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ON SOME TURAN-TYPE INEQUALITIES FOR DERIVATIVE OF A POLYNOMIAL

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Abstract. If $P(z) = a_n \prod_{\nu=1}^{n}(z - z_\nu)$ is a complex polynomial of degree n having all its zeros in $|z| \leq K$, $K \geq 1$ then Aziz (Proc Am Math Soc 89:259-266, 1983) proved that

$$\max_{|z|=1}|P'(z)| \geq \frac{2}{1+K^n}\sum_{\nu=1}^{n}\frac{K}{K+|z_\nu|}\max_{|z|=1}|P(z)|. \tag{0.1}$$

This paper presents a comprehensive analysis that encompasses the refinement of inequality (0.1) while also extending the well-established Turan's inequality. Furthermore, we broaden the scope of our findings by applying them to the polar derivative of a polynomial. Our investigation reveals that the bounds derived from our results exhibit a significantly enhanced level of precision compared to inequality (0.1). To illustrate this, we provide a numerical example to underscore the superior performance of our findings.

1. Introduction and Main Results

According to well known inequality of Bernstein [2] on the derivative of a polynomial $P(z)$ of degree n, we have

$$\max_{|z|=1}|P'(z)| \leq n\max_{|z|=1}|P(z)|.$$

The result is best possible and equality holds for a polynomial having all its zeros at the origin.

P. Turan [14] showed that, if a polynomial $P(z)$ of degree n has all its zeros in $|z| \leq 1$, then

$$\max_{|z|=1}|P'(z)| \geq \frac{n}{2}\max_{|z|=1}|P(z)|. \tag{1.1}$$

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Aziz [1] considered the modulus of each zero of the underlying polynomial in the bound and generalized the inequality (1.1) to the class of polynomials having all their zeros in a closed disk of finite radius greater than or equal to unit length by proving that, if $P(z) = a_n \prod_{\nu=1}^{n}(z - z_\nu)$ is a complex polynomial of degree n with $|z_\nu| \leq K$, $K \geq 1$, then

$$\max_{|z|=1} |P'(z)| \geq \frac{2}{1+K^n} \sum_{\nu=1}^{n} \frac{K}{K+|z_\nu|} \max_{|z|=1} |P(z)|. \tag{1.2}$$

In this paper, we are interested in estimating the lower bound for the maximum modulus of $P'(z)$ on $|z| = 1$ for polynomials of degree n having all its zeros in $|z| \leq K$, $K \geq 1$ and establish some refinements and generalisations of the inequalities (1.1) and (1.2), by proving the following result.

Theorem 1.1. *If $P(z) = a_0 + a_1 z + ... + a_{n-1} z^{n-1} + a_n z^n = a_n \prod_{\nu=1}^{n}(z - z_\nu)$ is a polynomial of degree $n \geq 2$ which has all its zeros in the disk $|z| \leq K$, $K \geq 1$, then*

$$\max_{|z|=1} |P'(z)| \geq \frac{2}{1+K^n} \sum_{\nu=1}^{n} \frac{K}{K+|z_\nu|} \left(\max_{|z|=1} |P(z)| + \frac{|a_{n-1}|\phi(K)}{K} \right) + |a_1|\psi(K) \tag{1.3}$$

where $\phi(K) = \left(\frac{K^n-1}{n} - \frac{K^{n-2}-1}{n-2} \right)$ or $\frac{(K-1)^2}{2}$ and $\psi(K) = \left(1 - \frac{1}{K^2}\right)$ or $\left(1 - \frac{1}{K}\right)$ according as $n > 2$ or $n = 2$. The result is best possible and equality holds in (1.3) for the polynomial $P(z) = z^n + K^n$.

Remark 1.2. Since $\phi(K)$ and $\psi(K)$ are non-negative, it clearly follows that inequality (1.3) refines Aziz's inequality (1.2). Further for $K = 1$, inequality (1.3) reduces to Turan's inequality (1.1).

Theorem 1.1 in general provides much better information than Aziz's inequality (1.2) regarding $\max_{|z|=1} |P'(z)|$, in case when $P(z)$ has all its zeros in $|z| \leq K$, $K \geq 1$. We illustrate this with the help of following example.

Example 1.3. Consider

$$P(z) = z^3 - z^2 + z - 1$$

which is polynomial of degree 3. Clearly, $P(z)$ has all its zeros in $|z| \leq 1$. We take $K = 2$ and find that

$$\max_{|z|=1} |P(z)| = 4.$$

Also, for $K = 2$, we obtain

$$\phi(K) = 4/3 \quad \textit{and} \quad \psi(K) = 3/4.$$

From inequality (1.2), we find that

$$\max_{|z|=1} |P'(z)| \geq 1.33$$

and by Theorem 1.1, for this polynomial, we obtain

$$\max_{|z|=1} |P'(z)| \geq 2.30$$

which is much better than the bound given by Aziz's inequality (1.2).

Definition 1.4. Let $p(z)$ be a polynomial of degree n with complex coefficients and $\alpha \in \mathbb{C}$ be a complex number, then the polynomial

$$D_\alpha p(z) = np(z) + (\alpha - z)p'(z)$$

is called polar derivative of $p(z)$ with pole α. Note that $D_\alpha p(z)$ is a polynomial of degree $n-1$ and it is a generalisation of the ordinary derivative in the sense that

$$\lim_{\alpha\to\infty} \frac{D_\alpha p(z)}{\alpha} = p'(z)$$

uniformly with respect to z for $|z| \leq R$, $R > 0$. For more information on polar derivatives of polynomials one can refer to monographs by Rahman and Schmeisser [13] or Milovanović et al. [14].

Bernstein-type inequalities on complex polynomials have been extended extensively from 'ordinary derivative' to 'polar derivative' of complex polynomials. In this context it is quite natural to seek an extension of Theorem 1.1 involving ordinary derivative of a restricted polynomial to the one in more generalized form involving polar derivative of a polynomial with the same restrictions which is stated below.

Theorem 1.5. *If* $P(z) = a_0 + a_1 z + ... + a_{n-1}z^{n-1} + a_n z^n = a_n \prod_{\nu=1}^{n}(z - z_\nu)$ *is a polynomial of degree* n *which has all its zeros in the disk* $|z| \leq K$, $K \geq 1$, *then for any complex number* α *with* $|\alpha| \geq K$,

$$\max_{|z|=1} |D_\alpha P(z)| \geq \frac{2(|\alpha| - K)}{1+K^n} \sum_{\nu=1}^{n} \frac{K}{K+|z_\nu|} \left(\max_{|z|=1} |P(z)| + \frac{|a_{n-1}|\phi(K)}{K} \right)$$

$$+ |na_0 + \alpha a_1|\psi(K) \tag{1.4}$$

where $\phi(K)$ *and* $\psi(K)$ *are same as defined in Theorem 1.1.*

Remark 1.6. If we divide (1.4) by $|\alpha|$ and take $|\alpha| \to \infty$, we get inequality (1.3) and thus Theorem 1.5 contains Theorem 1.1.

2. Lemmas

The following result is due to Frappier, Rahman and Ruscheweyh [3].

Lemma 2.1. *If* $P(z)$ *is a polynomial of degree* $n \geq 1$, *then for* $R \geq 1$,

$$\max_{|z|=R} |P(z)| \leq R^n \max_{|z|=1} |P(z)| - (R^n - R^{n-2})|P(0)| \quad if \quad n > 1 \tag{1.5}$$

and

$$\max_{|z|=R} |P(z)| \leq R \max_{|z|=1} |P(z)| - (R-1)|P(0)| \quad if \quad n = 1. \tag{1.6}$$

From above lemma, N. A. Rather [15] deduced that

Lemma 2.2. *If $P(z) = a_0 + a_1 z + ... + a_{n-1}z^{n-1} + a_n z^n = a_n \prod_{\nu=1}^{n}(z - z_\nu)$ is a polynomial of degree $n \geq 2$, having no zero in $|z| < 1$, then for every $\alpha \in \mathbb{C}$ with $|\alpha| \leq 1$ and $R \geq 1$*

$$\max_{|z|=R}|P(z)| \leq \frac{R^n+1}{2}\max_{|z|=1}|P(z)| - |\alpha|\frac{R^n-1}{2}\min_{|z|=1}|P(z)| - \left(\frac{R^n-1}{n} - \frac{R^{n-2}-1}{n-2}\right)|P'(0)| \quad if \quad n > 2 \tag{1.7}$$

and

$$\max_{|z|=R}|P(z)| \leq \frac{R^2+1}{2}\max_{|z|=1}|P(z)| - |\alpha|\frac{R^2-1}{2}\min_{|z|=1}|P(z)| - \frac{(R-1)^2}{2}|P'(0)| \quad if \quad n = 2. \tag{1.8}$$

Lemma 2.3. *If $P(z) = a_0 + a_1 z + ... + a_{n-1}z^{n-1} + a_n z^n = a_n \prod_{\nu=1}^{n}(z - z_\nu)$ is a polynomial of degree $n > 2$, having all zeros in $|z| < K$, $K \geq 1$, then*

$$\max_{|z|=K}|P(z)| \geq \frac{2K^n}{1+K^n}\max_{|z|=1}|P(z)| + \frac{2K^{n-1}|a_{n-1}|}{1+K^n}\left(\frac{K^n-1}{n} - \frac{K^{n-2}-1}{n-2}\right).$$

Proof. Since all the zeros of polynomial $P(z)$ lie in $|z| < K$, $K \geq 1$, all the zeros of $g^*(z) = z^n\overline{g(1/\bar{z})} = z^n\overline{P(K/\bar{z})}$ lie in $|z| \geq 1$, applying (1.7) with $R \geq 1$ and $\alpha = 0$ to the polynomial $g^*(z)$, we get

$$\max_{|z|=K}|g^*(z)| \leq \frac{K^n+1}{n}\max_{|z|=1}|g^*(z)| - \frac{|a_{n-1}|}{K}\left(\frac{K^n-1}{n} - \frac{K^{n-2}-1}{n-2}\right) \quad if \quad n > 2.$$

That is,

$$K^n\max_{|z|=1}|P(z)| \leq \frac{K^n+1}{2}\max_{|z|=K}|P(z)| - |a_{n-1}|K^{n-1}\left(\frac{K^n-1}{n} - \frac{K^{n-2}-1}{n-2}\right) \quad if \quad n > 2$$

or equivalently, we have for $n > 2$,

$$\max_{|z|=K}|P(z)| \geq \frac{2K^n}{1+K^n}\max_{|z|=1}|P(z)| + \frac{2K^{n-1}|a_{n-1}|}{1+K^n}\left(\frac{K^n-1}{n} - \frac{K^{n-2}-1}{n-2}\right).$$

□

3. Proofs of Theorems

Proof of Theorem 1.1. Since $P(z)$ has all its zeros in $|z| \leq K$, $K \geq 1$, the polynomial $G(z) = P(Kz) = a_n K^n \prod_{\nu=1}^{n}\left(z - \frac{z_\nu}{K}\right)$ has all its zeros in the closed unit disk $|z| \leq 1$.

Since for all z on $|z| = 1$ for which $G(z) \neq 0$,

$$\frac{zG'(z)}{G(z)} = \sum_{\nu=1}^{n}\frac{z}{z - z_\nu/K},$$

we have

$$\Re\left(\frac{zG'(z)}{G(z)}\right) \geq \sum_{\nu=1}^{n} \frac{1}{1+|z_\nu/K|} = \sum_{\nu=1}^{n} \frac{K}{K+|z_\nu|}.$$

But then

$$\left|\frac{zG'(z)}{G(z)}\right| \geq \sum_{\nu=1}^{n} \frac{K}{K+|z_\nu|},$$

for all z on $|z|=1$ for which $G(z) \neq 0$. Therefore

$$\max_{|z|=1} |G'(z)| \geq \sum_{\nu=1}^{n} \frac{K}{K+|z_\nu|} \max_{|z|=1} |G(z)|, \tag{1.9}$$

which is equivalent to

$$K \max_{|z|=K} |P'(z)| \geq \sum_{\nu=1}^{n} \frac{K}{K+|z_\nu|} \max_{|z|=K} |P(z)|. \tag{1.10}$$

Since $P'(z)$ is a polynomial of degree $n-1$, by (1.5) with $R=K$, $K \geq 1$,

$$K^{n-1} \max_{|z|=1} |P'(z)| - (K^{n-1} - K^{n-3})|a_1| \geq \max_{|z|=K} |P'(z)| \quad if\ n > 2. \tag{1.11}$$

Combining this inequality with (1.10), we get for $n > 2$,

$$K^{n} \max_{|z|=1} |P'(z)| - (K^{n} - K^{n-2})|a_1| \geq \sum_{\nu=1}^{n} \frac{K}{K+|z_\nu|} \max_{|z|=K} |P(z)|. \tag{1.12}$$

Using Lemma 3 in (1.12), we get

$$\begin{aligned} K^{n} \max_{|z|=1} |P'(z)| - (K^{n} - K^{n-2})|a_1| &\geq \frac{2K^n}{1+K^n} \sum_{\nu=1}^{n} \frac{K}{K+|z_\nu|} \max_{|z|=1} |P(z)| \\ &\quad + 2\frac{K^{n-1}|a_{n-1}|}{K^n+1} \sum_{\nu=1}^{n} \frac{K}{K+|z_\nu|} \left(\frac{K^n-1}{n} - \frac{K^{n-2}-1}{n-2}\right), \end{aligned}$$

$$\begin{aligned} \max_{|z|=1} |P'(z)| &\geq \frac{2}{1+K^n} \sum_{\nu=1}^{n} \frac{K}{K+|z_\nu|} \max_{|z|=1} |P(z)| \\ &\quad + \left(1 - \frac{1}{K^2}\right)|a_1| + \frac{2|a_{n-1}|}{K(K^n+1)} \sum_{\nu=1}^{n} \frac{K}{K+|z_\nu|} \left(\frac{K^n-1}{n} - \frac{K^{n-2}-1}{n-2}\right) \end{aligned}$$

which proves inequality (1.3) for the case $n > 1$. For the case $n = 2$, the result follows on similar lines in view of part second of Lemma 1 and Lemma 2 with $\alpha = 0$. This completes the proof. □

Proof of Theorem 1.2. Since $P(z)$ has all its zeros in $|z| \leq K$, $K \geq 1$, the polynomial $G(z) = P(Kz)$ has all its zeros in the closed unit disk $|z| \leq 1$. Now therefore for $\alpha/K \geq 1$, it is easy to obtain

$$\max_{|z|=1} |D_{\alpha/K} G(z)| \geq \frac{(|\alpha| - K)}{K} \max_{|z|=1} |G'(z)|$$

which is nothing but

$$\max_{|z|=1} |nP(Kz) + (\alpha/K - z)KP'(Kz)| \geq \frac{(|\alpha| - K)}{K} \max_{|z|=1} |G'(z)|.$$

Using definition of polar derivative and the inequality (1.9), we have

$$\max_{|z|=K} |D_\alpha P(z)| \geq \frac{(|\alpha| - K)}{K} \sum_{\nu=1}^{n} \frac{K}{1 + |z_\nu|} \max_{|z|=1} |G(z)|$$

which is equivalent to

$$\max_{|z|=K} |D_\alpha P(z)| \geq \frac{(|\alpha| - K)}{K} \sum_{\nu=1}^{n} \frac{K}{K + |z_\nu|} \max_{|z|=K} |P(z)|. \tag{1.13}$$

Since $D_\alpha P(z)$ is a polynomial of degree $n-1$, by (1.5) of Lemma 1 with $R = K$, $K \geq 1$, we have

$$K^{n-1} \max_{|z|=1} |D_\alpha P(z)| - (K^{n-1} - K^{n-3})|na_0 + \alpha a_1| \geq \max_{|z|=K} |D_\alpha P(z)| \quad if n > 2.$$

Combining this inequality with (1.13), we get for $n > 2$

(1.14)

$$K^{n-1} \max_{|z|=1} |D_\alpha P(z)| - (K^{n-1} - K^{n-3})|na_0 + \alpha a_1| \geq \frac{(|\alpha| - K)}{K} \sum_{\nu=1}^{n} \frac{K}{K + |z_\nu|} \max_{|z|=K} |P(z)|.$$

Using Lemma 3 in (1.14), we get

$$\begin{aligned} K^{n-1} \max_{|z|=1} |D_\alpha P(z)| \geq \frac{(|\alpha| - K)}{K} \sum_{\nu=1}^{n} \frac{K}{K + |z_\nu|} \\ + \left[\frac{2K^n}{1 + K^n} \max_{|z|=1} |P(z)| + \frac{2K^n |a_{n-1}|}{1 + K^n} \left(\frac{K^n - 1}{n} - \frac{K^n - 2}{n - 2} \right) \right] \\ + (K^{n-1} - K^{n-3})|na_0 + \alpha a_1| \end{aligned}$$

which implies

$$\begin{aligned} \max_{|z|=1} |D_\alpha P(z)| \geq \frac{2(|\alpha| - K)}{1 + K^n} \sum_{\nu=1}^{n} \frac{K}{K + |z_\nu|} \left(\max_{|z|=1} |P(z)| + \frac{|a_{n-1}|\phi(K)}{K} \right) \\ + |na_0 + \alpha a_1| \psi(K). \end{aligned}$$

□

This completes the proof.

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