\mathcal{T}\|}{b}\}), \phi(\max\{\|\mathcal{T}-\mathfrak{L}\mathcal{T}\|, \frac{\|\mathcal{T}-\mathfrak{L}\mathcal{T}\|}{b}\})).$$

Implies that,

$$\psi(\|\mathcal{T}-\mathfrak{L}\mathcal{T}\|) \leq \mathfrak{F}(\psi(\|\mathcal{T}-\mathfrak{L}\mathcal{T}\|), \phi(\|\mathcal{T}-\mathfrak{L}\mathcal{T}\|)).$$

Thus either, $\psi(\|\mathcal{T}-\mathfrak{L}\mathcal{T}\|) = 0$ or $\phi(\|\mathcal{T}-\mathfrak{L}\mathcal{T}\|) = 0$, which gives that, $\mathcal{T} = \mathfrak{L}\mathcal{T}$, *i.e.*, $\beth\mathcal{T} = \mathfrak{L}\mathcal{T} = \mathcal{T}$. Hence $\mathcal{T}$ is a (CFP) of $\aleph, \beth, \Gamma$ and $\mathfrak{L}$.
To prove that $\mathcal{T}$ is the unique (CFP), let $\mathfrak{I}$ be another (CFP) of $\aleph, \beth, \Gamma$ and $\mathfrak{L}$. Using (3.1), $\mathfrak{h} = \mathcal{T}, \mathfrak{g} = \mathfrak{I}$, we get

$$\psi(\|\aleph\mathcal{T}-\beth\mathfrak{I}\|) \leq \mathfrak{F}(\psi(\daleth(\mathcal{T},\mathfrak{I})), \phi(\daleth(\mathcal{T},\mathfrak{I}))) \tag{3.8}$$

$$= \mathfrak{F}(\psi(\max\{\|\Gamma\mathcal{T}-\mathfrak{L}\mathfrak{I}\|, \frac{\|\aleph\mathcal{T}-\Gamma\mathcal{T}\|}{1+\|\aleph\mathcal{T}-\Gamma\mathcal{T}\|}, \frac{\|\beth\mathfrak{I}-\mathfrak{L}\mathfrak{I}\|}{1+\|\beth\mathfrak{I}-\mathfrak{L}\mathfrak{I}\|}, \frac{\|\aleph\mathcal{T}-\mathfrak{L}\mathfrak{I}\|+\|\Gamma\mathcal{T}-\beth\mathfrak{I}\|}{2b}, \frac{\|\aleph\mathcal{T}-\Gamma\mathcal{T}\|+\|\beth\mathfrak{I}-\mathfrak{L}\mathfrak{I}\|}{2b}\}), \phi(\max\{\|\Gamma\mathcal{T}-\mathfrak{L}\mathfrak{I}\|, \frac{\|\aleph\mathcal{T}-\Gamma\mathcal{T}\|}{1+\|\aleph\mathcal{T}-\Gamma\mathcal{T}\|}, \frac{\|\beth\mathfrak{I}-\mathfrak{L}\mathfrak{I}\|}{1+\|\beth\mathfrak{I}-\mathfrak{L}\mathfrak{I}\|}, \frac{\|\aleph\mathcal{T}-\mathfrak{L}\mathfrak{I}\|+\|\Gamma\mathcal{T}-\beth\mathfrak{I}\|}{2b}, \frac{\|\aleph\mathcal{T}-\Gamma\mathcal{T}\|+\|\beth\mathfrak{I}-\mathfrak{L}\mathfrak{I}\|}{2b}\}))$$

$$= \mathfrak{F}(\psi(\max\{\|\mathcal{T}-\mathfrak{I}\|, \frac{\|\mathcal{T}-\mathcal{T}\|}{1+\|\mathcal{T}-\mathcal{T}\|}, \frac{\|\mathfrak{I}-\mathfrak{I}\|}{1+\|\mathfrak{I}-\mathfrak{I}\|}, \frac{\|\mathcal{T}-\mathfrak{I}\|+\|\mathcal{T}-\mathfrak{I}\|}{2b}, \frac{\|\mathcal{T}-\mathcal{T}\|+\|\mathfrak{I}-\mathcal{T}\|}{2b}\}), \phi(\max\{\|\mathcal{T}-\mathfrak{I}\|, \frac{\|\mathcal{T}-\mathcal{T}\|}{1+\|\mathcal{T}-\mathcal{T}\|}, \frac{\|\mathfrak{I}-\mathfrak{I}\|}{1+\|\mathfrak{I}-\mathfrak{I}\|}, \frac{\|\mathcal{T}-\mathfrak{I}\|+\|\mathcal{T}-\mathfrak{I}\|}{2b}, \frac{\|\mathcal{T}-\mathcal{T}\|+\|\mathfrak{I}-\mathcal{T}\|}{2b}\}))$$

$$\leq \mathfrak{F}(\psi(\max\{\|\mathcal{T}-\mathfrak{I}\|, \|\mathcal{T}-\mathcal{T}\|, \|\mathfrak{I}-\mathfrak{I}\|, \frac{\|\mathcal{T}-\mathfrak{I}\|+\|\mathcal{T}-\mathfrak{I}\|}{2b}, \frac{\|\mathcal{T}-\mathcal{T}\|+\|\mathfrak{I}-\mathcal{T}\|}{2b}\}), \phi(\max\{\|\mathcal{T}-\mathfrak{I}\|, \|\mathcal{T}-\mathcal{T}\|, \|\mathfrak{I}-\mathfrak{I}\|, \frac{\|\mathcal{T}-\mathfrak{I}\|+\|\mathcal{T}-\mathfrak{I}\|}{2b}, \frac{\|\mathcal{T}-\mathcal{T}\|+\|\mathfrak{I}-\mathcal{T}\|}{2b}\}))$$

$$= \mathfrak{F}(\psi(\max\{\|\mathcal{T}-\mathfrak{I}\|, \|\mathcal{T}-\mathcal{T}\|, \|\mathfrak{I}-\mathfrak{I}\|, \frac{\|\mathcal{T}-\mathfrak{I}\|}{b}, \frac{\|\mathfrak{I}-\mathcal{T}\|\|}{2b}\}), \phi(\max\{\|\mathcal{T}-\mathfrak{I}\|, \|\mathcal{T}-\mathcal{T}\|, \|\mathfrak{I}-\mathfrak{I}\|, \frac{\|\mathcal{T}-\mathfrak{I}\|}{b}, \frac{\|\mathfrak{I}-\mathcal{T}\|\|}{2b}\})).$$

Which implies that

$$\psi(\|\mathcal{T}-\mathfrak{I}\|) \leq \mathfrak{F}(\psi(\|\mathcal{T}-\mathfrak{I}\|), \phi(\|\mathcal{T}-\mathfrak{I}\|)).$$

Thus either $\psi(\|\mathcal{T}-\mathfrak{I}\|)=0$ or $\phi(\|\mathcal{T}-\mathfrak{I}\|)=0$. which gives that $\|\mathcal{T}=\mathfrak{I}\|$ $i.e, \mathcal{T}=\mathfrak{I}$. Hence, $\mathcal{T}$ is the unique (CFP) of $\aleph, \gimel, \Gamma$ and $\mathfrak{L}$. □

Corollary 3.3. *Let $(M, \|.\|))$ be a complete $CNbMS$ with the norm $D_b^c(\mathfrak{h}, 0) = \|\mathfrak{h}\|$ and the co-efficient $b \geq 1$. Suppose that the mappings $\aleph, \mathfrak{L}$ be self-mapping of a $(CbNS)$ $(M, \|.\|)$ satisfying:*

$$\psi(\|\aleph\mathfrak{h}-\aleph\mathfrak{g}\|) \leq \mathfrak{F}(\psi(\daleth(\mathfrak{h},\mathfrak{g})), \phi(\daleth(\mathfrak{h},\mathfrak{g})))$$

for all $\mathfrak{h}, \mathfrak{g}$, where $\daleth(\mathfrak{h},\mathfrak{g}) = \max\{\|\mathfrak{L}\mathfrak{h}-\mathfrak{L}\mathfrak{g}\|, \|\aleph\mathfrak{h}-\mathfrak{L}\mathfrak{h}\|, \|\aleph\mathfrak{g}-\mathfrak{L}\mathfrak{g}\|, \frac{\|\aleph\mathfrak{h}-\mathfrak{L}\mathfrak{g}\|+\mathfrak{L}\mathfrak{h}-\aleph\mathfrak{g}\|}{2b}\}$. If the pairs $(\aleph, \mathfrak{L})$ satisfies (CLR)-property of $\mathfrak{L}$ then, $\aleph$ and $\mathfrak{L}$ have a common point of coincidence in M. Moreover if $\aleph$ and $\mathfrak{L}$ are (WC), then the pair $(\aleph, \mathfrak{L})$ has a unique (CFP).

Proof. By putting $\aleph = \gimel$ and $\mathfrak{L} = \Gamma$ in Theorem (3.2), the result can be proved. □

Corollary 3.4. *Let $(M, \|.\|)$ be a complete $(CbNS)$ with the norm $D_b^c(\mathfrak{h}, 0) = \|\mathfrak{h}\|$ and the co-efficient $b \geq 1$. Suppose that the mappings $\aleph, \gimel, \Gamma$, and $\mathfrak{L}$ be a self mapping of a $(CbNS)$ $(M, \|.\|)$ satisfying*

$$\psi(\|\aleph\mathfrak{h}-\gimel\mathfrak{g}\|) \leq \mathfrak{F}(\psi(\daleth(\mathfrak{h},\mathfrak{g})), \phi(\daleth(\mathfrak{h},\mathfrak{g}))),$$

for all $\mathfrak{h}, \mathfrak{g}$, where $\daleth(\mathfrak{h},\mathfrak{g}) = \max\{\|\Gamma\mathfrak{h}-\mathfrak{L}\mathfrak{g}\|, \|\aleph\mathfrak{h}-\Gamma\mathfrak{h}\|, \|\gimel\mathfrak{g}-\mathfrak{L}\mathfrak{g}\|, \frac{\|\aleph\mathfrak{h}-\mathfrak{L}\mathfrak{g}\|+\|\Gamma\mathfrak{h}-\gimel\mathfrak{g}\|}{2b}\}$. If the pairs $(\aleph, \Gamma)$ and $(\gimel, \mathfrak{L})$ satisfy the (CLR)-property of $\Gamma\mathfrak{L}$, then $(\aleph, \Gamma)$ and $(\gimel, \mathfrak{L})$ have a point of coincidence. Moreover, if $(\aleph, \Gamma)$ and $(\gimel, \mathfrak{L})$ are compatible, then $\aleph, \gimel, \Gamma$ and $\mathfrak{L}$ have a unique (CFP).

Proof. By taking $\psi(t) = t$ in Theorem (3.2), we complete the proof. □

Example 3.5. Let $M = (0,1]$, $E = R^2$ and $P = \{(\mathfrak{h},\mathfrak{g}) \in E : \mathfrak{h},\mathfrak{g} \geq 0\}$ and $D_b^c : M \times M \to E$ as $D_b^c(\mathfrak{h},\mathfrak{g}) = \|\ \mathfrak{h}-\mathfrak{g}\ \|^2$, for all $\mathfrak{h},\mathfrak{g} \in M$ and $\aleph$, $\gimel, \Gamma, \mathfrak{L} : M \to M$. Then $(M, \|\ .\ \|)$ is a complete $(CNbMS)$ with co-efficient $b = 2$.

$$\aleph(\mathfrak{h}) = \begin{cases} \frac{1}{2} & if\ \mathfrak{h} \in (0, \frac{1}{2}] \\ \frac{1}{5} & if\ \mathfrak{h} \in (\frac{1}{2}, 1] \end{cases}, \qquad \gimel(\mathfrak{h}) = \begin{cases} \frac{1}{2} & if\ \mathfrak{h} \in (0, \frac{1}{2}] \\ \frac{1}{3} & if\ \mathfrak{h} \in (\frac{1}{2}, 1] \end{cases}$$

$$\Gamma(\mathfrak{h}) = \begin{cases} \frac{1}{2} & if\ \mathfrak{h} \in (0, \frac{1}{2}] \\ \frac{1}{7} & if\ \mathfrak{h} \in (\frac{1}{2}, 1] \end{cases}, \qquad \mathfrak{L}(\mathfrak{h}) = \begin{cases} \frac{1}{2} & if\ \mathfrak{h} \in (0, \frac{1}{2}] \\ \frac{1}{9} & if\ \mathfrak{h} \in (\frac{1}{2}, 1] \end{cases}$$

First we verify condition (1) of Lemma (3.1)

Let $\{\mathfrak{h}_n\} = \{\frac{1}{1+2n}\}_{n\geq 1}$ and $\{\mathfrak{g}_n\} = \{\frac{1}{3+1n}\}_{n\geq 1}$ be two sequence in M. Then

$$\lim_{n\to+\infty} \aleph(\mathfrak{h}_n) = \lim_{n\to+\infty} \aleph\left(\frac{1}{1+2n}\right) = \frac{1}{2}$$

$$\lim_{n\to+\infty} \Gamma(\mathfrak{h}_n) = \lim_{n\to+\infty} \Gamma\left(\frac{1}{1+2n}\right) = \frac{1}{2}$$

$$\lim_{n\to+\infty} \beth(\mathfrak{g}_n) = \lim_{n\to+\infty} \beth\left(\frac{1}{3+1n}\right) = \frac{1}{2}$$

$$\lim_{n\to+\infty} \mathfrak{L}(\mathfrak{g}_n) = \lim_{n\to+\infty} \mathfrak{L}\left(\frac{1}{3+1n}\right) = \frac{1}{2}$$

Thus $\lim_{n\to+\infty} \aleph(\mathfrak{h}_n) = \lim_{n\to+\infty} \Gamma(\mathfrak{h}_n) = \lim_{n\to+\infty} \beth(\mathfrak{g}_n) = \lim_{n\to+\infty} \mathfrak{L}(\mathfrak{g}_n) = \dfrac{1}{2} \in \Gamma(M) \cap \mathfrak{L}(M)$.

That is, $(\aleph, \Gamma)$ and $(\beth, \mathfrak{L})$ satisfies the common $(CLR_{\Gamma\mathfrak{L}})$-property.

Next, to verify inequality (3.1). Let us define the function $\psi, \phi : R_0^+ \to R_0^+$ by $\psi(t) = 2t$ and $\phi(t) = \frac{t}{4}$.
Case (i): Let $\mathfrak{h}, \mathfrak{g} \in (0, \frac{1}{2}]$. Then $\aleph(\mathfrak{h}) = \Gamma(\mathfrak{h}) = \beth(\mathfrak{g}) = \mathfrak{L}(\mathfrak{g}) = \frac{1}{2}$ and from the inequality (3.1)

$$\text{L.H.S} = \psi(\| \aleph\mathfrak{h} - \beth\mathfrak{g} \|^2) = \psi(\| \tfrac{1}{2} - \tfrac{1}{2} \|^2) = \psi(0) = 2(0) = 0$$

$$\begin{aligned}
\text{R.H.S} = \ & \mathfrak{F}(\psi\left(\daleth(\mathfrak{h},\mathfrak{g})\right), \phi\left(\daleth(\mathfrak{h},\mathfrak{g})\right)) \\
= \ & \mathfrak{F}(\psi(\max\{\| \Gamma\mathfrak{h} - \mathfrak{L}\mathfrak{g} \|^2, \frac{\| \aleph\mathfrak{h} - \Gamma\mathfrak{h} \|^2}{1 + \| \aleph\mathfrak{h} - \Gamma\mathfrak{h} \|^2}, \frac{\| \beth\mathfrak{g} - \mathfrak{L}\mathfrak{g} \|^2}{1 + \| \beth\mathfrak{g} - \mathfrak{L}\mathfrak{g} \|^2}, \\
& \frac{\| \aleph\mathfrak{h} - \mathfrak{L}\mathfrak{g} \|^2 + \| \Gamma\mathfrak{h} - \beth\mathfrak{g} \|^2}{2b}, \frac{\| \aleph\mathfrak{h} - \Gamma\mathfrak{h} \|^2 + \| \beth\mathfrak{g} - \mathfrak{L}\mathfrak{g} \|^2}{2b}\}), \\
& \phi(\max\{\| \Gamma\mathfrak{h} - \mathfrak{L}\mathfrak{g} \|^2, \frac{\| \aleph\mathfrak{h} - \Gamma\mathfrak{h} \|^2}{1 + \| \aleph\mathfrak{h} - \Gamma\mathfrak{h} \|^2}, \frac{\| \beth\mathfrak{g} - \mathfrak{L}\mathfrak{g} \|^2}{1 + \| \beth\mathfrak{g} - \mathfrak{L}\mathfrak{g} \|^2}, \\
& \frac{\| \aleph\mathfrak{h} - \mathfrak{L}\mathfrak{g} \|^2 + \| \Gamma\mathfrak{h} - \beth\mathfrak{g} \|^2}{2b}, \frac{\| \aleph\mathfrak{h} - \Gamma\mathfrak{h} \|^2 + \| \beth\mathfrak{g} - \mathfrak{L}\mathfrak{g} \|^2}{2b}\})) \\
= \ & \mathfrak{F}(\psi(\max\{\| \frac{1}{2} - \frac{1}{2} \|^2, \frac{\| \frac{1}{2} - \frac{1}{2} \|^2}{1 + \| \frac{1}{2} - \frac{1}{2} \|^2}, \frac{\| \frac{1}{2} - \frac{1}{2} \|^2}{1 + \| \frac{1}{2} - \frac{1}{2} \|^2}, \\
& \frac{\| \frac{1}{2} - \frac{1}{2} \|^2 + \| \frac{1}{2} - \frac{1}{2} \|^2}{4}, \frac{\| \frac{1}{2} - \frac{1}{2} \|^2 + \| \frac{1}{2} - \frac{1}{2} \|^2}{4}\}), \\
& \phi(\max\{\| \frac{1}{2} - \frac{1}{2} \|^2, \frac{\| \frac{1}{2} - \frac{1}{2} \|^2}{1 + \| \frac{1}{2} - \frac{1}{2} \|^2}, \frac{\| \frac{1}{2} - \frac{1}{2} \|^2}{1 + \| \frac{1}{2} - \frac{1}{2} \|^2}, \\
& \frac{\| \frac{1}{2} - \frac{1}{2} \|^2 + \| \frac{1}{2} - \frac{1}{2} \|^2}{4}, \frac{\| \frac{1}{2} - \frac{1}{2} \|^2 + \| \frac{1}{2} - \frac{1}{2} \|^2}{4}\}))
\end{aligned}$$

R.H.S $= \psi(0) - \phi(0) = 2(0) - \frac{0}{4} = 0$. Therefore, L.H.S = R.H.S
Case (ii): Let $\mathfrak{h}, \mathfrak{g} \in (\frac{1}{2}, 1]$

$$\text{L.H.S} = \psi(\| \aleph\mathfrak{h} - \beth\mathfrak{g} \|^2) = \psi\left(\| \tfrac{1}{5} - \tfrac{1}{3} \|^2\right) = \psi(\tfrac{4}{225}) = (\tfrac{8}{225}) = 0.03555$$

$$\begin{aligned}
\text{R.H.S} = \;& \mathfrak{F}(\psi(\daleth(\mathfrak{h},\mathfrak{g})), \phi(\daleth(\mathfrak{h},\mathfrak{g})))\\
= \;& \mathfrak{F}(\psi(\max\{\| \Gamma\mathfrak{h} - \mathfrak{L}\mathfrak{g} \|^2, \frac{\| \aleph\mathfrak{h} - \Gamma\mathfrak{h} \|2}{1 + \| \aleph\mathfrak{h} - \Gamma\mathfrak{h} \|^2}, \frac{\| \beth\mathfrak{g} - \mathfrak{L}\mathfrak{g} \|^2}{1 + \| \beth\mathfrak{g} - \mathfrak{L}\mathfrak{g} \|^2},\\
& \frac{\| \aleph\mathfrak{h} - \mathfrak{L}\mathfrak{g} \|^2 + \| \Gamma\mathfrak{h} - \beth\mathfrak{g} \|^2}{2b}, \frac{\| \aleph\mathfrak{h} - \Gamma\mathfrak{h} \|^2 + \| \beth\mathfrak{g} - \mathfrak{L}\mathfrak{g} \|^2}{2b} \}),\\
& \phi(\max\{\| \Gamma\mathfrak{h} - \mathfrak{L}\mathfrak{g} \|^2, \frac{\| \aleph\mathfrak{h} - \Gamma\mathfrak{h} \|2}{1 + \| \aleph\mathfrak{h} - \Gamma\mathfrak{h} \|^2}, \frac{\| \beth\mathfrak{g} - \mathfrak{L}\mathfrak{g} \|^2}{1 + \| \beth\mathfrak{g} - \mathfrak{L}\mathfrak{g} \|^2},\\
& \frac{\| \aleph\mathfrak{h} - \mathfrak{L}\mathfrak{g} \|^2 + \| \Gamma\mathfrak{h} - \beth\mathfrak{g} \|^2}{2b}, \frac{\| \aleph\mathfrak{h} - \Gamma\mathfrak{h} \|^2 + \| \beth\mathfrak{g} - \mathfrak{L}\mathfrak{g} \|^2}{2b} \}))\\
= \;& \mathfrak{F}\Bigg(\psi\Bigg(\max\Bigg\{\| \frac{1}{7} - \frac{1}{9} \|^2, \frac{\| \frac{1}{5} - \frac{1}{7} \|^2}{1 + \| \frac{1}{5} - \frac{1}{7} \|^2}, \frac{\| \frac{1}{3} - \frac{1}{9} \|^2}{1 + \| \frac{1}{3} - \frac{1}{9} \|^2},\\
& \frac{\| \frac{1}{5} - \frac{1}{9} \|^2 + \| \frac{1}{7} - \frac{1}{3} \|^2}{4}, \frac{\| \frac{1}{5} - \frac{1}{7} \|^2 + \| \frac{1}{3} - \frac{1}{9} \|^2}{4} \Bigg\}\Bigg),\\
& \phi\Bigg(\max\Bigg\{\| \frac{1}{7} - \frac{1}{9} \|^2, \frac{\| \frac{1}{5} - \frac{1}{7} \|^2}{1 + \| \frac{1}{5} - \frac{1}{7} \|^2}, \frac{\| \frac{1}{3} - \frac{1}{9} \|^2}{1 + \| \frac{1}{3} - \frac{1}{9} \|^2},\\
& \frac{\| \frac{1}{5} - \frac{1}{9} \|^2 + \| \frac{1}{7} - \frac{1}{3} \|^2}{4}, \frac{\| \frac{1}{5} - \frac{1}{7} \|^2 + \| \frac{1}{3} - \frac{1}{9} \|^2}{4} \Bigg\}\Bigg)\Bigg)\\
= \;& \mathfrak{F}\Bigg(\psi\Bigg(\max\Bigg\{\| \frac{9}{63} - \frac{7}{63} \|^2, \frac{\| \frac{7}{35} - \frac{5}{35} \|^2}{1 + \| \frac{7}{35} - \frac{5}{35} \|^2}, \frac{\| \frac{3}{9} - \frac{1}{9} \|^2}{1 + \| \frac{3}{9} - \frac{1}{9} \|^2},\\
& \frac{\| \frac{9}{45} - \frac{5}{45} \|^2 + \| \frac{3}{21} - \frac{7}{21} \|^2}{4}, \frac{\| \frac{7}{35} - \frac{5}{35} \|^2 + \| \frac{3}{9} - \frac{1}{9} \|^2}{4} \Bigg\}\Bigg),\\
& \phi\Bigg(\max\Bigg\{\| \frac{9}{63} - \frac{7}{63} \|^2, \frac{\| \frac{7}{35} - \frac{5}{35} \|^2}{1 + \| \frac{7}{35} - \frac{5}{35} \|^2}, \frac{\| \frac{3}{9} - \frac{1}{9} \|^2}{1 + \| \frac{3}{9} - \frac{1}{9} \|^2},\\
& \frac{\| \frac{9}{45} - \frac{5}{45} \|^2 + \| \frac{3}{21} - \frac{7}{21} \|^2}{4}, \frac{\| \frac{7}{35} - \frac{5}{35} \|^2 + \| \frac{3}{9} - \frac{1}{9} \|^2}{4} \Bigg\}\Bigg)\Bigg)\\
= \;& \mathfrak{F}\Bigg(\psi\Bigg(\max\Bigg\{\| \frac{2}{63} \|^2, \frac{\| \frac{2}{35} \|^2}{1 + \| \frac{2}{35} \|^2}, \frac{\| \frac{2}{9} \|^2}{1 + \| \frac{2}{9} \|^2}, \frac{\| \frac{4}{45} \|^2 + \| \frac{4}{21} \|^2}{4}, \frac{\| \frac{2}{35} \|^2 + \| \frac{2}{9} \|^2}{4} \Bigg\}\Bigg),\\
& \phi\Bigg(\max\Bigg\{\| \frac{2}{63} \|^2, \frac{\| \frac{2}{35} \|^2}{1 + \| \frac{2}{35} \|^2}, \frac{\| \frac{2}{9} \|^2}{1 + \| \frac{2}{9} \|^2}, \frac{\| \frac{4}{45} \|^2 + \| \frac{4}{21} \|^2}{4}, \frac{\| \frac{2}{35} \|^2 + \| \frac{2}{9} \|^2}{4} \Bigg\}\Bigg)\Bigg)\\
= \;& \mathfrak{F}\Bigg(\psi\Bigg(\max\Bigg\{\left(\frac{4}{3969}\right), \frac{(\frac{4}{1225})}{1 + (\frac{4}{1225})}, \frac{(\frac{4}{81})}{1 + (\frac{4}{81})}, \frac{(\frac{16}{2025}) + (\frac{16}{441})}{4}, \frac{(\frac{4}{1225}) + (\frac{4}{81})}{4} \Bigg\}\Bigg),\\
& \phi\Bigg(\max\Bigg\{\left(\frac{4}{3969}\right), \frac{(\frac{4}{1225})}{1 + (\frac{4}{1225})}, \frac{(\frac{4}{81})}{1 + (\frac{4}{81})}, \frac{(\frac{16}{2025}) + (\frac{16}{441})}{4}, \frac{(\frac{4}{1225}) + (\frac{4}{81})}{4} \Bigg\}\Bigg)\Bigg)\\
= \;& \mathfrak{F}\Bigg(\psi\Bigg(\max\Bigg\{\left(\frac{4}{3969}\right), \frac{(\frac{4}{1225})}{(\frac{1229}{1225})}, \frac{(\frac{4}{81})}{(\frac{85}{81})}, \frac{(\frac{7056}{893025}) + (\frac{32400}{893025})}{4}, \frac{(\frac{324}{99225}) + (\frac{4900}{99225})}{4} \Bigg\}\Bigg),\\
& \phi\Bigg(\max\Bigg\{\left(\frac{4}{3969}\right), \frac{(\frac{4}{1225})}{(\frac{1229}{1225})}, \frac{(\frac{4}{81})}{(\frac{85}{81})}, \frac{(\frac{7056}{893025}) + (\frac{32400}{893025})}{4}, \frac{(\frac{324}{99225}) + (\frac{4900}{99225})}{4} \Bigg\}\Bigg)\Bigg)
\end{aligned}$$

$$\begin{aligned}
&= \mathfrak{F}\Bigg(\psi\Bigg(\max\Bigg\{\left(\frac{4}{3969}\right), \left(\frac{4}{1225}\right)\times\left(\frac{1225}{1229}\right), \left(\frac{4}{81}\right)\times\left(\frac{81}{85}\right), \\
&\quad \left(\frac{39546}{893025}\right)\times\left(\frac{1}{4}\right), \left(\frac{5224}{99225}\right)\times\left(\frac{1}{4}\right)\Bigg\}\Bigg), \\
&\quad \phi\Bigg(\max\Bigg\{\left(\frac{4}{3969}\right), \left(\frac{4}{1225}\right)\times\left(\frac{1225}{1229}\right), \left(\frac{4}{81}\right)\times\left(\frac{81}{85}\right), \\
&\quad \left(\frac{39546}{893025}\right)\times\left(\frac{1}{4}\right), \left(\frac{5224}{99225}\right)\times\left(\frac{1}{4}\right)\Bigg\}\Bigg)\Bigg) \\
&= \mathfrak{F}\left(\psi\left(\max\left\{\left(\frac{4}{3969}\right), \left(\frac{4}{1229}\right), \left(\frac{4}{85}\right), \left(\frac{39456}{3572100}\right), \left(\frac{5224}{396900}\right)\right\}\right),\right. \\
&\quad \left.\phi\left(\max\left\{\left(\frac{4}{3969}\right), \left(\frac{4}{1229}\right), \left(\frac{4}{85}\right), \left(\frac{39456}{3572100}\right), \left(\frac{5224}{396900}\right)\right\}\right)\right) \\
&= \mathfrak{F}(\psi(\max\{(0.001), (0.003), (0.047), (0.011), (0.013)\}), \\
&\quad \phi(\max\{(0.001), (0.003), (0.047), (0.011), (0.013)\})) \\
&= \mathfrak{F}(\psi((0.047)), \phi((0.047)))
\end{aligned}$$

R.H.S $= \psi(0.047) - \phi(0.047) = (0.094) - (0.01175) = (0.08225)$
Therefore, the inequality (3.1) holds. $(0.03555) < (0.08225)$. L.H.S < R.H.S

$$\psi(\| \aleph\mathfrak{h} - \beth\mathfrak{g} \|) \leq \mathfrak{F}(\psi(\urcorner(\mathfrak{h},\mathfrak{g})), \phi(\urcorner(\mathfrak{h},\mathfrak{g})))$$

Therefore from Theorem (3.2) the mappings $\aleph, \beth, \Gamma$ and $\mathfrak{L}$ have a unique common fixed point $\mathfrak{h} = \frac{1}{2}$.

Example 3.6. Let $(M, \| \, . \, \|)$ be a $(CNbMS)$ with co-efficient $b = 2$ and $E = R^2$, $P = \{(\mathfrak{h},\mathfrak{g}) \in E | \mathfrak{h},\mathfrak{g} > 0\} \subset R^2$, $M = [0, +\infty)$, $D_b^c : M \times M \to E$, such that $D_b^c(\mathfrak{h},\mathfrak{g}) = (\| \mathfrak{h} - \mathfrak{g} \|^2, \alpha \| \mathfrak{h} - \mathfrak{g} \|^2)$, for all $\mathfrak{h},\mathfrak{g} \in M$, where $\alpha \geq 0$ is a constant and $\aleph, \beth, \Gamma, \mathfrak{L} : M \to M$. Let us define the function $\psi, \phi : R_0^+ \to R_0^+$ by $\psi(t) = t$ and $\phi(t) = kt$ for some $k > 0$.

$$\aleph(\mathfrak{h}) = \beth(\mathfrak{h}) = \begin{cases} \frac{\mathfrak{h}}{12}, & if\ \mathfrak{h} \in [0,1] \\ \frac{\mathfrak{h}+11}{12}, & if\ \mathfrak{h} \in (1,+\infty) \end{cases}, \quad \Gamma(\mathfrak{h}) = \mathfrak{L}(\mathfrak{h}) = \begin{cases} \frac{\mathfrak{h}}{6}, & if\ \mathfrak{h} \in [0,1] \\ \frac{\mathfrak{h}+5}{6}, & if\ \mathfrak{h} \in (1,+\infty) \end{cases}$$

First we verify that condition (1) of Lemma (3.1)

Let $\{\mathfrak{h}_n\} = \left\{\frac{1}{n}\right\}_{n\geq 1}$ and $\{\mathfrak{g}_n\} = \left\{\frac{1}{n+1}\right\}_{n\geq 1}$ be two sequence in M. Then we have

$$\begin{aligned}
\lim_{n\to+\infty}\aleph\{\mathfrak{h}_n\} &= \lim_{n\to+\infty}\aleph\left\{\tfrac{1}{n}\right\} = \lim_{n\to+\infty}\left\{\tfrac{1}{12n}\right\} = 0\\
\lim_{n\to+\infty}\Gamma\{\mathfrak{h}_n\} &= \lim_{n\to+\infty}\Gamma\left\{\tfrac{1}{n}\right\} = \lim_{n\to+\infty}\left\{\tfrac{1}{6n}\right\} = 0\\
\lim_{n\to+\infty}\daleth\{\mathfrak{g}_n\} &= \lim_{n\to+\infty}\daleth\left\{\tfrac{1}{n+1}\right\} = \lim_{n\to+\infty}\left\{\tfrac{1}{12(n+1)}\right\} = 0\\
\lim_{n\to+\infty}\mathfrak{L}\{\mathfrak{g}_n\} &= \lim_{n\to+\infty}\mathfrak{L}\left\{\tfrac{1}{n+1}\right\} = \lim_{n\to+\infty}\left\{\tfrac{1}{6(n+1)}\right\} = 0
\end{aligned}$$

Since $\mathfrak{h}(0) = \mathfrak{L}(0) = 0$ we have $0 \in \Gamma(M) \cap \mathfrak{L}(M)$. Therefore there exists sequence $\{\mathfrak{h}_n\}$ and $\{\mathfrak{g}_n\}$ in M such that

$$\lim_{n\to+\infty}\aleph\{\mathfrak{h}_n\} = \lim_{n\to+\infty}\Gamma\{\mathfrak{h}_n\} = \lim_{n\to+\infty}\daleth\{\mathfrak{g}_n\} = \lim_{n\to+\infty}\mathfrak{L}\{\mathfrak{g}_n\}$$

That is, $(\aleph, \Gamma)$ and $(\daleth, \mathfrak{L})$ satisfies the common $(CLR_{\Gamma\mathfrak{L}})$-property. Next, to verify inequality (3.1).

$$\psi(\parallel \aleph\mathfrak{h} - \daleth\mathfrak{g} \parallel) \leq \mathfrak{F}(\psi(\urcorner(\mathfrak{h},\mathfrak{g})), \phi(\urcorner(\mathfrak{h},\mathfrak{g})))$$

$$\urcorner(\mathfrak{h},\mathfrak{g}) = \max\left\{\parallel \Gamma\mathfrak{h} - \mathfrak{L}\mathfrak{g} \parallel,\ \tfrac{\|\aleph\mathfrak{h}-\Gamma\mathfrak{h}\|}{1+\|\aleph\mathfrak{h}-\Gamma\mathfrak{h}\|}, \tfrac{\|\daleth\mathfrak{g}-\mathfrak{L}\mathfrak{g}\|}{1+\|\daleth\mathfrak{g}-\mathfrak{L}\mathfrak{g}\|}, \tfrac{\|\aleph\mathfrak{h}-\mathfrak{L}\mathfrak{g}\|+\|\Gamma\mathfrak{h}-\daleth\mathfrak{g}\|}{2b},\ \tfrac{\|\aleph\mathfrak{h}-\Gamma\mathfrak{h}\|+\|\daleth\mathfrak{g}-\mathfrak{L}\mathfrak{g}\|}{2b}\right\}$$

Case (i): When $\mathfrak{h},\ \mathfrak{g} \in [0,1]$

$$\begin{aligned}
\text{L.H.S} &= \psi\left(\parallel \aleph\mathfrak{h} - \daleth\mathfrak{g} \parallel^2,\ \alpha \parallel \aleph\mathfrak{h} - \daleth\mathfrak{g} \parallel^2\right)\\
&= \psi\left(\parallel \tfrac{\mathfrak{h}}{12} - \tfrac{\mathfrak{g}}{12} \parallel^2,\ \alpha \parallel \tfrac{\mathfrak{h}}{12} - \tfrac{\mathfrak{g}}{12} \parallel^2\right)\\
&= \psi\left(\tfrac{1}{144} \parallel \mathfrak{h} - \mathfrak{g} \parallel^2, \tfrac{1}{144}\,\alpha \parallel \mathfrak{h} - \mathfrak{g} \parallel^2\right)\\
&= \left(\tfrac{1}{144}, \tfrac{1}{144}\alpha\right)\psi\left(\parallel \mathfrak{h} - \mathfrak{g} \parallel^2\right)
\end{aligned}$$

$$
\begin{aligned}
\text{R.H.S} = \ & \mathfrak{F}(\psi(\daleth(\mathfrak{h},\mathfrak{g})),\phi(\daleth(\mathfrak{h},\mathfrak{g})))\\
= \ & \mathfrak{F}(\psi(\max\{(\| \frac{\mathfrak{h}}{6}-\frac{\mathfrak{g}}{6}\|^2,\ \alpha\|\frac{\mathfrak{h}}{6}-\frac{\mathfrak{g}}{6}\|^2),\\
& \frac{(\|\frac{\mathfrak{g}}{12}-\frac{\mathfrak{g}}{6}\|^2,\ \alpha\|\frac{\mathfrak{g}}{12}-\frac{\mathfrak{g}}{6}\|^2)}{1+(\|\frac{\mathfrak{g}}{12}-\frac{\mathfrak{g}}{6}\|^2,\ \alpha\|\frac{\mathfrak{g}}{12}-\frac{\mathfrak{g}}{6}\|^2)},\frac{\left(\|\frac{\mathfrak{h}}{12}-\frac{\mathfrak{h}}{6}\|^2,\ \alpha\|\frac{\mathfrak{h}}{12}-\frac{\mathfrak{h}}{6}\|^2\right)}{1+\left(\|\frac{\mathfrak{h}}{12}-\frac{\mathfrak{h}}{6}\|^2,\ \alpha\|\frac{\mathfrak{h}}{12}-\frac{\mathfrak{h}}{6}\|^2\right)},\\
& \frac{\left(\left(\|\frac{\mathfrak{h}}{12}-\frac{\mathfrak{g}}{6}\|^2,\ \alpha\|\frac{\mathfrak{h}}{12}-\frac{\mathfrak{g}}{6}\|^2\right)+\left(\|\frac{\mathfrak{h}}{6}-\frac{\mathfrak{g}}{12}\|^2,\ \alpha\|\frac{\mathfrak{h}}{6}-\frac{\mathfrak{g}}{12}\|^2\right)\right)}{4},\\
& \frac{\left(\left(\|\frac{\mathfrak{h}}{12}-\frac{\mathfrak{h}}{6}\|^2,\ \alpha\|\frac{\mathfrak{h}}{12}-\frac{\mathfrak{h}}{6}\|^2\right)+(\|\frac{\mathfrak{g}}{12}-\frac{\mathfrak{g}}{6}\|^2,\ \alpha\|\frac{\mathfrak{g}}{12}-\frac{\mathfrak{g}}{6}\|^2)\right)}{4}\}),\\
& \phi(\max\{(\| \frac{\mathfrak{h}}{6}-\frac{\mathfrak{g}}{6}\|^2,\ \alpha\|\frac{\mathfrak{h}}{6}-\frac{\mathfrak{g}}{6}\|^2),\\
& \frac{(\|\frac{\mathfrak{g}}{12}-\frac{\mathfrak{g}}{6}\|^2,\ \alpha\|\frac{\mathfrak{g}}{12}-\frac{\mathfrak{g}}{6}\|^2)}{1+(\|\frac{\mathfrak{g}}{12}-\frac{\mathfrak{g}}{6}\|^2,\ \alpha\|\frac{\mathfrak{g}}{12}-\frac{\mathfrak{g}}{6}\|^2)},\frac{\left(\|\frac{\mathfrak{h}}{12}-\frac{\mathfrak{h}}{6}\|^2,\ \alpha\|\frac{\mathfrak{h}}{12}-\frac{\mathfrak{h}}{6}\|^2\right)}{1+\left(\|\frac{\mathfrak{h}}{12}-\frac{\mathfrak{h}}{6}\|^2,\ \alpha\|\frac{\mathfrak{h}}{12}-\frac{\mathfrak{h}}{6}\|^2\right)},\\
& \frac{\left(\left(\|\frac{\mathfrak{h}}{12}-\frac{\mathfrak{g}}{6}\|^2,\ \alpha\|\frac{\mathfrak{h}}{12}-\frac{\mathfrak{g}}{6}\|^2\right)+\left(\|\frac{\mathfrak{h}}{6}-\frac{\mathfrak{g}}{12}\|^2,\ \alpha\|\frac{\mathfrak{h}}{6}-\frac{\mathfrak{g}}{12}\|^2\right)\right)}{4},\\
& \frac{\left(\left(\|\frac{\mathfrak{h}}{12}-\frac{\mathfrak{h}}{6}\|^2,\ \alpha\|\frac{\mathfrak{h}}{12}-\frac{\mathfrak{h}}{6}\|^2\right)+(\|\frac{\mathfrak{g}}{12}-\frac{\mathfrak{g}}{6}\|^2,\ \alpha\|\frac{\mathfrak{g}}{12}-\frac{\mathfrak{g}}{6}\|^2)\right)}{4}\}))\\
= \ & \mathfrak{F}(\psi(\max\{(\|\frac{\mathfrak{h}-\mathfrak{g}}{6}\|^2,\ \alpha\|\frac{\mathfrak{h}-\mathfrak{g}}{6}\|^2),\\
& \frac{\left(\|\frac{\mathfrak{g}-2\mathfrak{g}}{12}\|^2,\ \alpha\|\frac{\mathfrak{g}-2\mathfrak{g}}{12}\|^2\right)}{1+\left(\|\frac{\mathfrak{g}-2\mathfrak{g}}{12}\|^2,\ \alpha\|\frac{\mathfrak{g}-2\mathfrak{g}}{12}\|^2\right)},\frac{\left(\|\frac{\mathfrak{h}-2\mathfrak{h}}{12}\|^2,\ \alpha\|\frac{\mathfrak{h}-2\mathfrak{h}}{12}\|^2\right)}{1+\left(\|\frac{\mathfrak{h}-2\mathfrak{h}}{12}\|^2,\ \alpha\|\frac{\mathfrak{h}-2\mathfrak{h}}{12}\|^2\right)},\\
& \frac{\left(\left(\|\frac{\mathfrak{h}-2\mathfrak{g}}{12}\|^2,\ \alpha\|\frac{\mathfrak{h}-2\mathfrak{g}}{12}\|^2\right)+\left(\|\frac{2\mathfrak{h}-\mathfrak{g}}{12}\|^2,\ \alpha\|\frac{2\mathfrak{h}-\mathfrak{g}}{12}\|^2\right)\right)}{4},\\
& \frac{\left(\left(\|\frac{\mathfrak{h}-2\mathfrak{h}}{12}\|^2,\ \alpha\|\frac{\mathfrak{h}-2\mathfrak{h}}{12}\|^2\right)+\left(\|\frac{\mathfrak{g}-2\mathfrak{g}}{12}\|^2,\ \alpha\|\frac{\mathfrak{g}-2\mathfrak{g}}{12}\|^2\right)\right)}{4}\}),\\
& \phi(\max\{(\|\frac{\mathfrak{h}-\mathfrak{g}}{6}\|^2,\ \alpha\|\frac{\mathfrak{h}-\mathfrak{g}}{6}\|^2),\\
& \frac{\left(\|\frac{\mathfrak{g}-2\mathfrak{g}}{12}\|^2,\ \alpha\|\frac{\mathfrak{g}-2\mathfrak{g}}{12}\|^2\right)}{1+\left(\|\frac{\mathfrak{g}-2\mathfrak{g}}{12}\|^2,\ \alpha\|\frac{\mathfrak{g}-2\mathfrak{g}}{12}\|^2\right)},\frac{\left(\|\frac{\mathfrak{h}-2\mathfrak{h}}{12}\|^2,\ \alpha\|\frac{\mathfrak{h}-2\mathfrak{h}}{12}\|^2\right)}{1+\left(\|\frac{\mathfrak{h}-2\mathfrak{h}}{12}\|^2,\ \alpha\|\frac{\mathfrak{h}-2\mathfrak{h}}{12}\|^2\right)},\\
& \frac{\left(\left(\|\frac{\mathfrak{h}-2\mathfrak{g}}{12}\|^2,\ \alpha\|\frac{\mathfrak{h}-2\mathfrak{g}}{12}\|^2\right)+\left(\|\frac{2\mathfrak{h}-\mathfrak{g}}{12}\|^2,\ \alpha\|\frac{2\mathfrak{h}-\mathfrak{g}}{12}\|^2\right)\right)}{4},\\
& \frac{\left(\left(\|\frac{\mathfrak{h}-2\mathfrak{h}}{12}\|^2,\ \alpha\|\frac{\mathfrak{h}-2\mathfrak{h}}{12}\|^2\right)+\left(\|\frac{\mathfrak{g}-2\mathfrak{g}}{12}\|^2,\ \alpha\|\frac{\mathfrak{g}-2\mathfrak{g}}{12}\|^2\right)\right)}{4}\}))
\end{aligned}
$$

$$= \mathfrak{F}(\psi(\max\{\left(\frac{1}{36}\parallel \mathfrak{h}-\mathfrak{g}\parallel^2, \frac{1}{36}\alpha\parallel \mathfrak{h}-\mathfrak{g}\parallel^2\right),$$
$$\frac{\left(\frac{1}{144}\parallel \mathfrak{g}-2\mathfrak{g}\parallel^2, \frac{1}{144}\alpha\parallel \mathfrak{g}-2\mathfrak{g}\parallel^2\right)}{1+\left(\frac{1}{144}\parallel \mathfrak{g}-2\mathfrak{g}\parallel^2, \frac{1}{144}\alpha\parallel \mathfrak{g}-2\mathfrak{g}\parallel^2\right)}, \frac{\left(\frac{1}{144}\parallel \mathfrak{h}-2\mathfrak{h}\parallel^2, \frac{1}{144}\alpha\parallel \mathfrak{h}-2\mathfrak{h}\parallel^2\right)}{1+\left(\frac{1}{144}\parallel \mathfrak{h}-2\mathfrak{h}\parallel^2, \frac{1}{144}\alpha\parallel \mathfrak{h}-2\mathfrak{h}\parallel^2\right)},$$
$$\frac{\left(\left(\frac{1}{144}\parallel \mathfrak{h}-2\mathfrak{g}\parallel^2, \frac{1}{144}\alpha\parallel \mathfrak{h}-2\mathfrak{g}\parallel^2\right)+\left(\frac{1}{144}\parallel 2\mathfrak{h}-\mathfrak{g}\parallel^2, \frac{1}{144}\alpha\parallel 2\mathfrak{h}-\mathfrak{g}\parallel^2\right)\right)}{4},$$
$$\frac{\left(\left(\frac{1}{144}\parallel \mathfrak{h}-2\mathfrak{h}\parallel^2, \frac{1}{144}\alpha\parallel \mathfrak{h}-2\mathfrak{h}\parallel^2\right)+\left(\frac{1}{144}\parallel \mathfrak{g}-2\mathfrak{g}\parallel^2, \frac{1}{144}\alpha\parallel \mathfrak{g}-2\mathfrak{g}\parallel^2\right)\right)}{4}\}),$$
$$\phi(\max\{\left(\frac{1}{36}\parallel \mathfrak{h}-\mathfrak{g}\parallel^2, \frac{1}{36}\alpha\parallel \mathfrak{h}-\mathfrak{g}\parallel^2\right),$$
$$\frac{\left(\frac{1}{144}\parallel \mathfrak{g}-2\mathfrak{g}\parallel^2, \frac{1}{144}\alpha\parallel \mathfrak{g}-2\mathfrak{g}\parallel^2\right)}{1+\left(\frac{1}{144}\parallel \mathfrak{g}-2\mathfrak{g}\parallel^2, \frac{1}{144}\alpha\parallel \mathfrak{g}-2\mathfrak{g}\parallel^2\right)}, \frac{\left(\frac{1}{144}\parallel \mathfrak{h}-2\mathfrak{h}\parallel^2, \frac{1}{144}\alpha\parallel \mathfrak{h}-2\mathfrak{h}\parallel^2\right)}{1+\left(\frac{1}{144}\parallel \mathfrak{h}-3\mathfrak{h}\parallel^2, \frac{1}{144}\alpha\parallel \mathfrak{h}-3\mathfrak{h}\parallel^2\right)},$$
$$\frac{\left(\left(\frac{1}{144}\parallel \mathfrak{h}-2\mathfrak{g}\parallel^2, \frac{1}{144}\alpha\parallel \mathfrak{h}-2\mathfrak{g}\parallel^2\right)+\left(\frac{1}{144}\parallel 2\mathfrak{h}-\mathfrak{g}\parallel^2, \frac{1}{144}\alpha\parallel 2\mathfrak{h}-\mathfrak{g}\parallel^2\right)\right)}{4},$$
$$\frac{\left(\left(\frac{1}{144}\parallel \mathfrak{h}-2\mathfrak{h}\parallel^2, \frac{1}{144}\alpha\parallel \mathfrak{h}-2\mathfrak{h}\parallel^2\right)+\left(\frac{1}{144}\parallel \mathfrak{g}-2\mathfrak{g}\parallel^2, \frac{1}{144}\alpha\parallel \mathfrak{g}-2\mathfrak{g}\parallel^2\right)\right)}{4}\}))$$

$$\text{R.H.S} = \mathfrak{F}(\psi(\max\{\left(\left(\frac{1}{36}, \frac{1}{36}\alpha\right)\parallel \mathfrak{h}-\mathfrak{g}\parallel^2\right),$$
$$\frac{\left(\left(\frac{1}{144}, \frac{1}{144}\alpha\right)\parallel \mathfrak{g}-2\mathfrak{g}\parallel^2\right)}{1+\left(\left(\frac{1}{144}, \frac{1}{144}\alpha\right)\parallel \mathfrak{g}-2\mathfrak{g}\parallel^2\right)}, \frac{\left(\frac{1}{144}, \frac{1}{144}\alpha\right)\left(\parallel \mathfrak{h}-2\mathfrak{h}\parallel^2\right)}{1+\left(\left(\frac{1}{144}, \frac{1}{144}\alpha\right)\parallel \mathfrak{h}-2\mathfrak{h}\parallel^2\right)},$$
$$\frac{\left(\left(\frac{1}{144}, \frac{1}{144}\alpha\right)\parallel \mathfrak{h}-2\mathfrak{g}\parallel^2\right)+\left(\left(\frac{1}{144}, \frac{1}{144}\alpha\right)\parallel 2\mathfrak{h}-\mathfrak{g}\parallel^2\right)}{4},$$
$$\frac{\left(\left(\frac{1}{144}, \frac{1}{144}\alpha\right)\parallel \mathfrak{h}-2\mathfrak{h}\parallel^2\right)+\left(\left(\frac{1}{144}, \frac{1}{144}\alpha\right)\parallel \mathfrak{g}-2\mathfrak{g}\parallel^2\right)}{4}\}),$$
$$\phi(\max\{\left(\left(\frac{1}{36}, \frac{1}{36}\alpha\right)\parallel \mathfrak{h}-\mathfrak{g}\parallel^2\right),$$
$$\frac{\left(\left(\frac{1}{144}, \frac{1}{144}\alpha\right)\parallel \mathfrak{g}-2\mathfrak{g}\parallel^2\right)}{1+\left(\left(\frac{1}{144}, \frac{1}{144}\alpha\right)\parallel \mathfrak{g}-2\mathfrak{g}\parallel^2\right)}, \frac{\left(\frac{1}{144}, \frac{1}{144}\alpha\right)\left(\parallel \mathfrak{h}-2\mathfrak{h}\parallel^2\right)}{1+\left(\left(\frac{1}{144}, \frac{1}{144}\alpha\right)\parallel \mathfrak{h}-2\mathfrak{h}\parallel^2\right)},$$
$$\frac{\left(\left(\frac{1}{144}, \frac{1}{144}\alpha\right)\parallel \mathfrak{h}-2\mathfrak{g}\parallel^2\right)+\left(\left(\frac{1}{144}, \frac{1}{144}\alpha\right)\parallel 2\mathfrak{h}-\mathfrak{g}\parallel^2\right)}{4},$$
$$\frac{\left(\left(\frac{1}{144}, \frac{1}{144}\alpha\right)\parallel \mathfrak{h}-2\mathfrak{h}\parallel^2\right)+\left(\left(\frac{1}{144}, \frac{1}{144}\alpha\right)\parallel \mathfrak{g}-2\mathfrak{g}\parallel^2\right)}{4}\})),$$
$$= \mathfrak{F}\left(\psi\left(\left(\frac{1}{36}, \frac{1}{36}\alpha\right)\parallel \mathfrak{h}-\mathfrak{g}\parallel^2\right), \phi\left(\left(\frac{1}{36}, \frac{1}{36}\alpha\right)\parallel \mathfrak{h}-\mathfrak{g}\parallel^2\right)\right)$$
$$= \psi\left(\left(\frac{1}{36}, \frac{1}{36}\alpha\right)\parallel \mathfrak{h}-\mathfrak{g}\parallel^2\right), \phi\left(\left(\frac{1}{36}, \frac{1}{36}\alpha\right)\parallel \mathfrak{h}-\mathfrak{g}\parallel^2\right)$$
$$= \left(\frac{1}{36}, \frac{1}{36}\alpha\right)\left(\psi\left(\parallel \mathfrak{h}-\mathfrak{g}\parallel^2\right)-\phi\left(\parallel \mathfrak{h}-\mathfrak{g}\parallel^2\right)\right)$$

$$\left(\frac{1}{144}, \frac{1}{144}\alpha\right)\psi\left(\parallel \mathfrak{h}-\mathfrak{g}\parallel^2\right) \leq \left(\frac{1}{36}, \frac{1}{36}\alpha\right)\left(\psi\left(\parallel \mathfrak{h}-\mathfrak{g}\parallel^2\right)-\phi\left(\parallel \mathfrak{h}-\mathfrak{g}\parallel^2\right)\right)$$

Therefore, L.H.S $\leq$ R.H.S. Clearly inequality (3.1) is satisfied for each $\mathfrak{h}, \mathfrak{g} \in [0, 1]$ and $\alpha \geq 0$.

Case (ii): When $\mathfrak{h} \in [0,1]$ and $\mathfrak{g} \in (1,+\infty)$

$$\begin{aligned}
\text{L.H.S} &= \psi\left(\| \aleph\mathfrak{h} - \beth\mathfrak{g} \|^2,\ \alpha \| \aleph\mathfrak{h} - \beth\mathfrak{g} \|^2\ \right) = \psi\left(\| \tfrac{\mathfrak{h}}{12} - \tfrac{\mathfrak{g}+11}{12} \|^2,\ \alpha \| \tfrac{\mathfrak{h}}{12} - \tfrac{\mathfrak{g}+11}{12} \|^2\ \right) \\
&= \psi\left(\tfrac{1}{144} \| \mathfrak{h} - (\mathfrak{g}+11) \|^2, \tfrac{1}{144}\ \alpha \| \mathfrak{h} - (\mathfrak{g}+11) \|^2\ \right) \\
&= \left(\tfrac{1}{144}, \tfrac{1}{144}\alpha\right) \psi\left(\| \mathfrak{h} - (\mathfrak{g}+11) \|^2\right)
\end{aligned}$$

$$\begin{aligned}
\text{R.H.S} = \ & \mathfrak{F}(\psi\left(\daleth(\mathfrak{h},\mathfrak{g})\right), \phi\left(\daleth(\mathfrak{h},\mathfrak{g})\right)) \\
= \ & \mathfrak{F}(\psi(\max\{\left(\| \frac{\mathfrak{h}}{6} - \frac{\mathfrak{g}+5}{6} \|^2,\ \alpha \| \frac{\mathfrak{h}}{6} - \frac{\mathfrak{g}+5}{6} \|^2\right), \\
& \frac{\left(\| \frac{\mathfrak{h}}{12} - \frac{\mathfrak{h}}{6} \|^2,\ \alpha \| \frac{\mathfrak{h}}{12} - \frac{\mathfrak{h}}{6} \|^2\right)}{1 + \left(\| \frac{\mathfrak{h}}{12} - \frac{\mathfrak{h}}{6} \|^2,\ \alpha \| \frac{\mathfrak{h}}{12} - \frac{\mathfrak{h}}{6} \|^2\right)}, \frac{\left(\| \frac{\mathfrak{g}+11}{12} - \frac{\mathfrak{g}+5}{6} \|^2,\ \alpha \| \frac{\mathfrak{g}+11}{12} - \frac{\mathfrak{g}+5}{6} \|^2\right)}{1 + \left(\| \frac{\mathfrak{g}+11}{12} - \frac{\mathfrak{g}+5}{6} \|^2,\ \alpha \| \frac{\mathfrak{g}+11}{12} - \frac{\mathfrak{g}+5}{6} \|^2\right)}, \\
& \frac{\left(\left(\| \frac{\mathfrak{h}}{12} - \frac{\mathfrak{g}+5}{6} \|^2,\ \alpha \| \frac{\mathfrak{h}}{12} - \frac{\mathfrak{g}+5}{6} \|^2\right) + \left(\| \frac{\mathfrak{h}}{6} - \frac{\mathfrak{g}+11}{12} \|^2,\ \alpha \| \frac{\mathfrak{h}}{6} - \frac{\mathfrak{g}+11}{12} \|^2\right)\right)}{4}, \\
& \frac{\left(\left(\| \frac{\mathfrak{h}}{12} - \frac{\mathfrak{h}}{6} \|^2,\ \alpha \| \frac{\mathfrak{h}}{12} - \frac{\mathfrak{h}}{6} \|^2\right) + \left(\| \frac{\mathfrak{g}+11}{12} - \frac{\mathfrak{g}+5}{6} \|^2,\ \alpha \| \frac{\mathfrak{g}+11}{12} - \frac{\mathfrak{g}+5}{6} \|^2\right)\right)}{4}\}), \\
& \phi(\max\{\left(\| \frac{\mathfrak{h}}{6} - \frac{\mathfrak{g}+5}{6} \|^2,\ \alpha \| \frac{\mathfrak{h}}{6} - \frac{\mathfrak{g}+5}{6} \|^2\right), \\
& \frac{\left(\| \frac{\mathfrak{h}}{12} - \frac{\mathfrak{h}}{6} \|^2,\ \alpha \| \frac{\mathfrak{h}}{12} - \frac{\mathfrak{h}}{6} \|^2\right)}{1 + \left(\| \frac{\mathfrak{h}}{12} - \frac{\mathfrak{h}}{6} \|^2,\ \alpha \| \frac{\mathfrak{h}}{12} - \frac{\mathfrak{h}}{6} \|^2\right)}, \frac{\left(\| \frac{\mathfrak{g}+11}{12} - \frac{\mathfrak{g}+5}{6} \|^2,\ \alpha \| \frac{\mathfrak{g}+11}{12} - \frac{\mathfrak{g}+5}{6} \|^2\right)}{1 + \left(\| \frac{\mathfrak{g}+11}{12} - \frac{\mathfrak{g}+5}{6} \|^2,\ \alpha \| \frac{\mathfrak{g}+11}{12} - \frac{\mathfrak{g}+5}{6} \|^2\right)}, \\
& \frac{\left(\left(\| \frac{\mathfrak{h}}{12} - \frac{\mathfrak{g}+5}{6} \|^2,\ \alpha \| \frac{\mathfrak{h}}{12} - \frac{\mathfrak{g}+5}{6} \|^2\right) + \left(\| \frac{\mathfrak{h}}{6} - \frac{\mathfrak{g}+11}{12} \|^2,\ \alpha \| \frac{\mathfrak{h}}{6} - \frac{\mathfrak{g}+11}{12} \|^2\right)\right)}{4}, \\
& \frac{\left(\left(\| \frac{\mathfrak{h}}{12} - \frac{\mathfrak{h}}{6} \|^2,\ \alpha \| \frac{\mathfrak{h}}{12} - \frac{\mathfrak{h}}{6} \|^2\right) + \left(\| \frac{\mathfrak{g}+11}{12} - \frac{\mathfrak{g}+5}{6} \|^2,\ \alpha \| \frac{\mathfrak{g}+11}{12} - \frac{\mathfrak{g}+5}{6} \|^2\right)\right)}{4}\})) \\
= \ & \mathfrak{F}(\psi(\max\{\left(\| \frac{\mathfrak{h}-(\mathfrak{g}+5)}{6} \|^2,\ \alpha \| \frac{\mathfrak{h}-(\mathfrak{g}+5)}{6} \|^2\right), \\
& \frac{\left(\| \frac{\mathfrak{h}-2\mathfrak{h}}{12} \|^2,\ \alpha \| \frac{\mathfrak{h}-2\mathfrak{h}}{12} \|^2\right)}{1 + \left(\| \frac{\mathfrak{h}-2\mathfrak{h}}{12} \|^2,\ \alpha \| \frac{\mathfrak{h}-2\mathfrak{h}}{12} \|^2\right)}, \frac{\left(\| \frac{(\mathfrak{g}+11)-2(\mathfrak{g}+5)}{12} \|^2,\ \alpha \| \frac{(\mathfrak{g}+11)-2(\mathfrak{g}+5)}{12} \|^2\right)}{1 + \left(\| \frac{(\mathfrak{g}+11)-2(\mathfrak{g}+5)}{12} \|^2,\ \alpha \| \frac{(\mathfrak{g}+11)-2(\mathfrak{g}+5)}{12} \|^2\right)}, \\
& \frac{\left(\left(\| \frac{\mathfrak{h}-2(\mathfrak{g}+5)}{12} \|^2,\ \alpha \| \frac{\mathfrak{h}-2(\mathfrak{g}+5)}{12} \|^2\right) + \left(\| \frac{2\mathfrak{h}-(\mathfrak{g}+11)}{12} \|^2,\ \alpha \| \frac{2\mathfrak{h}-(\mathfrak{g}+11)}{12} \|^2\right)\right)}{4}, \\
& \frac{\left(\left(\| \frac{\mathfrak{h}-2\mathfrak{h}}{12} \|^2,\ \alpha \| \frac{\mathfrak{h}-2\mathfrak{h}}{12} \|^2\right) + \left(\| \frac{(\mathfrak{g}+11)-2(\mathfrak{g}+5)}{12} \|^2,\ \alpha \| \frac{(\mathfrak{g}+11)-2(\mathfrak{g}+5)}{12} \|^2\right)\right)}{4}\}), \\
& \phi(\max\{\left(\| \frac{\mathfrak{h}-(\mathfrak{g}+5)}{6} \|^2,\ \alpha \| \frac{\mathfrak{h}-(\mathfrak{g}+5)}{6} \|^2\right), \\
& \frac{\left(\| \frac{\mathfrak{h}-2\mathfrak{h}}{12} \|^2,\ \alpha \| \frac{\mathfrak{h}-2\mathfrak{h}}{12} \|^2\right)}{1 + \left(\| \frac{\mathfrak{h}-2\mathfrak{h}}{12} \|^2,\ \alpha \| \frac{\mathfrak{h}-2\mathfrak{h}}{12} \|^2\right)}, \frac{\left(\| \frac{(\mathfrak{g}+11)-2(\mathfrak{g}+5)}{12} \|^2,\ \alpha \| \frac{(\mathfrak{g}+11)-2(\mathfrak{g}+5)}{12} \|^2\right)}{1 + \left(\| \frac{(\mathfrak{g}+11)-2(\mathfrak{g}+5)}{12} \|^2,\ \alpha \| \frac{(\mathfrak{g}+11)-2(\mathfrak{g}+5)}{12} \|^2\right)}, \\
& \frac{\left(\left(\| \frac{\mathfrak{h}-2(\mathfrak{g}+5)}{12} \|^2,\ \alpha \| \frac{\mathfrak{h}-2(\mathfrak{g}+5)}{12} \|^2\right) + \left(\| \frac{2\mathfrak{h}-(\mathfrak{g}+11)}{12} \|^2,\ \alpha \| \frac{2\mathfrak{h}-(\mathfrak{g}+11)}{12} \|^2\right)\right)}{4}, \\
& \frac{\left(\left(\| \frac{\mathfrak{h}-2\mathfrak{h}}{12} \|^2,\ \alpha \| \frac{\mathfrak{h}-2\mathfrak{h}}{12} \|^2\right) + \left(\| \frac{(\mathfrak{g}+11)-2(\mathfrak{g}+5)}{12} \|^2,\ \alpha \| \frac{(\mathfrak{g}+11)-2(\mathfrak{g}+5)}{12} \|^2\right)\right)}{4}\}))
\end{aligned}$$

$$
\begin{aligned}
&= \mathfrak{F}(\psi(\max\{\left(\left(\frac{1}{36},\frac{1}{36}\ \alpha\right) \parallel \mathfrak{h}-(\mathfrak{g}+5) \parallel^2\right),\\
&\quad \frac{\left(\left(\frac{1}{144},\frac{1}{144}\ \alpha\right) \parallel \mathfrak{h}-2\mathfrak{h} \parallel^2\right)}{1+\left(\left(\frac{1}{144},\frac{1}{144}\alpha\right) \parallel \mathfrak{h}-2\mathfrak{h} \parallel^2\right)}, \frac{\left(\left(\frac{1}{144},\frac{1}{144}\ \alpha\right) \parallel (\mathfrak{g}+11)-2(\mathfrak{g}+5) \parallel^2\right)}{1+\left(\left(\frac{1}{144},\frac{1}{144}\ \alpha\right) \parallel (\mathfrak{g}+11)-2(\mathfrak{g}+5) \parallel^2\right)},\\
&\quad \frac{\left(\left(\frac{1}{144},\frac{1}{144}\alpha\right) \parallel \mathfrak{h}-2(\mathfrak{g}+5) \parallel^2\right)+\left(\left(\frac{1}{144},\frac{1}{144}\ \alpha\right) \parallel 2\mathfrak{h}-(\mathfrak{g}+11) \parallel^2\right)}{4},\\
&\quad \frac{\left(\left(\frac{1}{144},\frac{1}{144}\alpha\right) \parallel \mathfrak{h}-2\mathfrak{h} \parallel^2\right)+\left(\left(\frac{1}{144},\frac{1}{144}\alpha\right) \parallel (\mathfrak{g}+11)-2(\mathfrak{g}+5) \parallel^2\right)}{4}\}),\\
&\quad \phi(\max\{\left(\left(\frac{1}{36},\frac{1}{36}\ \alpha\right) \parallel \mathfrak{h}-(\mathfrak{g}+5) \parallel^2\right),\\
&\quad \frac{\left(\left(\frac{1}{144},\frac{1}{144}\ \alpha\right) \parallel \mathfrak{h}-2\mathfrak{h} \parallel^2\right)}{1+\left(\left(\frac{1}{144},\frac{1}{144}\alpha\right) \parallel \mathfrak{h}-2\mathfrak{h} \parallel^2\right)}, \frac{\left(\left(\frac{1}{144},\frac{1}{144}\ \alpha\right) \parallel (\mathfrak{g}+11)-2(\mathfrak{g}+5) \parallel^2\right)}{1+\left(\left(\frac{1}{144},\frac{1}{144}\ \alpha\right) \parallel (\mathfrak{g}+11)-2(\mathfrak{g}+5) \parallel^2\right)},\\
&\quad \frac{\left(\left(\frac{1}{144},\frac{1}{144}\alpha\right) \parallel \mathfrak{h}-2(\mathfrak{g}+5) \parallel^2\right)+\left(\left(\frac{1}{144},\frac{1}{144}\ \alpha\right) \parallel 2\mathfrak{h}-(\mathfrak{g}+11) \parallel^2\right)}{4},\\
&\quad \frac{\left(\left(\frac{1}{144},\frac{1}{144}\alpha\right) \parallel \mathfrak{h}-2\mathfrak{h} \parallel^2\right)+\left(\left(\frac{1}{144},\frac{1}{144}\alpha\right) \parallel (\mathfrak{g}+11)-2(\mathfrak{g}+5) \parallel^2\right)}{4}\}))\\
&= \mathfrak{F}\left(\psi(\left(\frac{1}{36},\frac{1}{36}\ \alpha\right) \parallel \mathfrak{h}-(\mathfrak{g}+5) \parallel^2), \phi(\left(\frac{1}{36},\frac{1}{36}\ \alpha\right) \parallel \mathfrak{h}-(\mathfrak{g}+5) \parallel^2)\right)\\
&= \psi\left(\left(\frac{1}{36},\frac{1}{36}\ \alpha\right) \parallel \mathfrak{h}-(\mathfrak{g}+5) \parallel^2\right) - \phi\left(\left(\frac{1}{36},\frac{1}{36}\ \alpha\right) \parallel \mathfrak{h}-(\mathfrak{g}+5) \parallel^2\right)\\
\text{R.H.S} &= \left(\frac{1}{36},\frac{1}{36}\ \alpha\right)\left(\psi\left(\parallel \mathfrak{h}-(\mathfrak{g}+5) \parallel^2\right) - \phi\left(\parallel \mathfrak{h}-(\mathfrak{g}+5) \parallel^2\right)\right)
\end{aligned}
$$

Therefore, L.H.S $\leq$ R.H.S. Calculating the same as in case (i), we conclude that inequality (3.1) is satisfied for each $\mathfrak{h} \in [0,1]$ and $\mathfrak{g} \in (1,+\infty)$ and $\alpha \geq 0$.

Case (iii): When $\mathfrak{h} \in (1,+\infty)$ and $\mathfrak{g} \in [0,1]$

$$
\begin{aligned}
\text{L.H.S} &= \psi\left(\parallel \aleph\mathfrak{h} - \daleth\mathfrak{g} \parallel^2,\ \alpha \parallel \aleph\mathfrak{h} - \daleth\mathfrak{g} \parallel^2\ \right)\\
&= \psi\left(\parallel \tfrac{\mathfrak{h}+11}{12} - \tfrac{\mathfrak{g}}{12} \parallel^2,\ \alpha \parallel \tfrac{\mathfrak{h}+11}{12} - \tfrac{\mathfrak{g}}{12} \parallel^2\ \right)\\
&= \psi\left(\tfrac{1}{144} \parallel (\mathfrak{h}+11) - \mathfrak{g} \parallel^2, \tfrac{1}{144}\ \alpha \parallel (\mathfrak{h}+11) - \mathfrak{g} \parallel^2\ \right)\\
&= \left(\tfrac{1}{144},\tfrac{1}{144}\alpha\right)\psi\left(\parallel (\mathfrak{h}+11) - \mathfrak{g} \parallel^2\right)
\end{aligned}
$$

$$
\begin{aligned}
\text{R.H.S} = \;& \mathfrak{F}(\psi\left(\daleth(\mathfrak{h},\mathfrak{g})\right), \phi\left(\daleth(\mathfrak{h},\mathfrak{g})\right)) \\
= \;& \mathfrak{F}(\psi(\max\{\left(\| \frac{\mathfrak{h}+5}{6} - \frac{\mathfrak{g}}{6} \|^2,\ \alpha \| \frac{\mathfrak{h}+5}{6} - \frac{\mathfrak{g}}{6} \|^2\right), \\
& \frac{\left(\| \frac{\mathfrak{h}+11}{12} - \frac{\mathfrak{h}+5}{6} \|^2,\ \alpha \| \frac{\mathfrak{h}+11}{12} - \frac{\mathfrak{h}+5}{6} \|^2\right)}{1+\left(\| \frac{\mathfrak{h}+11}{12} - \frac{\mathfrak{h}+5}{6} \|^2,\ \alpha \| \frac{\mathfrak{h}+11}{12} - \frac{\mathfrak{h}+5}{6} \|^2\right)}, \frac{\left(\| \frac{\mathfrak{g}}{12} - \frac{\mathfrak{g}}{6} \|^2,\ \alpha \| \frac{\mathfrak{g}}{12} - \frac{\mathfrak{g}}{6} \|^2\right)}{1+\left(\| \frac{\mathfrak{g}}{12} - \frac{\mathfrak{g}}{6} \|^2,\ \alpha \| \frac{\mathfrak{g}}{12} - \frac{\mathfrak{g}}{6} \|^2\right)}, \\
& \frac{\left(\left(\| \frac{\mathfrak{h}+11}{12} - \frac{\mathfrak{g}}{6} \|^2,\ \alpha \| \frac{\mathfrak{h}+11}{12} - \frac{\mathfrak{g}}{6} \|^2\right) + \left(\| \frac{\mathfrak{h}+5}{6} - \frac{\mathfrak{g}}{12} \|^2,\ \alpha \| \frac{\mathfrak{h}+5}{6} - \frac{\mathfrak{g}}{12} \|^2\right)\right)}{4}, \\
& \frac{\left(\left(\| \frac{\mathfrak{h}+11}{12} - \frac{\mathfrak{h}+5}{6} \|^2,\ \alpha \| \frac{\mathfrak{h}+11}{12} - \frac{\mathfrak{h}+5}{6} \|^2\right) + \left(\| \frac{\mathfrak{g}}{12} - \frac{\mathfrak{g}}{6} \|^2,\ \alpha \| \frac{\mathfrak{g}}{12} - \frac{\mathfrak{g}}{6} \|^2\right)\right)}{4}\}), \\
& \phi(\max\{\left(\| \frac{\mathfrak{h}+5}{6} - \frac{\mathfrak{g}}{6} \|^2,\ \alpha \| \frac{\mathfrak{h}+5}{6} - \frac{\mathfrak{g}}{6} \|^2\right), \\
& \frac{\left(\| \frac{\mathfrak{h}+11}{12} - \frac{\mathfrak{h}+5}{6} \|^2,\ \alpha \| \frac{\mathfrak{h}+11}{12} - \frac{\mathfrak{h}+5}{6} \|^2\right)}{1+\left(\| \frac{\mathfrak{h}+11}{12} - \frac{\mathfrak{h}+5}{6} \|^2,\ \alpha \| \frac{\mathfrak{h}+11}{12} - \frac{\mathfrak{h}+5}{6} \|^2\right)}, \frac{\left(\| \frac{\mathfrak{g}}{12} - \frac{\mathfrak{g}}{6} \|^2,\ \alpha \| \frac{\mathfrak{g}}{12} - \frac{\mathfrak{g}}{6} \|^2\right)}{1+\left(\| \frac{\mathfrak{g}}{12} - \frac{\mathfrak{g}}{6} \|^2,\ \alpha \| \frac{\mathfrak{g}}{12} - \frac{\mathfrak{g}}{6} \|^2\right)}, \\
& \frac{\left(\left(\| \frac{\mathfrak{h}+11}{12} - \frac{\mathfrak{g}}{6} \|^2,\ \alpha \| \frac{\mathfrak{h}+11}{12} - \frac{\mathfrak{g}}{6} \|^2\right) + \left(\| \frac{\mathfrak{h}+5}{6} - \frac{\mathfrak{g}}{12} \|^2,\ \alpha \| \frac{\mathfrak{h}+5}{6} - \frac{\mathfrak{g}}{12} \|^2\right)\right)}{4}, \\
& \frac{\left(\left(\| \frac{\mathfrak{h}+11}{12} - \frac{\mathfrak{h}+5}{6} \|^2,\ \alpha \| \frac{\mathfrak{h}+11}{12} - \frac{\mathfrak{h}+5}{6} \|^2\right) + \left(\| \frac{\mathfrak{g}}{12} - \frac{\mathfrak{g}}{6} \|^2,\ \alpha \| \frac{\mathfrak{g}}{12} - \frac{\mathfrak{g}}{6} \|^2\right)\right)}{4}\})) \\
= \;& \mathfrak{F}(\psi(\max\{\left(\| \frac{(\mathfrak{h}+5)-\mathfrak{g}}{6} \|^2,\ \alpha \| \frac{(\mathfrak{h}+5)-\mathfrak{g}}{6} \|^2\right), \\
& \frac{\left(\| \frac{(\mathfrak{h}+11)-2(\mathfrak{h}+5)}{12} \|^2,\ \alpha \| \frac{(\mathfrak{h}+11)-2(\mathfrak{h}+5)}{12} \|^2\right)}{1+\left(\| \frac{(\mathfrak{h}+11)-2(\mathfrak{h}+5)}{12} \|^2,\ \alpha \| \frac{(\mathfrak{h}+11)-2(\mathfrak{h}+5)}{12} \|^2\right)}, \frac{\left(\| \frac{\mathfrak{g}-2\mathfrak{g}}{12} \|^2,\ \alpha \| \frac{\mathfrak{g}-2\mathfrak{g}}{12} \|^2\right)}{1+\left(\| \frac{\mathfrak{g}-2\mathfrak{g}}{12} \|^2,\ \alpha \| \frac{\mathfrak{g}-2\mathfrak{g}}{12} \|^2\right)}, \\
& \frac{\left(\left(\| \frac{(\mathfrak{h}+11)-2\mathfrak{g}}{12} \|^2,\ \alpha \| \frac{(\mathfrak{h}+11)-2\mathfrak{g}}{12} \|^2\right) + \left(\| \frac{2(\mathfrak{h}+5)-\mathfrak{g}}{12} \|^2,\ \alpha \| \frac{2(\mathfrak{h}+5)-\mathfrak{g}}{12} \|^2\right)\right)}{4}, \\
& \frac{\left(\left(\| \frac{(\mathfrak{h}+11)-2(\mathfrak{h}+5)}{12} \|^2,\ \alpha \| \frac{(\mathfrak{h}+11)-2(\mathfrak{h}+5)}{12} \|^2\right) + \left(\| \frac{\mathfrak{g}-2\mathfrak{g}}{12} \|^2,\ \alpha \| \frac{\mathfrak{g}-2\mathfrak{g}}{12} \|^2\right)\right)}{4}\}), \\
& \phi(\max\{\left(\| \frac{(\mathfrak{h}+5)-\mathfrak{g}}{6} \|^2,\ \alpha \| \frac{(\mathfrak{h}+5)-\mathfrak{g}}{6} \|^2\right), \\
& \frac{\left(\| \frac{(\mathfrak{h}+11)-2(\mathfrak{h}+5)}{12} \|^2,\ \alpha \| \frac{(\mathfrak{h}+11)-2(\mathfrak{h}+5)}{12} \|^2\right)}{1+\left(\| \frac{(\mathfrak{h}+11)-2(\mathfrak{h}+5)}{12} \|^2,\ \alpha \| \frac{(\mathfrak{h}+11)-2(\mathfrak{h}+5)}{12} \|^2\right)}, \frac{\left(\| \frac{\mathfrak{g}-2\mathfrak{g}}{12} \|^2,\ \alpha \| \frac{\mathfrak{g}-2\mathfrak{g}}{12} \|^2\right)}{1+\left(\| \frac{\mathfrak{g}-2\mathfrak{g}}{12} \|^2,\ \alpha \| \frac{\mathfrak{g}-2\mathfrak{g}}{12} \|^2\right)}, \\
& \frac{\left(\left(\| \frac{(\mathfrak{h}+11)-2\mathfrak{g}}{12} \|^2,\ \alpha \| \frac{(\mathfrak{h}+11)-2\mathfrak{g}}{12} \|^2\right) + \left(\| \frac{2(\mathfrak{h}+5)-\mathfrak{g}}{12} \|^2,\ \alpha \| \frac{2(\mathfrak{h}+5)-\mathfrak{g}}{12} \|^2\right)\right)}{4}, \\
& \frac{\left(\left(\| \frac{(\mathfrak{h}+11)-2(\mathfrak{h}+5)}{12} \|^2,\ \alpha \| \frac{(\mathfrak{h}+11)-2(\mathfrak{h}+5)}{12} \|^2\right) + \left(\| \frac{\mathfrak{g}-2\mathfrak{g}}{12} \|^2,\ \alpha \| \frac{\mathfrak{g}-2\mathfrak{g}}{12} \|^2\right)\right)}{4}\}))
\end{aligned}
$$

$$
\begin{aligned}
&= \mathfrak{F}(\psi(\max\{\left(\left(\frac{1}{36},\frac{1}{36}\ \alpha\right)\parallel(\mathfrak{h}+5)-\mathfrak{g}\parallel^2\right),\\
&\quad \frac{\left(\left(\frac{1}{144},\frac{1}{144}\ \alpha\right)\parallel(\mathfrak{h}+11)-2(\mathfrak{h}+5)\parallel^2\right)}{1+\left(\left(\frac{1}{144},\frac{1}{144}\alpha\right)\parallel(\mathfrak{h}+11)-2(\mathfrak{h}+5)\parallel^2\right)},\frac{\left(\left(\frac{1}{144},\frac{1}{144}\ \alpha\right)\parallel\mathfrak{g}-2\mathfrak{g}\parallel^2\right)}{1+\left(\left(\frac{1}{144},\frac{1}{144}\ \alpha\right)\parallel\mathfrak{g}-2\mathfrak{g}\parallel^2\right)},\\
&\quad \frac{\left(\left(\frac{1}{144},\frac{1}{144}\alpha\right)\parallel(\mathfrak{h}+11)-2\mathfrak{g}\parallel^2\right)+\left(\left(\frac{1}{144},\frac{1}{144}\ \alpha\right)\parallel 2(\mathfrak{h}+5)-\mathfrak{g}\parallel^2\right)}{4},\\
&\quad \frac{\left(\left(\frac{1}{144},\frac{1}{144}\alpha\right)\parallel(\mathfrak{h}+15)-2(\mathfrak{h}+5)\parallel^2\right)+\left(\left(\frac{1}{144},\frac{1}{144}\alpha\right)\parallel\mathfrak{g}-2\mathfrak{g}\parallel^2\right)}{4}\}),\\
&\quad \phi(\max\{\left(\left(\frac{1}{36},\frac{1}{36}\ \alpha\right)\parallel(\mathfrak{h}+5)-\mathfrak{g}\parallel^2\right),\\
&\quad \frac{\left(\left(\frac{1}{144},\frac{1}{144}\ \alpha\right)\parallel(\mathfrak{h}+11)-2(\mathfrak{h}+5)\parallel^2\right)}{1+\left(\left(\frac{1}{144},\frac{1}{144}\alpha\right)\parallel(\mathfrak{h}+11)-2(\mathfrak{h}+5)\parallel^2\right)},\frac{\left(\left(\frac{1}{144},\frac{1}{144}\ \alpha\right)\parallel\mathfrak{g}-2\mathfrak{g}\parallel^2\right)}{1+\left(\left(\frac{1}{144},\frac{1}{144}\ \alpha\right)\parallel\mathfrak{g}-2\mathfrak{g}\parallel^2\right)},\\
&\quad \frac{\left(\left(\frac{1}{144},\frac{1}{144}\alpha\right)\parallel(\mathfrak{h}+11)-2\mathfrak{g}\parallel^2\right)+\left(\left(\frac{1}{144},\frac{1}{144}\ \alpha\right)\parallel 2(\mathfrak{h}+5)-\mathfrak{g}\parallel^2\right)}{4},\\
&\quad \frac{\left(\left(\frac{1}{144},\frac{1}{144}\alpha\right)\parallel(\mathfrak{h}+15)-2(\mathfrak{h}+5)\parallel^2\right)+\left(\left(\frac{1}{144},\frac{1}{144}\alpha\right)\parallel\mathfrak{g}-2\mathfrak{g}\parallel^2\right)}{4}\}))\\
&= \mathfrak{F}\left(\psi(\left(\frac{1}{36},\frac{1}{36}\ \alpha\right)\parallel(\mathfrak{h}+5)-\mathfrak{g}\parallel^2),\phi(\left(\frac{1}{36},\frac{1}{36}\ \alpha\right)\parallel(\mathfrak{h}+5)-\mathfrak{g}\parallel^2)\right)\\
&= \psi\left(\left(\frac{1}{36},\frac{1}{36}\ \alpha\right)\parallel(\mathfrak{h}+5)-\mathfrak{g}\parallel^2\right)-\phi\left(\left(\frac{1}{36},\frac{1}{36}\ \alpha\right)\parallel(\mathfrak{h}+5)-\mathfrak{g}\parallel^2\right)\\
\text{R.H.S} &= \left(\frac{1}{36},\frac{1}{36}\ \alpha\right)\left(\psi\left(\parallel(\mathfrak{h}+5)-\mathfrak{g}\parallel^2\right)-\phi\left(\parallel(\mathfrak{h}+5)-\mathfrak{g}\parallel^2\right)\right)
\end{aligned}
$$

Therefore, L.H.S $\leq$ R.H.S. Inequality (3.1) is satisfied for each $\mathfrak{h}\in(1,+\infty)$ and $\mathfrak{g}\in[0,1]$ and $\alpha\geq 0$.

Case (iv): When $\mathfrak{h},\mathfrak{g}\in(1,+\infty)$

$$
\begin{aligned}
\text{L.H.S} &= \psi\left(\parallel\aleph\mathfrak{h}-\daleth\mathfrak{g}\parallel^2,\ \alpha\parallel\aleph\mathfrak{h}-\daleth\mathfrak{g}\parallel^2\ \right)\\
&= \psi\left(\parallel\frac{\mathfrak{h}+11}{12}-\frac{\mathfrak{g}+11}{12}\parallel^2,\ \alpha\parallel\frac{\mathfrak{h}+11}{12}-\frac{\mathfrak{g}+11}{12}\parallel^2\ \right)\\
&= \psi\left(\frac{1}{144}\parallel(\mathfrak{h}+11)-(\mathfrak{g}+11)\parallel^2,\frac{1}{144}\ \alpha\parallel(\mathfrak{h}+11)-(\mathfrak{g}+11)\parallel^2\ \right)\\
&= \left(\frac{1}{144},\frac{1}{144}\alpha\right)\psi\left(\parallel(\mathfrak{h}+11)-(\mathfrak{g}+11)\parallel^2\right)
\end{aligned}
$$

$$\begin{aligned}
\text{R.H.S} = {} & \mathfrak{F}(\psi\left(\daleth(\mathfrak{h},\mathfrak{g})\right), \phi\left(\daleth(\mathfrak{h},\mathfrak{g})\right)) \\
= {} & \mathfrak{F}(\psi(\max\{\left(\| \frac{\mathfrak{h}+5}{6} - \frac{\mathfrak{g}+5}{6} \|^2,\ \alpha \| \frac{\mathfrak{h}+5}{6} - \frac{\mathfrak{g}+5}{6} \|^2\right), \\
& \frac{\left(\| \frac{\mathfrak{h}+11}{12} - \frac{\mathfrak{h}+5}{6} \|^2,\ \alpha \| \frac{\mathfrak{h}+11}{12} - \frac{\mathfrak{h}+5}{6} \|^2\right)}{1+\left(\| \frac{\mathfrak{h}+11}{12} - \frac{\mathfrak{h}+5}{6} \|^2,\ \alpha \| \frac{\mathfrak{h}+11}{12} - \frac{\mathfrak{h}+5}{6} \|^2\right)}, \frac{\left(\| \frac{\mathfrak{g}+11}{12} - \frac{\mathfrak{g}+5}{6} \|^2,\ \alpha \| \frac{\mathfrak{g}+11}{12} - \frac{\mathfrak{g}+5}{6} \|^2\right)}{1+\left(\| \frac{\mathfrak{g}+11}{12} - \frac{\mathfrak{g}+5}{6} \|^2,\ \alpha \| \frac{\mathfrak{g}+11}{12} - \frac{\mathfrak{g}+5}{6} \|^2\right)}, \\
& \frac{\left(\left(\| \frac{\mathfrak{h}+11}{12} - \frac{\mathfrak{g}+5}{6} \|^2,\ \alpha \| \frac{\mathfrak{h}+11}{12} - \frac{\mathfrak{g}+5}{6} \|^2\right) + \left(\| \frac{\mathfrak{h}+5}{6} - \frac{\mathfrak{g}+11}{12} \|^2,\ \alpha \| \frac{\mathfrak{h}+5}{6} - \frac{\mathfrak{g}+11}{12} \|^2\right)\right)}{4}, \\
& \frac{\left(\left(\| \frac{\mathfrak{h}+11}{12} - \frac{\mathfrak{h}+5}{6} \|^2,\ \alpha \| \frac{\mathfrak{h}+11}{12} - \frac{\mathfrak{h}+5}{6} \|^2\right) + \left(\| \frac{\mathfrak{g}+11}{12} - \frac{\mathfrak{g}+5}{6} \|^2,\ \alpha \| \frac{\mathfrak{g}+11}{12} - \frac{\mathfrak{g}+5}{6} \|^2\right)\right)}{4}\}), \\
& \phi(\max\{\left(\| \frac{\mathfrak{h}+5}{6} - \frac{\mathfrak{g}+5}{6} \|^2,\ \alpha \| \frac{\mathfrak{h}+5}{6} - \frac{\mathfrak{g}+5}{6} \|^2\right), \\
& \frac{\left(\| \frac{\mathfrak{h}+11}{12} - \frac{\mathfrak{h}+5}{6} \|^2,\ \alpha \| \frac{\mathfrak{h}+11}{12} - \frac{\mathfrak{h}+5}{6} \|^2\right)}{1+\left(\| \frac{\mathfrak{h}+11}{12} - \frac{\mathfrak{h}+5}{6} \|^2,\ \alpha \| \frac{\mathfrak{h}+11}{12} - \frac{\mathfrak{h}+5}{6} \|^2\right)}, \frac{\left(\| \frac{\mathfrak{g}+11}{12} - \frac{\mathfrak{g}+5}{6} \|^2,\ \alpha \| \frac{\mathfrak{g}+11}{12} - \frac{\mathfrak{g}+5}{6} \|^2\right)}{1+\left(\| \frac{\mathfrak{g}+11}{12} - \frac{\mathfrak{g}+5}{6} \|^2,\ \alpha \| \frac{\mathfrak{g}+11}{12} - \frac{\mathfrak{g}+5}{6} \|^2\right)}, \\
& \frac{\left(\left(\| \frac{\mathfrak{h}+11}{12} - \frac{\mathfrak{g}+5}{6} \|^2,\ \alpha \| \frac{\mathfrak{h}+11}{12} - \frac{\mathfrak{g}+5}{6} \|^2\right) + \left(\| \frac{\mathfrak{h}+5}{6} - \frac{\mathfrak{g}+11}{12} \|^2,\ \alpha \| \frac{\mathfrak{h}+5}{6} - \frac{\mathfrak{g}+11}{12} \|^2\right)\right)}{4}, \\
& \frac{\left(\left(\| \frac{\mathfrak{h}+11}{12} - \frac{\mathfrak{h}+5}{6} \|^2,\ \alpha \| \frac{\mathfrak{h}+11}{12} - \frac{\mathfrak{h}+5}{6} \|^2\right) + \left(\| \frac{\mathfrak{g}+11}{12} - \frac{\mathfrak{g}+5}{6} \|^2,\ \alpha \| \frac{\mathfrak{g}+11}{12} - \frac{\mathfrak{g}+5}{6} \|^2\right)\right)}{4}\})) \\
= {} & \mathfrak{F}(\psi(\max\{\left(\| \frac{(\mathfrak{h}+5)-(\mathfrak{g}+5)}{6} \|^2,\ \alpha \| \frac{(\mathfrak{h}+5)-(\mathfrak{g}+5)}{6} \|^2\right), \\
& \frac{\left(\| \frac{(\mathfrak{h}+11)-2(\mathfrak{h}+5)}{12} \|^2, \alpha \| \frac{(\mathfrak{h}+11)-2(\mathfrak{h}+5)}{12} \|^2\right)}{1+\left(\| \frac{(\mathfrak{h}+11)-2(\mathfrak{h}+5)}{12} \|^2, \alpha \| \frac{(\mathfrak{h}+11)-2(\mathfrak{h}+5)}{12} \|^2\right)}, \frac{\left(\| \frac{(\mathfrak{g}+11)-2(\mathfrak{g}+5)}{12} \|^2, \alpha \| \frac{(\mathfrak{g}+11)-2(\mathfrak{g}+5)}{12} \|^2\right)}{1+\left(\| \frac{(\mathfrak{g}+11)-2(\mathfrak{g}+5)}{12} \|^2, \alpha \| \frac{(\mathfrak{g}+11)-2(\mathfrak{g}+5)}{12} \|^2\right)}, \\
& \frac{\left(\left(\| \frac{(\mathfrak{h}+11)-2(\mathfrak{g}+5)}{12} \|^2,\ \alpha \| \frac{(\mathfrak{h}+11)-2(\mathfrak{g}+5)}{12} \|^2\right) + \left(\| \frac{2(\mathfrak{h}+5)-(\mathfrak{g}+11)}{12} \|^2,\ \alpha \| \frac{2(\mathfrak{h}+5)-(\mathfrak{g}+11)}{12} \|^2\right)\right)}{4}, \\
& \frac{\left(\left(\| \frac{(\mathfrak{h}+11)-2(\mathfrak{h}+5)}{12} \|^2,\ \alpha \| \frac{(\mathfrak{h}+11)-2(\mathfrak{h}+5)}{12} \|^2\right) + \left(\| \frac{(\mathfrak{g}+11)-2(\mathfrak{g}+5)}{12} \|^2,\ \alpha \| \frac{(\mathfrak{g}+11)-2(\mathfrak{g}+5)}{12} \|^2\right)\right)}{4}\}), \\
& \phi(\max\{\left(\| \frac{(\mathfrak{h}+5)-(\mathfrak{g}+5)}{6} \|^2,\ \alpha \| \frac{(\mathfrak{h}+5)-(\mathfrak{g}+5)}{6} \|^2\right), \\
& \frac{\left(\| \frac{(\mathfrak{h}+11)-2(\mathfrak{h}+5)}{12} \|^2, \alpha \| \frac{(\mathfrak{h}+11)-2(\mathfrak{h}+5)}{12} \|^2\right)}{1+\left(\| \frac{(\mathfrak{h}+11)-2(\mathfrak{h}+5)}{12} \|^2, \alpha \| \frac{(\mathfrak{h}+11)-2(\mathfrak{h}+5)}{12} \|^2\right)}, \frac{\left(\| \frac{(\mathfrak{g}+11)-2(\mathfrak{g}+5)}{12} \|^2, \alpha \| \frac{(\mathfrak{g}+11)-2(\mathfrak{g}+5)}{12} \|^2\right)}{1+\left(\| \frac{(\mathfrak{g}+11)-2(\mathfrak{g}+5)}{12} \|^2, \alpha \| \frac{(\mathfrak{g}+11)-2(\mathfrak{g}+5)}{12} \|^2\right)}, \\
& \frac{\left(\left(\| \frac{(\mathfrak{h}+11)-2(\mathfrak{g}+5)}{12} \|^2,\ \alpha \| \frac{(\mathfrak{h}+11)-2(\mathfrak{g}+5)}{12} \|^2\right) + \left(\| \frac{2(\mathfrak{h}+5)-(\mathfrak{g}+11)}{12} \|^2,\ \alpha \| \frac{2(\mathfrak{h}+5)-(\mathfrak{g}+11)}{12} \|^2\right)\right)}{4}, \\
& \frac{\left(\left(\| \frac{(\mathfrak{h}+11)-2(\mathfrak{h}+5)}{12} \|^2,\ \alpha \| \frac{(\mathfrak{h}+11)-2(\mathfrak{h}+5)}{12} \|^2\right) + \left(\| \frac{(\mathfrak{g}+11)-2(\mathfrak{g}+5)}{12} \|^2,\ \alpha \| \frac{(\mathfrak{g}+11)-2(\mathfrak{g}+5)}{12} \|^2\right)\right)}{4}\}))
\end{aligned}$$

$$
\begin{aligned}
= \ & \mathfrak{F}(\psi(\max\{\left(\left(\frac{1}{36},\frac{1}{36}\ \alpha\right) \parallel (\mathfrak{h}+5)-(\mathfrak{g}+5) \parallel^2\right), \\
& \frac{((\frac{1}{144},\frac{1}{144}\ \alpha) \parallel (\mathfrak{h}+11)-2(\mathfrak{h}+5) \parallel^2)}{1+((\frac{1}{144},\frac{1}{144}\alpha) \parallel (\mathfrak{h}+11)-2(\mathfrak{h}+5) \parallel^2)}, \frac{((\frac{1}{144},\frac{1}{144}\ \alpha) \parallel (\mathfrak{g}+11)-2(\mathfrak{g}+5) \parallel^2)}{1+((\frac{1}{144},\frac{1}{144}\ \alpha) \parallel (\mathfrak{g}+11)-2(\mathfrak{g}+5) \parallel^2)}, \\
& \frac{((\frac{1}{144},\frac{1}{144}\alpha) \parallel (\mathfrak{h}+11)-2(\mathfrak{g}+5) \parallel^2)+((\frac{1}{144},\frac{1}{144}\ \alpha) \parallel 2(\mathfrak{h}+5)-(\mathfrak{g}+11) \parallel^2)}{4}, \\
& \frac{((\frac{1}{144},\frac{1}{144}\alpha) \parallel (\mathfrak{h}+15)-2(\mathfrak{h}+5) \parallel^2)+((\frac{1}{144},\frac{1}{144}\alpha) \parallel (\mathfrak{g}+11)-2(\mathfrak{g}+5) \parallel^2)}{4}\}), \\
& \phi(\max\{\left(\left(\frac{1}{36},\frac{1}{36}\ \alpha\right) \parallel (\mathfrak{h}+5)-(\mathfrak{g}+5) \parallel^2\right), \\
& \frac{((\frac{1}{144},\frac{1}{144}\ \alpha) \parallel (\mathfrak{h}+11)-2(\mathfrak{h}+5) \parallel^2)}{1+((\frac{1}{144},\frac{1}{144}\alpha) \parallel (\mathfrak{h}+11)-2(\mathfrak{h}+5) \parallel^2)}, \frac{((\frac{1}{144},\frac{1}{144}\ \alpha) \parallel (\mathfrak{g}+11)-2(\mathfrak{g}+5) \parallel^2)}{1+((\frac{1}{144},\frac{1}{144}\ \alpha) \parallel (\mathfrak{g}+11)-2(\mathfrak{g}+5) \parallel^2)}, \\
& \frac{((\frac{1}{144},\frac{1}{144}\alpha) \parallel (\mathfrak{h}+11)-2(\mathfrak{g}+5) \parallel^2)+((\frac{1}{144},\frac{1}{144}\ \alpha) \parallel 2(\mathfrak{h}+5)-(\mathfrak{g}+11) \parallel^2)}{4}, \\
& \frac{((\frac{1}{144},\frac{1}{144}\alpha) \parallel (\mathfrak{h}+15)-2(\mathfrak{h}+5) \parallel^2)+((\frac{1}{144},\frac{1}{144}\alpha) \parallel (\mathfrak{g}+11)-2(\mathfrak{g}+5) \parallel^2)}{4}\})) \\
= \ & \mathfrak{F}\left(\psi\left(\left(\frac{1}{36},\frac{1}{36}\ \alpha\right) \parallel (\mathfrak{h}+5)-(\mathfrak{g}+5) \parallel^2\right), \phi\left(\left(\frac{1}{36},\frac{1}{36}\ \alpha\right) \parallel (\mathfrak{h}+5)-(\mathfrak{g}+5) \parallel^2\right)\right) \\
= \ & \left(\frac{1}{36},\frac{1}{36}\ \alpha\right)\left(\psi\left(\parallel (\mathfrak{h}+5)-(\mathfrak{g}+5) \parallel^2\right)-\phi\left(\parallel (\mathfrak{h}+5)-(\mathfrak{g}+5) \parallel^2\right)\right)
\end{aligned}
$$

Therefore, L.H.S $\leq$ R.H.S. Inequality (3.1) is satisfied for each $\mathfrak{h},\mathfrak{g} \in (1,+\infty)$ and $\alpha \geq 0$. Thus all the conditions of Theorem (3.2) are satisfied and 0 remains the unique common fixed point of the mappings $\aleph$, $\beth, \Gamma$ and $\mathfrak{L}$.

4. Application

We have find the existence and uniqueness of common solutions for a system of functional equations arising in dynamic programming, it was formed by Bellman and Lee [7] with the help of our main Theorem (3.2).

Let $(P, \|.\|)$ and $(Q, \|.\|)$ are nored liner spaces, $M \subseteq P$ and $D \subseteq Q$. Taking M and D signify the state and decision spaces, respectively. Let $B(M)$ denotes the set of all real-valued functions on M. It is easy to verify that $B(M)$ is a linear space over R under usual definitions of addition and scalar multiplication, and with the norm $\|.\|$ for an arbitrary $\mathfrak{h} \in B(M)$, define

$$\|\mathfrak{h}\| = \sup\{|\mathfrak{h}(\mathfrak{y})| : \mathfrak{y} \in M\}.$$

Let $\|\mathfrak{h}-\mathfrak{g}\| = \sup|\mathfrak{h}(\mathfrak{y})-\mathfrak{g}(\mathfrak{y})|$ for all $\mathfrak{h},\mathfrak{g} \in B(M)$. Then $(B(M), \|.\|)$ is a normed liner space. As proposed in Bellman and Lee [7], the fundamental form of the functional equation in dynamic programming is

$$\aleph(\mathfrak{y}) = opt_{\mathfrak{z}\in D}\mathfrak{H}(\mathfrak{y},\mathfrak{z},\aleph(\beth(\mathfrak{y},\mathfrak{z}))), \text{for all } \ \mathfrak{y} \in M,$$

where $\mathfrak{y}$ and $\mathfrak{z}$ denote the state and decision vectors, respectively. $\beth$ denotes the transformation of the process, $\aleph(\mathfrak{y})$ denotes the optimal return function with the initial state $\mathfrak{y}$ and opt represents sup or inf. For the existence and uniqueness of a solution to the following functional equation arising in dynamic programming, Liu et al. [19] proved fixed point theorems providing a contractive condition of integral type.

$$\aleph(\mathfrak{y}) = \text{ opt }_{\mathfrak{z}\in D} \left\{\mathfrak{J}(\mathfrak{y},\mathfrak{z}) + \mathfrak{H}_1\left(\mathfrak{y},\mathfrak{z},\aleph\left(\beth(\mathfrak{y},\mathfrak{z})\right)\right)\right\}, \text{for all } \mathfrak{y}\in M,$$

Further, Liu and colleagues [20] developed common fixed point theorems that meet contractive conditions of the integral type, and they used these findings to verify the existence and uniqueness of common solutions to the following system of functional equations occurring in dynamic programming.

$$\begin{aligned}\mathfrak{P}_i(\mathfrak{y}) =&\text{opt}_{\mathfrak{z}\in D} \left\{\mathfrak{J}(\mathfrak{y},\mathfrak{z}) + \mathfrak{H}_i\left(\mathfrak{y},\mathfrak{z},\mathfrak{P}_i\left(\beth(\mathfrak{y},\mathfrak{z})\right)\right)\right\}, \text{for all } \mathfrak{y}\in M, i=1,2;\\ \mathfrak{T}_i(\mathfrak{y}) =&\text{opt}_{\mathfrak{z}\in D} \left\{\mathfrak{J}(\mathfrak{y},\mathfrak{z}) + \mathfrak{T}_i\left(\mathfrak{y},\mathfrak{z},\mathfrak{T}_i\left(\beth(\mathfrak{y},\mathfrak{z})\right)\right)\right\}, \text{for all } \mathfrak{y}\in M, i=1,2;\end{aligned} \tag{4.1}$$

where $\mathfrak{J}: M\times D\to\mathcal{R}, \beth: M\times D\to M$ and $\mathfrak{H}_i,\mathfrak{T}_i : M\times D\times\mathcal{R}\to\mathcal{R}$.

THEOREM 4.1. *Let $\mathfrak{H}_i,\mathfrak{T}_i : M\times D\times R\to R$ for $i=1,2$ be four bounded functions and let $\mathfrak{S}_i,\mathfrak{Q}_i : B(M)\to B(M)$ be four operators defined by*

$$\mathfrak{S}_i\mathfrak{h}(\mathfrak{y}) = \text{opt}_{\mathfrak{z}\in D} \left\{\mathfrak{J}(\mathfrak{y},\mathfrak{z}) + \mathfrak{H}_i\left(\mathfrak{y},\mathfrak{z},\mathfrak{h}\left(\beth(\mathfrak{y},\mathfrak{z})\right)\right)\right\}, \textit{for all } (\mathfrak{y},\mathfrak{h})\in M\times B(M), i=1,2 \tag{4.2}$$

$$\mathfrak{Q}_i\mathfrak{g}(\mathfrak{y}) = \text{opt}_{\mathfrak{z}\in D} \left\{\mathfrak{J}(\mathfrak{y},\mathfrak{z}) + \mathfrak{T}_i\left(\mathfrak{y},\mathfrak{z},\mathfrak{g}\left(\beth(\mathfrak{y},\mathfrak{z})\right)\right)\right\}, \textit{for all } (\mathfrak{y},\mathfrak{g})\in M\times B(M), i=1,2. \tag{4.3}$$

Assume that the following conditions hold:

1. *there exists $\mathfrak{S}_i = \{\mathfrak{S}_1,\mathfrak{S}_2\}$such that for all sequence $\{\mathfrak{h}_n\}_{n\geq 1}\subset B(M)$,*

$$\lim_{n\to+\infty}\mathfrak{S}_1\{\mathfrak{h}_n\} = \lim_{n\to+\infty}\mathfrak{S}_2\{\mathfrak{h}_n\} = \mathfrak{h}^*\in B(M)$$

and

$$\lim_{n\to+\infty}\sup_{\mathfrak{y}\in M}\|\mathfrak{S}_1\mathfrak{S}_2\{\mathfrak{h}_n\} - \mathfrak{S}_2\mathfrak{S}_1\{\mathfrak{h}_n\}\| = 0$$

2. *there exists $\mathfrak{Q}_i = \{\mathfrak{Q}_1,\mathfrak{Q}_2\}$such that for all sequence $\{\mathfrak{g}_n\}_{n\geq 1}\subset B(M)$,*

$$\lim_{n\to+\infty}\mathfrak{Q}_1\{\mathfrak{g}_n\} = \lim_{n\to+\infty}\mathfrak{Q}_2\{\mathfrak{g}_n\} = \mathfrak{g}^*\in B(M)$$

and

$$\lim_{n\to+\infty}\sup_{\mathfrak{y}\in M}\|\mathfrak{Q}_1\mathfrak{Q}_2\{\mathfrak{g}_n\} - \mathfrak{Q}_2\mathfrak{Q}_1\{\mathfrak{g}_n\}\| = 0$$

3. *$\mathfrak{S}_1(B(M))\subseteq\mathfrak{Q}_2(B(M))$ and $\mathfrak{S}_2(B(M))\subseteq\mathfrak{T}_1(B(M))$*

4.

$$\psi\left(\int_0^{|\mathfrak{H}_1(\mathfrak{y},\mathfrak{z},\mathfrak{h}(\beth(\mathfrak{y},\mathfrak{z})))-\mathfrak{T}_1(\mathfrak{y},\mathfrak{z},\mathfrak{g}(\beth(\mathfrak{y},\mathfrak{z})))|}\varphi(t)dt\right)\leq\mathfrak{F}\left(\psi\left(\int_0^{\urcorner(\mathfrak{h},\mathfrak{g})}\varphi(t)dt\right),\phi\left(\int_0^{\urcorner(\mathfrak{h},\mathfrak{g})}\varphi(t)dt\right)\right), \tag{4.4}$$

where

$$\daleth(\mathfrak{h},\mathfrak{g}) = \max\{\ \| \mathfrak{Q}_1\mathfrak{h} - \mathfrak{S}_2\mathfrak{g} \|,\ \frac{\| \mathfrak{S}_1\mathfrak{h} - \mathfrak{Q}_1\mathfrak{h} \|}{1+ \| \mathfrak{S}_1\mathfrak{h} - \mathfrak{Q}_1\mathfrak{h} \|}, \frac{\| \mathfrak{S}_2\mathfrak{g} - \mathfrak{Q}_2\mathfrak{g} \|}{1+ \| \mathfrak{S}_2\mathfrak{g} - \mathfrak{Q}_2\mathfrak{g} \|},$$

$$\frac{\| \mathfrak{S}_1\mathfrak{h} - \mathfrak{Q}_2\mathfrak{g} \| + \| \mathfrak{Q}_1\mathfrak{h} - \mathfrak{S}_2\mathfrak{g} \|}{2b}, \ \frac{\| \mathfrak{S}_1\mathfrak{h} - \mathfrak{Q}_1\mathfrak{h} \| + \| \mathfrak{S}_2\mathfrak{g} - \mathfrak{Q}_2\mathfrak{g} \|}{2b}\ \}$$

where $\mathfrak{h},\mathfrak{g} \in B(M), \mathfrak{y} \in M, \mathfrak{z} \in D, \psi \in \Psi, \phi \in \Phi$ *and* $\varphi : R_+ \to R_+$ *is a non-negative summable Lebesgue integrable function such that*

$$\int_0^{\epsilon} \varphi(t)\ dt > 0,$$

for each $\epsilon > 0$*. Then the system of functional equations (4.1) has a unique bounded solution.*

Proof. Then the system of functional equations (4.1) have a unique bounded solution if and only if the operator (4.2) and (4.3) have a common fixed point. Now, since $\mathfrak{H}_i$ and $\mathfrak{T}_i$ are bounded for $i = 1, 2$, there exists a positive number $\Omega > 0$ such that

$$\sup\{| |\mathfrak{H}_i(\mathfrak{y},\mathfrak{z},t)|, |\mathfrak{T}_i(\mathfrak{y},\mathfrak{z},t)| : (\mathfrak{y},\mathfrak{z},t) \in M \times D \times R, i = 1, 2\} \leq \Omega,$$

from (4.2) and (4.3) we obtain $\mathfrak{S}(\mathfrak{h})$ and $\mathfrak{Q}(\mathfrak{g})$ are bounded for each $\mathfrak{h},\mathfrak{g} \in B(M)$, which yields that $\mathfrak{S}$ and $\mathfrak{Q}$ are self mappings in $B(M)$. By using a property from the integration theory [10] and the properties of φ, we conclude that for each positive number ω, there exists a positive number $\beta(\omega)$. it follows from $\varphi : R_+ \to R_+$, for all $\epsilon > 0\ \exists\ \omega > 0$ such that

$$\int_{\Upsilon} \varphi(t)dt < \epsilon, \text{ for all } \ \Upsilon \subset \mathcal{R}^+ \text{with } m(\Upsilon) \leq \beta(\omega),$$

where $m(\Upsilon)$ denotes the Lebesgue measure of Υ. Let $\mathfrak{y} \in M,\ \mathfrak{h}_1, \mathfrak{h}_2 \in B(M)$. Suppose that opt $t_{\mathfrak{z}\in D} = \sup_{\mathfrak{z}\in D}$. Then using (4.2) and (4.3) we can find $\mathfrak{z}_1, \mathfrak{z}_2 \in D$ such that

$$\mathfrak{S}_1\mathfrak{h}(\mathfrak{y}) < \mathfrak{J}(\mathfrak{y},\mathfrak{z}_1) + \mathfrak{H}_1(\mathfrak{y},\mathfrak{z}_1,\mathfrak{h}_1(\sqsupset(\mathfrak{y},\mathfrak{z}_1))) + \beta(\omega), \tag{4.5}$$

$$\mathfrak{Q}_1\mathfrak{g}(\mathfrak{y}) < \mathfrak{J}(\mathfrak{y},\mathfrak{z}_2) + \mathfrak{T}_1(\mathfrak{y},\mathfrak{z}_2,\mathfrak{g}(\sqsupset(\mathfrak{y},\mathfrak{z}_2))) + \beta(\omega), \tag{4.6}$$

$$\mathfrak{S}_1\mathfrak{h}(\mathfrak{y}) \geq \mathfrak{J}(\mathfrak{y},\mathfrak{z}_2) + \mathfrak{H}_1(\mathfrak{y},\mathfrak{z}_2,\mathfrak{h}(\sqsupset(\mathfrak{y},\mathfrak{z}_2))), \tag{4.7}$$

$$\mathfrak{Q}_1\mathfrak{g}(\mathfrak{y}) \geq \mathfrak{J}(\mathfrak{y},\mathfrak{z}_1) + \mathfrak{T}_1(\mathfrak{y},\mathfrak{z}_1,\mathfrak{g}(\sqsupset(\mathfrak{y},\mathfrak{z}_1))). \tag{4.8}$$

From (4.5) and (4.8), we get

$$\begin{aligned}\mathfrak{S}_1\mathfrak{h}(\mathfrak{y}) - \mathfrak{Q}_1\mathfrak{g}(\mathfrak{y}) &< \mathfrak{H}_1(\mathfrak{y},\mathfrak{z}_1,\mathfrak{h}(\sqsupset(\mathfrak{y},\mathfrak{z}_1))) - \mathfrak{T}_1(\mathfrak{y},\mathfrak{z}_1,\mathfrak{g}(\sqsupset(\mathfrak{y},\mathfrak{z}_1))) + \beta(\omega) \\ &\leq |\mathfrak{H}_1(\mathfrak{y},\mathfrak{z}_1,\mathfrak{h}(\sqsupset(\mathfrak{y},\mathfrak{z}_1))) - \mathfrak{T}_1(\mathfrak{y},\mathfrak{z}_1,\mathfrak{g}(\sqsupset(\mathfrak{y},\mathfrak{z}_1)))| + \beta(\omega).\end{aligned}$$

Hence we get

$$\mathfrak{S}_1\mathfrak{h}(\mathfrak{y}) - \mathfrak{Q}_1\mathfrak{g}(\mathfrak{y}) < |\mathfrak{H}_1(\mathfrak{y},\mathfrak{z}_1,\mathfrak{h}(\sqsupset(\mathfrak{y},\mathfrak{z}_1))) - \mathfrak{T}_1(\mathfrak{y},\mathfrak{z}_1,\mathfrak{g}(\sqsupset(\mathfrak{y},\mathfrak{z}_1)))| + \beta(\omega) \tag{4.9}$$

From (4.6) and (4.7), we get

$$\begin{aligned}\mathfrak{Q}_1\mathfrak{g}(\mathfrak{y}) - \mathfrak{S}_1\mathfrak{h}(\mathfrak{y}) &< \mathfrak{T}_1(\mathfrak{y},\mathfrak{z}_2,\mathfrak{g}(\sqsupset(\mathfrak{y},\mathfrak{z}_2))) - \mathfrak{H}_1(\mathfrak{y},\mathfrak{z}_2,\mathfrak{h}(\sqsupset(\mathfrak{y},\mathfrak{z}_2))) + \beta(\omega) \\ &\leq |\mathfrak{T}_1(\mathfrak{y},\mathfrak{z}_2,\mathfrak{g}(\sqsupset(\mathfrak{y},\mathfrak{z}_2))) - \mathfrak{H}_1(\mathfrak{y},\mathfrak{z}_2,\mathfrak{h}(\sqsupset(\mathfrak{y},\mathfrak{z}_2)))| + \beta(\omega).\end{aligned} \tag{4.10}$$

From (4.9) and (4.10), we get

$$|\mathfrak{S}_1\mathfrak{h}(\mathfrak{y}) - \mathfrak{Q}_1\mathfrak{g}(\mathfrak{y})| < \max\{\mathfrak{L}_1, \mathfrak{L}_2\} + \beta(\omega),$$

where

$$\begin{aligned} \mathfrak{L}_1 &= \left|\mathfrak{H}_1\left(\mathfrak{y}, \mathfrak{z}_1, \mathfrak{h}\left(\sqsupset\left(\mathfrak{y}, \mathfrak{z}_1\right)\right)\right) - \mathfrak{T}_1\left(\mathfrak{y}, \mathfrak{z}_1, \mathfrak{g}\left(\sqsupset\left(\mathfrak{y}, \mathfrak{z}_1\right)\right)\right)\right|, \\ \mathfrak{L}_2 &= \left|\mathfrak{T}_1\left(\mathfrak{y}, \mathfrak{z}_2, \mathfrak{g}\left(\sqsupset\left(\mathfrak{y}, \mathfrak{z}_2\right)\right)\right) - \mathfrak{H}_1\left(\mathfrak{y}, \mathfrak{z}_2, \mathfrak{h}\left(\sqsupset\left(\mathfrak{y}, \mathfrak{z}_2\right)\right)\right)\right|. \end{aligned} \tag{4.11}$$

It follows from (4.4) and (4.11)

$$\begin{aligned} \psi\left(\int_0^{|\mathfrak{S}\mathfrak{h}(\mathfrak{y})-\mathfrak{Q}\mathfrak{g}(\mathfrak{y})|}\phi(t)dt\right) &\leq \psi\left(\int_0^{\max|\mathfrak{L}_1,\mathfrak{L}_2|+\beta(\omega)}\varphi(t)dt\right) \\ &= \max\left\{\psi\left(\int_0^{\mathfrak{L}_1+\beta(\omega)}\varphi(t)dt\right), \psi\left(\int_0^{\mathfrak{L}_2+\beta(\omega)}\varphi(t)dt\right)\right\} \\ &= \max\left\{\psi\left(\int_0^{\mathfrak{L}_1}\varphi(t)dt + \int_{\mathfrak{L}_1}^{\mathfrak{L}_1+\beta(\omega)}\varphi(t)dt\right),\right. \\ &\left.\psi\left(\int_0^{\mathfrak{L}_2}\varphi(t)dt + \int_{\mathfrak{L}_2}^{\mathfrak{L}_2+\beta(\omega)}\varphi(t)dt\right)\right\} \\ &\leq \max\left\{\psi\left(\int_0^{\mathfrak{L}_1}\varphi(t)dt\right) + \psi\left(\int_{\mathfrak{L}_1}^{\mathfrak{L}_1+\beta(\omega)}\varphi(t)dt\right),\right. \\ &\left.\psi\left(\int_0^{\mathfrak{L}_2}\varphi(t)dt\right) + \psi\left(\int_{\mathfrak{L}_2}^{\mathfrak{L}_2+\beta(\omega)}\varphi(t)dt\right)\right\} \\ &\leq \max\left\{\psi\left(\int_0^{\mathfrak{L}_1}\varphi(t)dt\right), \psi\left(\int_0^{\mathfrak{L}_2}\varphi(t)dt\right)\right\} \\ &+ \max\left\{\psi\left(\int_{\mathfrak{L}_1}^{\mathfrak{L}_1+\beta(\omega)}\varphi(t)dt\right), \psi\left(\int_{\mathfrak{L}_2}^{\mathfrak{L}_2+\beta(\omega)}\varphi(t)dt\right)\right\} \\ &\leq \mathfrak{F}\left(\psi\left(\int_0^{\urcorner(\mathfrak{h},w)}\varphi(t)dt\right), \phi\left(\int_0^{\urcorner(\mathfrak{h},w)}\varphi(t)dt\right)\right) + (\epsilon) \end{aligned}$$

Taking $\epsilon \to 0^+$in the above inequality and using $\psi \in \Psi$, we get

$$\psi\left(\int_0^{\|\mathfrak{S}\mathfrak{h}-\mathfrak{Q}\mathfrak{g}\|}\varphi(t)dt\right) \leq \mathfrak{F}\left(\psi\left(\int_0^{\urcorner(\mathfrak{h},\mathfrak{g})}\varphi(t)dt\right), \phi\left(\int_0^{\urcorner(\mathfrak{h},\mathfrak{g})}\varphi(t)dt\right)\right).$$

where

$$\urcorner(\mathfrak{h},\mathfrak{g}) = \max\{\parallel \mathfrak{Q}_1\mathfrak{h} - \mathfrak{S}_2\mathfrak{g} \parallel, \frac{\parallel \mathfrak{S}_1\mathfrak{h} - \mathfrak{Q}_1\mathfrak{h} \parallel}{1+ \parallel \mathfrak{S}_1\mathfrak{h} - \mathfrak{Q}_1\mathfrak{h} \parallel}, \frac{\parallel \mathfrak{S}_2\mathfrak{g} - \mathfrak{Q}_2\mathfrak{g} \parallel}{1+ \parallel \mathfrak{S}_2\mathfrak{g} - \mathfrak{Q}_2\mathfrak{g} \parallel},$$

$$\frac{\parallel \mathfrak{S}_1\mathfrak{h} - \mathfrak{Q}_2\mathfrak{g} \parallel + \parallel \mathfrak{Q}_1\mathfrak{h} - \mathfrak{S}_2\mathfrak{g} \parallel}{2b}, \frac{\parallel \mathfrak{S}_1\mathfrak{h} - \mathfrak{Q}_1\mathfrak{h} \parallel + \parallel \mathfrak{S}_2\mathfrak{g} - \mathfrak{Q}_2\mathfrak{g} \parallel}{2b}\}$$

If $\aleph = \mathfrak{S}_1,\ \daleth = \mathfrak{S}_2,\ \Gamma = \mathfrak{T}_1,$ and $\mathfrak{L} = \mathfrak{T}_2$, it is easy to observe that all the hypotheses of Theorem (3.2) are satisfied. Hence the mappings $\mathfrak{S}_1$ and $\mathfrak{Q}_1$ have a unique common fixed point in $B(M)$, and hence the system of functional equations (4.1) has a unique bounded common solution.

□

5. Conclusion

This article illustrates the work establishing the (CLR)- properties for four self-maps via ultra-altering and altering distance mapping within the framework of common fixed points theorems for cone b-normed spaces. We use this application to discover the existence and uniqueness of a common solution for a set of functional equations that arise in dynamic programming. An example is given in support of our main result. Our findings provide up new possibilities for the researchers working in the area.

References

[1] M. Abbas and G. Jungck, *Common fixed point results for noncommuting mappings without continuity in cone metric spaces*, J. Math. Appl. **341** (2008), 1.3, 1.8.
https://dx.doi.org/10.1016/j.jmaa.2007.09.070

[2] S. Aleksic, Z. Kadelburg, Z.D. Mitrovic and S. Radenovic, *A new survey: Cone metric spaces*, J. Inter. Math. Virtual Institute **9** (2019), 93–121.

[3] A.H. Ansari, *Note on $\phi-\psi$-contractive type mappings and related fixed points*, The 2nd Regional Conference on Mathematics and Applications PNU (2014), 377–380.

[4] A.H. Ansari, S. Chandok, N. Hussin and L. Paunovic, *Fixed points of (ψ, ϕ)-weak contractions in regular cone metric spaces via new function*, J. Adv. Math. Stud. **9** (1) (2016), 72–82.
https://www.isroset.org/pub_paper/IJSRMSS/32-IJSRMSS-01061.pdf

[5] A. Aghajani, M. Abbas and J.R. Roshan, *Common fixed point of generalized weak contractive mappings in partially ordered b-metric spaces*, Math. Slovaca **64** (4) (2014), 941–960.
https://dx.doi.org/10.2478/s12175-014-0250-6

[6] I.A. Bakhtin, *The contraction mapping principle in almost metric space*, Funct. Anal. Gos. Ped. Inst. Unianowsk **30** (1989), 26–37.
https://www.scirp.org/reference/referencespapers?referenceid=2207100

[7] R. Bellman and E.S. Lee, *Functional equations in dynamic programming*, Aequationes Mathematicae **17** (1) (1978), 1–18.
https://dx.doi.org/10.1007/BF01818535

[8] M. Boriceanu, M. Bota and A. Petrusel, *Mutivalued fractals in b-metric spaces*, Cent. Eur. J. Math. **8** (2) (2010), 367–377.
https://dx.doi.org/10.2478/s11533-010-0009-4

[9] S. Czerwik, *Contraction mappings in b-metric spaces*, Acta Math Inform Univ Ostrav **5** (1993), 5–11.
http://dml.cz/dmlcz/120469

[10] E. Hewitt and K. Stromberg, *Real and Abstract Analysis*, New York: Springer-Verlag, (1965).
https://link.springer.com/book/10.1007/978-3-642-88044-5

[11] L.G. Huang and X. Zhang, *Cone metric space and fixed point theorems of contractive mappings*, J. Mat. Anal. App. **332** (2) (2007), 1468–1476.
https://dx.doi.org/10.1016/j.jmaa.2005.03.087

[12] H. Huang and S. Xu, *Fixed point theorems of contractive mappings in cone b-metric spaces and applications*, Fixed Point Theory Appl. **112** (2013), 1–10.
https://dx.doi.org/10.1186/1687-1812-2014-55

[13] N. Hussain and M.H. Shah, *KKM mappings in cone b-metric space*, Comp. Math. Appl. **62** (2011), 1677–1684.
https://dx.doi.org/10.1016/j.camwa.2011.06.004

[14] S. Jankovic, Z. Kadelburg and S. Radenovic, *On cone metric spaces: a survey*, Nonlinear Analysis: Theory, Methods & Applications **74** (7) (2011), 2591–2601.
https://dx.doi.org/10.1016/j.na.2010.12.014

[15] Jitender Kumar, *Common Fixed Point Theorems Of Weakly Compatible Maps Satisfying (E.A.) and (CLR) Property*, International Journal of Pure and Applied Mathematics **88** (3) (2013),

363–376.
https://dx.doi.org/10.12732/ijpam.v88i3.4

[16] G. Jungck, *Compatible mappings and common fixed points*, Int. J. Math. Sci. **9** (4) (1986), 771–777.
https://dx.doi.org/10.1155/S0161171286000935

[17] E. Karapinar, *Fixed Point Theorems in cone Banach spaces*, Fixed Point Theory and Applications (2009), 1087–1093.
https://dx.doi.org/10.1155/2009/609281

[18] M.S. Khan, M. Swaleh and S. Sessa, *Fixed point theorems by altering distances between the points*, Bull. Austral. Math. Soc. **30** (1984), 1–9.
https://dx.doi.org/10.1017/S0004972700001659

[19] Z. Liu, X. Li, S.M. Kang and S.Y. Cho, *Fixed point theorems for mappings satisfying contractive conditions of integral type and applications*, Fixed Point Theory and Applications **64** (2011), 1–18.
https://dx.doi.org/10.1186/1687-1812-2011-64

[20] Z. Liu, X. Zou, S.M. Kang and J.S. Ume, *Common fixed points for a pair of mappings satisfying contractive conditions of integral type*, Journal of Inequalities and Applications **394** (2014), 1–19.
https://dx.doi.org/10.1186/1029-242X-2014-394

[21] H.K. Nashine, *Common fixed point theorems satisfying integral type rational contractive conditions and applications*, Miskolc Mathematical Notes **16** (1) (2015), 321–351.
https://dx.doi.org/10.18514/MMN.2015.1133

[22] H.K. Pathak, *Common Fixed Point Theorems Using Property (E.A) In Complex-Valued Metric Spaces*, Thai Journal of Mathematics **11** (2) (2013), 347–355.
https://thaijmath2.in.cmu.ac.th/index.php/thaijmath/article/view/382/378

[23] K.P.R. Rao, *Common Fixed Point Theorems For Mappings Under (Clrs)-Property In Partial Metric Spaces*, Demonstratio Mathematica **49** (3) (2016), 357–371.
https://dx.doi.org/10.1515/dema-2016-0030

[24] Sh. Rezapour and R. Hamlbarani, *Some notes on the paper: Cone metric spaces and fixed point theorems of contractive mappings*, Journal of Mathematical Analysis and Applications **345** (2) (2008), 719–724.
https://dx.doi.org/10.1016/j.jmaa.2008.04.049

[25] W. Sintunavarat and P. Kumam, *Common fixed point theorems for a pair of weakly compatible mappings in fuzzy metric spaces*, Journal of Applied Mathematics (2011), 1–14.
https://dx.doi.org/10.1155/2011/637958

[26] D. Turkoglu, M. Abuloha and T. Abdeljawad, *Some theorems and examples of cone metric spaces*, J. Comput. Anal. **12** (4) (2010), 739–753.

[27] C. Yang and X. Zhu, *The concept of cone b-metric space and fixed point theorems*, Open Mathematics **19** (2021), 1187–1196.
https://dx.doi.org/10.1515/math-2021-0068

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