Korean J. Math. **32** (2024), No. 4, pp. 639–646
https://dx.doi.org/10.11568/kjm.2024.32.4.639

EGODIC SHADOWABLE POINTS AND UNIFORM LIMITS

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Abstract. In this paper, we study some dynamical properties of ergodic shadowable points for dynamical systems on noncompact metric spaces. We also show that if a sequence of homeomorphisms on a metric space which converges uniformly to a homeomorphism has the ergodic shadowing property, then so does the uniform limit.

1. Introduction and preliminaries

Variant concepts related to the shadowing property play an important role in the qualitative theory of hyperbolic dynamical systems on compact metric spaces (see [2, 5, 15, 19]). Also, some pointwise versions of shadowing in dynamical systems have been studied by many authors(see [6, 7, 11–14, 16–18]). Morales [13] introduced the notion of shadowable points closely related to that of absolutely non-shadowable points and studied the relations between several invariant sets through pointwise dynamics. In [6], Kawaguchi further extended this notion by introducing the concept of quantitative shadowable points for continuous maps and studied shadowable points with a given shadowing accuracy. Koo *et al.* [12] introduced the notions of topologically stable points and finitely shadowable points for homeomorphisms on a compact metric space and investigated the connection between pointwise topological points and various shadowing properties. Then, Koo and Lee [8] investigated a pointwise version of Walters topological stability in the class of homeomorphisms on a compact metric space and the topological stability of the uniform limits of uniform expansive dynamical systems with the uniform shadowing property. Also, Koo *et al.* [11] studied some invariance and recurrence properties related to the set of periodic shadowable points for homeomorphisms on a compact metric space and the relations between the sets of shadowable points, periodic shadowable points and uniformly expansive points, respectively. Some of the results concerning above works have been extended to a flow version by Aponte and Villavicencio(see [3]).

Fakhari and Ghane [5] introduced the notion of the ergodic shadowing property for continuous maps on a compact metric space which is equivalent to the map being topologically mixing and has the ordinary shadowing property. Then, they defined some

Received July 3, 2024. Revised September 13, 2024. Accepted September 20, 2024.
2010 Mathematics Subject Classification: 37A25, 37B65, 37B20.
Key words and phrases: ergodic shadowing property, shadowing property, nonwandering, chain recurrence, ergodic shadowable point, uniform limit.
This work was supported by research fund of Chungnam National University.
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kind of specification property and investigated its relation to the ergodic shadowing property. Ahmadi [1] introduced and studied the topological concepts of ergodic shadowing, chain transitivity and topological ergodicity for uniformly continuous maps on uniform spaces. Also, the author showed that the topological ergodic shadowing property is preserved by higher iterations of the uniformly continuous map on a uniform space. Das and Das [4] introduced a new variant of shadowing, namely, the mean ergodic shadowing and established a relationship of this notion with other variants of shadowing. Furthermore, they gave a necessary and sufficient condition for an orbital limit function to have the mean ergodic shadowing property.

Recently, Rego and Arbieto [18] studied the problem of positive topological entropy for flows using pointwise dynamics. They also introduced the notion of ergodic shadowable points for continuous maps on a compact metric space and dealt with pointwise versions of some shadowing-type properties including the globalness of shadowable points with gaps(see [18, Theorem D]).

To present well known notions of dynamical systems and our main results that is to extend some results about the ergodic shadowing property of continuous maps on compact metric spaces to the class of homeomorphisms on noncompact metric spaces, let us recall basic notions of dynamical systems which is used in this paper.

Let X be a metric space equipped with a metric d and $f : X \to X$ be a homeomorphism. For $\delta > 0$, a sequence $\{x_i\}_{i\in\mathbb{Z}}$ of points in X is called a *δ-pseudo orbit* of f if $d(f(x_i), x_{i+1}) < \delta$ for all $i \in \mathbb{Z}$. Given $\varepsilon > 0$, a sequence $\{x_i\}_{i\in\mathbb{Z}}$ is called to be *ε-shadowed* by $x \in X$ if $d(f^i(x), x_i) < \varepsilon$ for all $i \in \mathbb{Z}$. A homeomorphism $f : X \to X$ is said to have the *shadowing property* if for every $\varepsilon > 0$ there is $\delta > 0$ such that any δ-pseudo orbit of f is ε-shadowed by a point of X.

Now, we shall introduce the notion of the ergodic shadowing property for the class of homeomorphisms of a metric space. Given a sequence $\xi = \{x_i\}_{i\in\mathbb{Z}}$ in X, put

$$\mathrm{Npo}(\xi, f, \delta) = \{i \in \mathbb{Z} : d(f(x_i), x_{i+1}) \geq \delta\}$$

and

$$\mathrm{Npo}_n(\xi, f, \delta) = \mathrm{Npo}(\xi, f, \delta) \cap [-(n-1), n-1],$$

where $[-(n-1), n-1] = \{i \in \mathbb{Z} : -(n-1) \leq i \leq n-1\}$. For a sequence ξ and a point $x \in X$, put

$$\mathrm{Ns}(\xi, x, f, \delta) = \{i \in \mathbb{Z} : d(f^i(x), x_i) \geq \delta\}$$

and

$$\mathrm{Ns}_n(\xi, x, f, \delta) = \mathrm{Ns}(\xi, x, f, \delta) \cap [-(n-1), n-1].$$

A sequence $\xi = \{x_i\}_{i\in\mathbb{Z}}$ in X is called a *δ-ergodic pseudo orbit* of f if $\mathrm{Npo}(\xi, f, \delta)$ has density zero, which means that

$$\lim_{n\to\infty} \frac{\sharp \mathrm{Npo}_n(\xi, f, \delta)}{2n-1} = 0, \tag{1.1}$$

where $\sharp$ denotes the cardinal number. A δ-ergodic pseudo orbit ξ of f is said to be *ε-ergodic shadowed* by a point x in X if

$$\lim_{n\to\infty} \frac{\sharp \mathrm{Ns}_n(\xi, x, f, \varepsilon)}{2n-1} = 0.$$

A homeomorphism $f : X \to X$ has the *ergodic shadowing property* if for any $\varepsilon > 0$ there is $\delta > 0$ such that any δ-ergodic pseudo orbit of f can be ε-ergodic shadowed by a point in X.

We define a pointwise version of the ergodic shadowing property due to Rego and Arbieto in [18]. We say that a point $x \in X$ is called *ergodic shadowable* of f if for each $\varepsilon > 0$ there is $\delta_x > 0$ such that every δ_x-ergodic pseudo orbit $\xi = \{x_i\}_{i\in\mathbb{Z}}$ of f through $x_0 = x$ is ε-ergodic shadowed by a point in X(see [18]). We denote by $\mathrm{Erg}(f)$ the set of all ergodic shadowable points in X.

We denote by $H(X)$ the set of all homeomorphisms from a metric space X to itself. We say that a homeomorphism $f : X \to X$ of a metric space is *bi-uniform* if f and f^{-1} are uniformly continuous. We say that a sequence (f_n) of $H(X)$ has the *ergodic shadowing property* if f_n has the ergodic shadowing property for each $n \in \mathbb{N}$.

We say that a point $x \in X$ is *nonwandering* of f if for any neighborhood U of x there exists $n \in \mathbb{Z} \setminus \{0\}$ such that $f^n(U) \cap U \neq \emptyset$. The set of all nonwandering points of f is called the nonwandering set denoted by $\Omega(f)$. We say that a homeomorphism $f : X \to X$ is *nonwandering* if $\Omega(f) = X$. A δ-*chain* from x to y of length n is a finite sequence $x_0 = x, x_1, \dots, x_n = y$ such that $d(f(x_i), x_{i+1}) < \delta$ for $i = 0, \dots, n-1$. We say that a point $x \in X$ is *chain recurrent* of f if for any $\delta > 0$ there is a δ-chain from x to itself. We denote by the set of all chain recurrent points of f as $CR(f)$. We say that a homeomorphism $f : X \to X$ is *chain recurrent* if $CR(f) = X$. We note that if a homeomorphism $f : X \to X$ of a compact metric space X has the shadowing property, then $\Omega(f) = CR(f)$(see [2, Theorem 3.1.2]).

In this paper, we investigate some dynamical properties of the ergodic shadowing property for the class of bi-uniform homeomorphisms on a metric space. We also show that if a sequence of $H(X)$ which converges uniformly to an $f \in H(X)$ has the ergodic shadowing property, then the uniform limit has the same property.

2. Proofs of the results

In this section, we present some results related variant shadowing properties and recurrences for the class of homeomorphisms of a noncompact metric space. Then we give proofs of our main results of this paper.

LEMMA 2.1. [10, Lemma 3.1] *Let $f : X \to X$ be a homeomorphism of a metric space that f is uniformly continuous. Then f has the ergodic shadowing property if and only if so does f^k for every $k \in \mathbb{N}$.*

LEMMA 2.2. [10, Lemma 3.2] *Let $f : X \to X$ be a bi-uniform homeomorphism of a metric space. Then f has the ergodic shadowing property if and only if so does f^{-1}.*

LEMMA 2.3. *If $f : X \to X$ is a bi-uniform homeomorphism of a metric space, then* $\mathrm{Erg}(f) = \mathrm{Erg}(f^{-1})$.

Proof. We claim that $\mathrm{Erg}(f) \subset \mathrm{Erg}(f^{-1})$ and $\mathrm{Erg}(f^{-1}) \subset \mathrm{Erg}(f)$. We only prove that if x is an ergodic shadowable point of f, then x is an ergodic shadowable point of f^{-1} as the same proof works for $\mathrm{Erg}(f^{-1}) \subset \mathrm{Erg}(f)$.

Let $x \in \mathrm{Erg}(f)$. Given $\varepsilon > 0$, then there is $\delta_x > 0$ such that any δ_x-ergodic pseudo orbit of f through x is ε-ergodic shadowed by a point in X. Since f is uniformly continuous, there is $\delta_1 > 0$ such that $d(x, z) < \delta_1$ implies $d(f(x), f(z)) < \delta_x$.

Let $\eta = \{y_i\}_{i\in\mathbb{Z}}$ be any δ_1-ergodic pseudo orbit of f^{-1} through $y_0 = x$, which means

$$\lim_{n\to\infty} \frac{\sharp\mathrm{Npo}_n(\eta, f^{-1}, \delta_1)}{2n-1} = 0. \tag{2.1}$$

Putting $J_{f^{-1}} := \mathbb{Z}\setminus \mathrm{Npo}(\eta, f^{-1}, \delta_1) = \{i \in \mathbb{Z} : d(f^{-1}(y_i), y_{i+1}) < \delta_1\}$. By applying the uniform continuity of f for each $i \in J_{f^{-1}}$, we get that $d(f^{-1}(y_i), y_{i+1}) < \delta_1$ implies $d(y_i, f(y_{i+1})) < \delta_x$ for each $i \in J_{f^{-1}}$. Define a sequence $\xi = \{x_i\}_{i\in\mathbb{Z}}$ of X by $x_i = y_{-i}$ for all $i \in \mathbb{Z}$. Then we can see that $d(f(x_i), x_{i+1}) < \delta_x$ for each $i \in J_f := -J_{f^{-1}}$. Here $-J_{f^{-1}} = \{-i \ : \ i \in J_{f^{-1}}\}$. From $\sharp J_f = \sharp J_{f^{-1}}$ and (2.1), we obtain that

$$\mathrm{Npo}_n(\xi, f, \delta_x) = \mathrm{Npo}_n(\eta, f^{-1}, \delta_1)$$

and

$$\lim_{n\to\infty} \frac{\sharp \mathrm{Npo}_n(\xi, f, \delta_x)}{2n-1} = 0.$$

Thus $\xi = \{x_i\}_{i\in\mathbb{Z}}$ is a δ_x-ergodic pseudo orbit of f through $x_0 = x$ and so is ε-ergodic shadowed by a point $y \in X$ from the ergodic shadowableness of x for f. This implies that $d(f^i(y), x_i) \geq \varepsilon$ for each $i \in \mathrm{Ns}(\xi, y, f, \varepsilon)$ and so $d((f^{-1})^{-i}(y), y_{-i}) = d(f^i(y), x_i) \geq \varepsilon$ for each $-i \in \mathrm{Ns}(\eta, y, f^{-1}, \varepsilon)$. Thus we can see that $\sharp\mathrm{Ns}(\xi, y, f, \varepsilon) = \sharp\mathrm{Ns}(\eta, y, f^{-1}, \varepsilon)$. Since $\lim_{n\to\infty} \frac{\sharp \mathrm{Ns}_n(\xi,y,f,\varepsilon)}{2n-1} = 0$, we immediately get that

$$\lim_{n\to\infty} \frac{\sharp \mathrm{Ns}_n(\eta, y, f^{-1}, \varepsilon)}{2n-1} = 0.$$

Thus x is an ergodic shadowable point of f^{-1}, i.e., $x \in \mathrm{Erg}(f^{-1})$. This completes the proof. □

We obtain some dynamical properties of ergodic shadowable points for homeomorphisms on a noncompact metric space.

THEOREM 2.4. *Let $f : X \to X$ be a bi-uniform homeomorphism of a metric space. Then the following statements hold:*

1. $\mathrm{Erg}(f)$ *is an invariant set.*
2. $\mathrm{Erg}(f) = \mathrm{Erg}(f^k)$ *for every* $k \in \mathbb{Z}\setminus\{0\}$.

Proof. First, we prove item (1). It suffices to show that $\mathrm{Erg}(f) = f(\mathrm{Erg}(f))$. That is, if x is an ergodic shadowable point of f, so are $f(x)$ and $f^{-1}(x)$. We only prove that $f(x)$ is an ergodic shadowable point as the same proof works for f^{-1}.

Fix $\varepsilon > 0$. Let $\delta_x > 0$ correspond to ε by the ergodic shadowableness of x. Say, take $\delta_{f(x)} = \delta_x$ and let $\eta = \{y_i\}_{i\in\mathbb{Z}}$ be any δ_x-ergodic pseudo orbit of f through $y_0 = f(x)$. From the sequence $\{y_i\}_{i\in\mathbb{Z}}$, we define a sequence $\xi = \{x_i\}_{i\in\mathbb{Z}}$ of points in X by

$$x_i = \begin{cases} y_{i-1} & \text{if } i \in \mathbb{Z}\setminus\{0\}, \\ x & \text{if } i = 0. \end{cases}$$

If $d(f(x_0), x_1) = d(f(x), y_0) < \delta_x$ and $d(f(x_{-1}), x_0) = d(f(y_{-2}), x) < \delta_x$, there is nothing to do. If not, we see that

$$\lim_{n\to\infty} \frac{\sharp \mathrm{Npo}_n(\xi, f, \delta_x)}{2n-1} = \lim_{n\to\infty} \frac{\sharp \mathrm{Npo}_n(\eta, f, \delta_x) + 2}{2n-1} = 0,$$

since $\{y_i\}_{i\in\mathbb{Z}}$ is a δ_x-ergodic pseudo orbit of f through $y_0 = f(x)$. Thus $\{x_i\}_{i\in\mathbb{Z}}$ is a δ_x-ergodic pseudo orbit of f through $x_0 = x$ and also is ε-ergodic shadowed by a point $z \in X$ by the ergodic shadowableness of x. Thus there exists $\hat{z} \in X$ with $\hat{z} = f(z)$

such that $\{y_i\}_{i\in\mathbb{Z}}$ is ε-ergodic shadowed by a point $\hat{z}\in X$ since

$$\lim_{n\to\infty}\frac{\sharp\mathrm{Ns}_n(\eta,f(z),f,\varepsilon)}{2n-1}=\begin{cases}\lim_{n\to\infty}\frac{\sharp\mathrm{Ns}_n(\eta,z,f,\varepsilon)}{2n-1}=0 & \text{if } d(f^{-1}(\hat{z}),y_{-1})=d(z,y_{-1})<\varepsilon,\\ \le\lim_{n\to\infty}\frac{\sharp\mathrm{Ns}_n(\eta,z,f,\varepsilon)+1}{2n-1}=0 & \text{if otherwise.}\end{cases}$$

Thus $f(x)$ is an ergodic shadowable point of f, which means that $f(x)\in\mathrm{Erg}(f)$, and so $x\in f^{-1}(\mathrm{Erg}(f))$. This prove $\mathrm{Erg}(f)\subset f^{-1}(\mathrm{Erg}(f))$, i.e., $f(\mathrm{Erg}(f))\subset\mathrm{Erg}(f)$. Similarly, we also can prove the converse implication.

Next, we can prove item (2) as in Lemmas 2.1 and 2.2. This completes the proof. □

The pointwise ergodic property is preserved by a topological conjugacy as the following result.

Proposition 2.5. *If $h:X\to Y$ is a bi-uniform homeomorphism of metric spaces, then $h(\mathrm{Erg}(f))=\mathrm{Erg}(h\circ f\circ h^{-1})$.*

Proof. Let X,Y be metric spaces equipped with metrics d_X and d_Y, respectively and $g=h\circ f\circ h^{-1}$. Then, it is enough to show that $\mathrm{Erg}(g)\subset h(\mathrm{Erg}(f))$. Let $y\in\mathrm{Erg}(g)$ and $\varepsilon>0$ be given. We claim that $f^{-1}(y)\in\mathrm{Erg}(f)$ such that $h(x)=y$ for some $x\in X$. Since h^{-1} is uniformly continuous, we find $\varepsilon_1>0$ such that the inequality $d_Y(x,y)<\varepsilon_1$ with $x,y\in Y$ implies $d_X(h^{-1}(x),h^{-1}(y))<\varepsilon$. Since y is an ergodic shadowable point of g, we take $\delta_1=\delta_1(y)>0$ such that any δ_1-ergodic pseudo orbit $\eta=\{y_k\}_{k\in\mathbb{Z}}$ of g through $y_0=y$ in Y can be ε_1-ergodic shadowed by a point in Y. From the uniform continuity of h, we find $\delta>0$ such that $d_X(x,y)<\delta$ implies $d_Y(h(x),h(y))<\delta_1$.

Let $\xi=\{x_k\}_{k\in\mathbb{Z}}$ be any δ-ergodic pseudo orbit of f through $x_0=f^{-1}(y)$ in X. Set $J_f=\{i\in\mathbb{Z}:d_X(f(x_i),x_{i+1})<\delta\}$ and $y_k=h(x_k)$ for each $k\in\mathbb{Z}$. Then $J_f=\mathbb{Z}\setminus\mathrm{Npo}(\xi,f,\delta)$. Since $d_X(f(x_k),x_{k+1})<\delta$ for each $k\in J_f$ and $g\circ h=h\circ f$, we see that

$$d_Y(g(y_k),y_{k+1})=d_Y(g(h(x_k)),h(x_{k+1}))=d_Y(h(f(x_k)),h(x_{k+1}))<\delta_1,\ \forall k\in J_f$$

and so $\mathrm{Npo}(\eta,g,\delta_1)\subset\mathrm{Npo}(\xi,f,\delta)$. It follows from the density zero of $\mathrm{Npo}(\xi,f,\delta)$ that

$$\lim_{n\to\infty}\frac{\sharp\mathrm{Npo}_n(\eta,g,\delta_1)}{2n-1}=0,$$

which means that $\eta=\{y_k\}_{k\in\mathbb{Z}}$ is a δ_1-ergodic pseudo orbit of g through $y_0=y$ in Y. Hence there exists $z\in Y$ such that

$$\lim_{n\to\infty}\frac{\sharp\mathrm{Ns}_n(\eta,z,g,\varepsilon_1)}{2n-1}=0. \tag{2.2}$$

Setting $w=h^{-1}(z)$ and $J_g=\{i\in\mathbb{Z}:d_Y(g^i(y),y_i)<\varepsilon_1\}$. Then $J_g=\mathbb{Z}\setminus\mathrm{Ns}(\eta,g,\varepsilon_1)$. From the uniform continuity of h^{-1}, we have

$$d_X(f^i(w),x_i)=d_X(f^i(h^{-1}(z)),h^{-1}(y_i))=d_X(h^{-1}(g^i(z)),h^{-1}(y_i))<\varepsilon,\ \forall i\in J_g.$$

Thus we have that $J_g\subset\{i\in\mathbb{Z}:d_X(f^i(w),x_i)<\varepsilon\}$ and so $\mathrm{Ns}_n(\xi,w,f,\varepsilon)\subset\mathrm{Ns}_n(\eta,z,g,\varepsilon_1)$. It follows from (2.2) that

$$\lim_{n\to\infty}\frac{\sharp\mathrm{Ns}_n(\xi,w,f,\varepsilon)}{2n-1}\le\lim_{n\to\infty}\frac{\sharp\mathrm{Ns}_n(\eta,z,g,\varepsilon_1)}{2n-1}=0,$$

which means that any δ-ergodic pseudo orbit $\xi = \{x_k\}_{k\in\mathbb{Z}}$ of f through $x_0 = f^{-1}(y)$ in X is ε-ergodic shadowed by a point $w \in X$. Thus $x = f^{-1}(y)$ is an ergodic shadowable point of f, i.e., $f^{-1}(y) \in \mathrm{Erg}(f)$. Hence $\mathrm{Erg}(g) \subset h(\mathrm{Erg}(f))$. Similarly, we also can prove the converse implication. This completes the proof. □

For the proof of Corollary 2.10, we need some results. Rego obtained some results about chain transitivity, chain mixing and chain recurrence for the uniform limit map of a sequence of continuous maps on a compact metric space(see [16, Theorem 4.1.4]). We also can obtain the similar result about chain recurrence of the uniform limit of a sequence in $H(X)$ when X is a metric space.

Lemma 2.6. *Let X be a metric space and let (f_n) be a sequence of homeomorphisms which converges uniformly to an $f \in H(X)$. If f_n is chain recurrent for every $n \in \mathbb{N}$, then f is chain recurrent.*

Proof. Let $x, y \in X$ and fix $\varepsilon > 0$. Since f_n is chain recurrent for every $n \in \mathbb{N}$, there exist finite $\frac{\varepsilon}{2}$-pseudo orbits $\{x_i^n\}_{i=0}^{k_n}$ of f_n starting on x and ending on x. Thus we can see that $\{x_i^{n_0}\}_{i=0}^{k_{n_0}}$ is an ε-pseudo orbit of f starting on x and ending on x if we take n_0 sufficiently large since

$$d(f(x_i^{n_0}), x_{i+1}^{n_0}) \le d(f_{n_0}(x_i^{n_0}), f(x_i^{n_0})) + d(f(x_i^{n_0}), x_{i+1}^{n_0}) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon,\ i = 0, \cdots, k_{n_0}.$$

Hence f is chain recurrent. This completes the proof. □

We also obtain the following result as an extension of [9, Theorem 1.1].

Proposition 2.7. *Let (f_n) be a sequence of $H(X)$ which converges uniformly to an $f \in H(X)$. If (f_n) has the ergodic shadowing property, then the uniform limit f has the ergodic shadowing property.*

Proof. Fix $\varepsilon > 0$. Since f_n has the ergodic shadowing property for each $n \in \mathbb{N}$, choose $\delta_n > 0$ such that for each $n \in \mathbb{N}$ we have that every δ_n-ergodic pseudo orbit of f_n is $\frac{\varepsilon}{2}$-ergodic shadowed by some point y_n. We claim that every $\frac{\delta_{n_0}}{2}$-ergodic pseudo orbit of f is ε-ergodic shadowed by a point in X. The rest of the proof is similar to that of Theorem 1.1 in [9] and we shall omit the proof. □

We say that a metric space X is *relatively compact* in the strong sense(or *s-relatively compact*) if every bounded subset of X is relatively compact. The following result says that the ergodic shadowing property is stronger than the shadowing property for dynamical systems on an s-relatively compact metric space, which is an extension of [5, Lemma 3.2] in the case of noncompact metric spaces.

Lemma 2.8. [10, Theorem 2.5] *Let $f : X \to X$ be a homeomorphism of an s-relatively compact metric space. If f has the ergodic shadowing property, then f has the shadowing property.*

From [20, Lemma 3.4.3], we obtain the following result for the class of homeomorphisms on a metric space X.

Lemma 2.9. *If a homeomorphism $f : X \to X$ of a metric space has the shadowing property, then $\Omega(f) = CR(f)$.*

Proof. The proof is almost analogous to the proof of Lemma 3.4.3 in [20] and we shall omit it. □

As a consequence of Proposition 2.7, we obtain the following result that slightly improves Theorem D in [16] for the class of homeomorphisms on a metric space.

Corollary 2.10. *Let X be an s-relatively compact metric space and (f_n) be a sequence of nonwandering homeomorphisms on X which converges uniformly to an $f \in H(X)$. If (f_n) has the ergodic shadowing property, then the uniform limit f is nonwandering.*

Proof. Suppose that f_n is nonwandering for each $n \in \mathbb{N}$. Then f_n is chain-recurrent for each $n \in \mathbb{N}$. From Lemma 2.6, we can see that f is chain recurrent. It also follows from Proposition 2.7 that the uniform limit f has the ergodic shadowing property. Thus f has the shadowing property by Lemma 2.8, and so $\Omega(f) = CR(f)$ by Lemma 2.9. Hence f is nonwandering. This completes the proof. □

We give an example to illustrate the notion of the ergodic shadowing property of dynamical systems on a noncompact metric space.

Example 2.11. [10, Example 4.1] Let $X = \bigcup_{n\in\mathbb{Z}} X_n$ be a subspace of the euclidean metrizable space $\mathbb{R}^2$ where $X_n = \{n\} \times [0, 2^{-|n|}]$ for every $n \in \mathbb{Z}$. Define a homeomorphism $f : X \to X$by

$$f(n, y) = \begin{cases} (n + 1, 2y), & \text{if } n \in \mathbb{Z}^-, \\ (n + 1, \frac{1}{2}y), & \text{if } n \in \mathbb{Z}^+, \end{cases}$$

where $\mathbb{Z}^+ = \{0\} \cup \mathbb{N}$ and $\mathbb{Z}^- = \mathbb{Z} \setminus \mathbb{Z}^+$. Then we can see that f has the ergodic shadowing property. But, the homeomorphism $g : Y \to Y$ on the noncompact space $Y = \bigcup_{n\in\mathbb{N}}\{n\} \times [0, 1]$ given by translation on the first coordinate and identity on the second does not have the ergodic shadowing property.

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