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A REMARK ON REGULAR TRANSFORMS OF POSITIVE CLOSED $(1,1)$-CURRENTS

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Abstract. In this note, we prove that every regular transform of a positive closed $(1,1)$-current on a compact Kähler manifold admits Lipschitz quasi-potentials. As an application, we obtain some regularity properties of the Dinh-Sibony approximation of positive closed currents in the case of bidegree $(1,1)$.

1. Introduction

One difference between the theory of currents on a general compact Kähler manifold and that on a projective space appears in approximating positive closed currents. As a projective space has many symmetries in the holomorphic category, using a convolution formula, we can regularize positive closed currents by smooth positive closed currents of the same bidegree (cf. [3]). However, this is no more true if we instead consider general compact Kähler manifolds. They do not have many symmetries. They might even have a finite number of automorphisms. In result, the same idea of regularization cannot be applied.

In [1], Demailly proved that on a general compact Kähler manifold, positive closed $(1,1)$-currents can be approximated by smooth closed $(1,1)$-forms with various estimates for approximating forms. In particular, one might say that the negative part is relatively small in a sense that approximating smooth currents are uniformly bounded below by a fixed smooth negative closed form. For general bidegrees in the case of compact Kähler manifolds, in [2], Dinh-Sibony proved that a positive closed current can be approximated by differences of two smooth positive closed currents of the same bidegree such that the masses of positive and negative parts are uniformly bounded in terms of the mass of the given positive closed current.

The aim of this article is to investigate some regularity properties of the Dinh-Sibony approximation in the bidegree $(1,1)$ case via studying regular transforms of positive closed $(1,1)$-currents.

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Let (X, ω) be a compact Kähler manifold of complex dimension $k(\geq 2)$. Let $\mathfrak{X} := X_1 \times X_2$ be the product of two copies of X and $\pi : \hat{\mathfrak{X}} \to \mathfrak{X}$ the blow-up of $\mathfrak{X}$ along the diagonal submanifold $\Delta \subset \mathfrak{X}$.

Theorem 1.1. *Let Φ be a smooth positive closed (k,k)-form on $\widehat{\mathfrak{X}}$ and $\mathscr{L}^\Phi$ its associated regular transform (for the definition, see Section 2). Let S be a positive closed $(1,1)$-current. Then, $\mathscr{L}^\Phi(S)$ admits Lipschitz quasi-potentials. More precisely, if S satisfies $S - \alpha_S = dd^c u$, we have $\mathscr{L}^\Phi(S) - \mathscr{L}^\Phi(\alpha_S) = dd^c \mathscr{L}^\Phi(u)$ where $\mathscr{L}^\Phi(\alpha_S)$ is a real closed smooth $(1,1)$-form and $\mathscr{L}^\Phi(u)$ is a Lipschitz quasi-plurisubharmonic function.*

If we apply Theorem 1.1 to the approximation of the identity transform $\mathscr{L}_\theta = \mathscr{L}_\theta^+ - \mathscr{L}^-$ (for the definition, see Section 2), we obtain the following corollary. Notice that in the case of bidegree $(1,1)$, we get C^1-coefficients after applying the transform three times. In general, using Young's inequality or Hölder's inequality, it is expected to get C^1 coefficients after the $(k+2)$-times applications of the transform.

Corollary 1.2. *Let S be a positive closed $(1,1)$-current on X. Let $\theta \in \mathbb{C} \setminus \{0\}$ be such that $|\theta| \ll 1$. Then, we have the following:*

1. *$\mathscr{L}_\theta^+(S)$ and $\mathscr{L}^-(S)$ are positive closed $(1,1)$-current with Lipschitz quasi-potentials.*
2. *The current $\mathscr{L}_\theta(\mathscr{L}_\theta(S))$ belongs to $\{S\}$ and is given by a difference of two positive closed $(1,1)$-forms with bounded coefficients: $\mathscr{L}_\theta^+(\mathscr{L}_\theta^+(S)) + \mathscr{L}^-(\mathscr{L}^-(S))$ and $\mathscr{L}_\theta^+(\mathscr{L}_-(S)) + \mathscr{L}^-(\mathscr{L}_\theta^+(S))$.*
3. *The current $\mathscr{L}_\theta(\mathscr{L}_\theta(\mathscr{L}_\theta))$ belongs to $\{S\}$ and can be written as a difference of two positive closed $(1,1)$-forms with C^1 coefficients. In particular, the current S can be written as $S = S^+ - S^-$ where $S\pm$ are positive closed $(1,1)$-current, the current S^- admits a Lipschitz quasi-potential and each of S^+ and S^- can be approximated by positive closed $(1,1)$-currents with C^1 coefficients.*

2. (Semi-)regular transforms

For the basics of the pluripotential theory and the theory of currents, we refer the reader to [1]. We start by recalling the notion of the (semi-)regular transform of currents in [4]. Recall the spaces X, $\mathfrak{X}$ and $\widehat{\mathfrak{X}}$ in Introduction.

$$\begin{tikzcd}
& \widehat{\mathfrak{X}} = \widehat{X \times X} \arrow[d, "\pi"] \arrow[ddl, bend right, "\Pi_1"'] \arrow[ddr, bend left, "\Pi_2"] & \\
& \mathfrak{X} = X \times X \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\
X & & X
\end{tikzcd}$$

Let $\pi_i : \mathfrak{X} \to X$ denote the canonical projection on its factor for $i = 1, 2$. Then, $\omega_{\mathfrak{X}} := \pi_1^*\omega + \pi_2^*\omega$ is a natural Kähler form on $\mathfrak{X}$. Let $\pi : \widehat{\mathfrak{X}} \to \mathfrak{X}$ be the blow-up of $\mathfrak{X}$ along the diagonal submanifold Δ in $\mathfrak{X} = X \times X$ and let $\widehat{\Delta} := \pi^{-1}(\Delta)$ denote the exceptional hypersurface. We define $\Pi_i := \pi_i \circ \pi$ for $i = 1, 2$. Then, Π_i and its restrcition to $\widehat{\Delta}$ are both submersions for $i = 1, 2$. In particular, for each $P \in X$, its inverse images $\Pi_1^{-1}(P)$ and $\Pi_2^{-1}(P)$ under Π_1 and Π_2 are biholomorphic to a blow-up of X at P. Hence, Π_1 and Π_2 are proper holomorphic submersions. Thus, their push-forwards $(\Pi_1)_*$, $(\Pi_2)_*$ on the space of positive closed currents $\widehat{\mathfrak{X}}$ and their pull-backs Π_1^* and Π_2^* on the space of positive closed currents on X are well defined and commute with operators d and dd^c.

Definition 2.1. Let $0 \le q \le k$ be an integer. Let $\mathcal{Q}$ be a form of bidegree (q,q) on $\widehat{\mathfrak{X}}$ which is smooth outside $\widehat{\Delta}$ and such that

$$|\mathcal{Q}| \lesssim -\log \operatorname{dist}(\cdot, \widehat{\Delta}) \quad \text{and} \quad |\nabla \mathcal{Q}| \lesssim \operatorname{dist}(\cdot, \widehat{\Delta})^{-1}$$

near $\widehat{\Delta}$. Let p be an integer such that $k-q \le p \le k$. A linear mapping $\mathscr{L}^{\mathcal{Q}}$ on the space of currents of bidegree (p,p) on X to the space of currents $(p+q-k, p+q-k)$ on X defined by

$$\mathscr{L}^{\mathcal{Q}}(S) := (\Pi_2)_*(\Pi_1^*(S) \wedge \mathcal{Q})$$

is called a semi-regular transform of bidegree $(q-k, q-k)$ associated with the form $\mathcal{Q}$.

If $\mathcal{Q}$ is smooth, then the transform $\mathscr{L}^{\mathcal{Q}}$ is said to be regular. If $\mathcal{Q}$ is positive, then the transform $\mathscr{L}^{\mathcal{Q}}$ is said to be positive. If $\mathcal{Q}$ is closed, then the transform $\mathscr{L}^{\mathcal{Q}}$ is said to be closed.

Here, $|\mathcal{Q}|$ and $|\nabla \mathcal{Q}|$ mean the sum of the absolute value of the coefficients of $\mathcal{Q}$ and the sum of the estimates of the gradients of their coefficients with respect to a fixed finite atlas of $\widehat{\mathfrak{X}}$, respectively.

Remark 2.2. A positive semi-regular transform maps positive currents to positive currents. A closed semi-regular transform $\mathscr{L}^{\mathcal{Q}}$ maps closed currents to closed currents and satisfies $\mathscr{L}^{\mathcal{Q}}(dd^c S) = dd^c \mathscr{L}^{\mathcal{Q}}(S)$ for every current S.

Young's inequality (or Hölder's inequality) gives the following proposition.

Proposition 2.3 (Lemma 2.1 in [2] or Proposition 2.3.2 in [4]). *Any semi-regular transform can be extended to a linear continuous operator from the space of currents of order* 0 *to the space of $L^{1+1/k}$-forms. It defines a linear continuous operator from the space of L^α-forms, $\alpha \ge 1$, to the space of $L^{\alpha'}$-forms where α' is given by $(\alpha')^{-1} + 1 = \alpha^{-1} + (1+1/k)^{-1}$ if $\alpha < k+1$ and $\alpha' = \infty$ if $\alpha \ge k+1$. It also defines a linear continuous operator from the space of L^∞-forms to the space of C^1-forms.*

In particular, we are interested in approximating the identity transform. The regularization of positive closed currents in this section was introduced in [2] and [4].

By a theorem of Blanchard, $\widehat{\mathfrak{X}}$ is a compact Kähler manifold. We choose a Kähler form $\omega_{\widehat{\mathfrak{X}}}$ on $\widehat{X \times X}$. We assume that $\omega_{\widehat{\mathfrak{X}}}$ is normalized as follows. The current $\pi_*([\widehat{\Delta}] \wedge \omega_{\widehat{\mathfrak{X}}}^{k-1})$ has support in Δ and is a positive closed current of bidimension (k,k). Hence, by the support theorem, $\pi_*([\widehat{\Delta}] \wedge \omega_{\widehat{\mathfrak{X}}}^{k-1})$ is a positive constant multiple of $[\Delta]$. By multiplying a proper positive constant to $\omega_{\widehat{\mathfrak{X}}}$, we may assume that $\pi_*([\widehat{\Delta}] \wedge \omega_{\widehat{\mathfrak{X}}}^{k-1}) = [\Delta]$.

Let $\alpha_{\widehat{\mathfrak{X}}}$ be a real smooth closed $(1,1)$ form in $\{[\widehat{\Delta}]\}$. Let u be the q-psh function on $\widehat{\mathfrak{X}}$ such that $[\Delta] - \alpha_{\widehat{\mathfrak{X}}} = dd^c u$. Let χ be a smooth convex increasing function on

$\mathbb{R} \cup \{-\infty\}$ such that $\chi(t) = t$ for $t \geq 0$, $\chi(t) = -1$ for $t \leq -2$ and $0 \leq \chi' \leq 1$. For $\theta \in \mathbb{C}$ such that $|\theta| < 1$, we define

$$\chi_\theta(t) := \chi(t - \log|\theta|) + \log|\theta| \quad \text{and} \quad u_\theta := \chi_\theta(u).$$

Then, $u_\theta = u_{|\theta|}$ and u_θ decreasingly converges to u as $|\theta| \to 0$. Let $m_{\widehat{\Delta}} > 0$ be sufficiently large so that $m_{\widehat{\Delta}}\omega_{\widehat{\mathfrak{X}}} - \alpha_{\widehat{\Delta}}$ is positive. Then, we have

$$dd^c u_\theta = (\chi''_\theta \circ u) du \wedge d^c u + (\chi'_\theta \circ u) dd^c u = -(\chi'_\theta \circ u)\alpha_{\widehat{\Delta}} \geq -m_{\widehat{\Delta}}\omega_{\widehat{\mathfrak{X}}}.$$

So, for $\theta \in \mathbb{C}$, the smooth closed $(1,1)$-current $\alpha_{\widehat{\Delta}} + dd^c u_\theta$ can be written as a difference of two smooth positive closed $(1,1)$-currents as $\alpha_{\widehat{\Delta}} + dd^c u_\theta = (m_{\widehat{\Delta}}\omega_{\widehat{\mathfrak{X}}} + dd^c u_\theta) - (m_{\widehat{\Delta}}\omega_{\widehat{\mathfrak{X}}} - \alpha_{\widehat{\Delta}})$. For notational convenience, we write

$$\mathscr{L}_\theta^+ := \mathscr{L}^{(m_{\widehat{\Delta}}\omega_{\widehat{\mathfrak{X}}} + dd^c u_\theta)\wedge\omega_{\widehat{\mathfrak{X}}}^{k-1}}, \quad \mathscr{L}^- := \mathscr{L}^{(m_{\widehat{\Delta}}\omega_{\widehat{\mathfrak{X}}} - \alpha_{\widehat{\Delta}})\wedge\omega_{\widehat{\mathfrak{X}}}^{k-1}} \quad \text{and} \quad \mathscr{L}_\theta := \mathscr{L}^{(\alpha_{\widehat{\Delta}} + dd^c u_\theta)\wedge\omega_{\widehat{\mathfrak{X}}}^{k-1}}.$$

Then, we obviously have $\mathscr{L}_\theta = \mathscr{L}_\theta^+ - \mathscr{L}^-$ and $\mathscr{L}_0 = \mathrm{id}$ on smooth forms (see [2, p.486]). The transforms $\mathscr{L}_\theta^+$ and $\mathscr{L}^-$ are positive closed and of bidegree $(0,0)$.

3. Proof of Theorem 1.1

We start by proving Lemma 3.2, which is related to singularities of quasi-plurisubharmonic functions.

The following so-called exponential estimate is classical. For instance, see [6] and [5].

Lemma 3.1. *Let u is a quasi-plurisubharmonic function on X, then there exist constants $\alpha > 0$ and $C > 0$ such that*

$$\int_X e^{-\alpha u}\omega^k < C.$$

Together with the Hölder inequality, we prove the following lemma:

Lemma 3.2. *Let u be a quasi-plurisubharmonic function such that $u \leq -1$. Then, for every $n \in N$, there exists a constant $M > 0$ such that for every $P \in X$, we have*

$$\int_X |u(z)|^n \mathrm{dist}(z,P)^{2-2k}\omega^k < M \quad \text{and} \quad \int_X |u(z)|^n \mathrm{dist}(z,P)^{1-2k}\omega^k < M.$$

Proof. It suffices to consider the second argument since X is compact. Note further that once we prove the above estimate locally, we can use the standard compactness argument to get a global estimate. If a local chart does not contain P, then the estimate is trivial due to the local integrability of plurisubharmonic functions. So, we replace X by a closed unit ball $\overline{B}$ in $\mathbb{C}^k$, assume $P \in B$ and prove that for each $n \in \mathbb{N}$, there exists $M > 0$ such that

$$\int_{\overline{B}} |u(z)|^n |z - P|^{1-2k} dV(z) < M$$

where u is a plurisubharmonic function in a neighborhood of $\overline{B}$ and V denotes the standard Lebesgue measure on $\mathbb{C}^k$.

From direct computations using the polar coordinates, we see that there exists $C_1 > 0$ such that

$$\int_{\overline{B}} |z-P|^{\frac{1-4k}{2}} dV(z) < C_1.$$

Let $p = \frac{4k-1}{4k-2}$ and $q = 4k-1$. Then, we have $p^{-1} + q^{-1} = 1$. According to Lemma 3.1, we can find $N \in \mathbb{N}$ and $C_2 > 0$ such that

$$\int_{\overline{B}} e^{-u/N} dV(z) < C_2.$$

By the Hölder inequality, we obtain

$$\begin{aligned}
\int_{\overline{B}} e^{-u/(qN)} |z-P|^{1-2k} dV(z) &\leq \left(\int_{\overline{B}} \left(|z-P|^{1-2k}\right)^p dV(z) \right)^{1/p} \left(\int_{\overline{B}} \left(e^{-u/(qN)}\right)^q dV(z) \right)^{1/q} \\
&= \left(\int_{\overline{B}} \left(|z-P|^{\frac{1-4k}{2}}\right) dV(z) \right)^{1/p} \left(\int_{\overline{B}} e^{-u/N} dV(z) \right)^{1/q} \\
&\leq C_1^{1/p} C_2^{1/q}.
\end{aligned}$$

Since $\frac{x^{nqN}}{(nqN)!} \leq e^x$ for $x > 0$, we have

$$\int_{\overline{B}} \frac{(-u)^n}{(nqN)!} |z-P|^{1-2k} dV(z) \leq \int_{\overline{B}} e^{-u/(qN)} |z-P|^{1-2k} dV(z) \leq C_1^{1/p} C_2^{1/q},$$

which implies our claim as desired. □

Proof of Theorem 1.1. By applying $\mathscr{L}^\Phi(\cdot)$ to $S - \alpha_S = dd^c u$, we obtain

$$\mathscr{L}^\Phi(S) - \mathscr{L}^\Phi(\alpha_S) = \mathscr{L}^\Phi(dd^c u) = (\Pi_2)_*(\Pi_1^*(dd^c u) \wedge \Phi) = dd^c (\Pi_2)_*(\Pi_1^*(u)\Phi).$$

Since $\mathscr{L}^\Phi(S)$ is positive and $\mathscr{L}^\Phi(\alpha_S)$ is of C^1-coefficient, there exists a constant $M_S > 0$ such that

$$dd^c(\Pi_2)_*(\Pi_1^*(u)\Phi) = \mathscr{L}^\Phi(S) - \mathscr{L}^\Phi(\alpha_S) = \mathscr{L}^\Phi(dd^c u) \geq -\mathscr{L}^\Phi(\alpha_S) \geq -M_S\omega.$$

So, quasi-plurisubharmonicity of $\mathscr{L}^\Phi(u) = (\Pi_2)_*(\Pi_1^*(u) \wedge \Phi)$ is proved.

Now, we prove that $\mathscr{L}^\Phi(u)$ is Lipschitz. Since Π_1 is holomoprhic map and u is quasi-plurisubharmonic, $\Pi_1^*(u)$ is also quasi-plurisubharmonic and not identically $-\infty$. This implies that the current $\Pi_1^*(u) \wedge \Phi$ has no mass on an analytic subset. In particular, it has no mass on $\widehat{\Delta}$. So, we can write

$$\mathscr{L}^\Phi(u) = (\Pi_2)_*(\Pi_1^*(u)\Phi) = (\pi_2)_* \left(\pi_1^*(u) \left(\pi|_{\widehat{\mathfrak{X}} \setminus \widehat{\Delta}} \right)_* \Phi \right).$$

Here, we understand $\left(\pi|_{\widehat{\mathfrak{X}} \setminus \widehat{\Delta}} \right)_* \Phi$ as the trivial extension across Δ of the current $\left(\pi|_{\widehat{\mathfrak{X}} \setminus \widehat{\Delta}} \right)_* \Phi$ on $\mathfrak{X} \setminus \Delta$. In particular, we have

$$\mathscr{L}^\Phi(u)(y) = (\Pi_2)_*(\Pi_1^*(u)\Phi)(y) = \int_{x \in (X \setminus \{y\}) \times \{y\}} \pi_1^*(u)(x) [(\pi|_{\widehat{\mathfrak{X}} \setminus \widehat{\Delta}})_* \Phi](x,y).$$

From the property of the regular transform, for each $x \in X \setminus \{y\}$, we have

$$\left|[(\pi|_{\widehat{\mathfrak{X}}\setminus\widehat{\Delta}})_*\Phi](x,y)\right| \lesssim \operatorname{dist}(x,y)^{2-2k} \quad \text{and} \tag{1}$$

$$\left|\nabla[(\pi|_{\widehat{\mathfrak{X}}\setminus\widehat{\Delta}})_*\Phi](x,y)\right| \lesssim \operatorname{dist}(x,y)^{1-2k}. \tag{2}$$

Together with (1), Lemma 3.2 implies that $\mathscr{L}^\Phi(u)$ is everwhere finite. Let $v \in \mathbb{C}^k$ be a direction vector such that $|v| = 1$. In a local coordinate chart near $y = P$, we have

$$\begin{aligned}
&\left|\mathscr{L}^\Phi(u)(P+hv) - \mathscr{L}^\Phi(u)(P)\right| \\
&= \left|\int_{x\in X\setminus\{P,P+hv\}} u(x)\left([(\pi|_{\widehat{\mathfrak{X}}\setminus\widehat{\Delta}})_*\Phi](x,P+hv) - [(\pi|_{\widehat{\mathfrak{X}}\setminus\widehat{\Delta}})_*\Phi](x,P)\right)\right|
\end{aligned}$$

Here, we identified $X \times \{P\}$ and $X \times \{P+hv\}$ with X and used the fact that the current $\pi_1^*(u)(x)[(\pi|_{\widehat{\mathfrak{X}}\setminus\widehat{\Delta}})_*\Phi](x,P)$ has no mass on the fiber $\pi_2^{-1}(y)$ for all $y \in X$.

As a regular transform, the form $[(\pi|_{\widehat{\mathfrak{X}}\setminus\widehat{\Delta}})_*\Phi](x,P+hv)$ is smooth in h outside $h = 0$ and (2) implies that its C^1-norm is bounded by $\operatorname{dist}(x,P+hv)^{1-2k}$ up to a constant multiple independent of h. Hence, by the mean value theorem, we have

$$\begin{aligned}
&\left|\mathscr{L}^\Phi(u)(P+hv) - \mathscr{L}^\Phi(u)(P)\right| \\
&\le |h|\left|\int_{x\in X\setminus\{P,P+hv\}} u(x)\left(v\cdot\nabla[(\pi|_{\widehat{\mathfrak{X}}\setminus\widehat{\Delta}})_*\Phi](x,P+h_cv)\right)\right| \\
&\lesssim |h|\left|\int_{x\in X\setminus\{P+h_cv\}} |u(x)|\operatorname{dist}(x,P+h_cv)^{1-2k}\right|
\end{aligned}$$

where $\cdot$ means the standard inner-product in $\mathbb{C}^k$ and h_c is a constant depending on h. Here, the inequality $\lesssim$ means that $\le$ up to a multiplicative constant independent of P, v and h_c. Then, due to Lemma 3.2, the last integral is uniformly bounded by a constant independent of P, v and h_c, which implies $\mathscr{L}^\Phi(u)$ is Lipschitz as desired. □

Before the proof of Corollary 1.2, we prove a proposition.

PROPOSITION 3.3. *Let S is a positive closed $(1,1)$-current with Lipschitz quasi-potential. Then, $\mathscr{L}^\Phi(S)$ is a positive closed $(1,1)$-form with bounded coefficients.*

Proof. Assume that u_S is a Lipschitz function on X where u_S is a function such that $S - \alpha_S = dd^c u_S$. Then, we have

$$\mathscr{L}^\Phi(S) = \mathscr{L}^\Phi(\alpha_S) + \mathscr{L}^\Phi(dd^c u_S).$$

Then, as $\mathscr{L}^\Phi(\alpha_S)$ has at least C^1-coefficients, we only consider the second term $\mathscr{L}^\Phi(dd^c u_S)$. Since $\pi_*\Phi$ is closed, the transform can be written as

$$\begin{aligned}
\mathscr{L}^\Phi(dd^c u_S) &= (\pi_2)_*\left[\pi_1^*(dd^c u_S) \wedge \left(\pi|_{\widehat{\mathfrak{X}}\setminus\widehat{\Delta}}\right)_*\Phi\right] = d(\pi_2)_*[\pi_1^*(d^c u_S) \wedge \pi_*\Phi] \\
&= \int_X d^c u_S(x) \wedge d_y(\pi_*\Phi(x,y)).
\end{aligned}$$

Here, $\pi_*\Phi$ is the trivial extension of $\left(\pi|_{\widehat{\mathfrak{X}}\setminus\widehat{\Delta}}\right)_*\Phi$ across Δ and we denote by d_y the exterior derivative with respect to the variable y. The last equality comes from the observation that the operator d in the second last expression is the exterior differentiation with respect to y and not x. Notice that $d^c u_S$ is a 1-form with bounded measurable coefficients as each coefficient is almost everywhere differentiable. We focus on coefficients. From the gradient condition of $\pi_*\Phi$ in (2), the singularities of each component of $d_y\left(\pi_*\Phi(x,y)\right)$ are bounded by $\operatorname{dist}(x,y)^{1-2k}$ up to a multiplicative constant independent of x, y. Hence, $\mathscr{L}^{\Phi}(dd^c u_S)(x)$ is bounded at each $x \in X$ by a uniform constant. $\square$

Proof of Corollary 1.2. We consider the first assertion. For the proof, it suffices to observe that $(m_{\widehat{\Delta}}\omega_{\widehat{\mathfrak{X}}} + dd^c u_\theta)$ and $(m_{\widehat{\Delta}}\omega_{\widehat{\mathfrak{X}}} - \alpha_{\widehat{\Delta}})$ are smooth positive closed (k,k)-form on $\widehat{\mathfrak{X}}$. So, our theorem implies the corollary. The second and last assertions are from Proposition 3.3 and Proposition 2.3. For the last expression of S, we set

$$S^+ := \sum_{i=0}^{3}\left(\mathscr{L}^{(m_{\widehat{\Delta}}\omega_{\widehat{\mathfrak{X}}}-\alpha_{\widehat{\Delta}})\wedge\omega_{\widehat{\mathfrak{X}}}^{k-1}}\right)^i(S) \quad \text{and}$$

$$S^- := \sum_{i=1}^{3}\left(\mathscr{L}^{(m_{\widehat{\Delta}}\omega_{\widehat{\mathfrak{X}}}-\alpha_{\widehat{\Delta}})\wedge\omega_{\widehat{\mathfrak{X}}}^{k-1}}\right)^i(S)$$

where the exponent i in the expressions means the i-times iterated composition of the operator $\mathscr{L}^{(m_{\widehat{\Delta}}\omega_{\widehat{\mathfrak{X}}}-\alpha_{\widehat{\Delta}})\wedge\omega_{\widehat{\mathfrak{X}}}^{k-1}}$ and by convention, $i = 0$ corresponds to the identity operator. $\square$

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